Approximation Algorithms
A Survey

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Discrete Optimization '99
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Approximation Algorithms Since 1977

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Recent Techniques in Approximation Algorithms

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Three Recent Techniques in Approximation Algorithms

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Two Recent Techniques in Approximation Algorithms

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What...?

An α-approximation algorithm is a polynomial-time algorithm which always produces a solution of value within α times the value of an optimal solution.

α is called the performance guarantee of the algorithm: for min problems, $\alpha > 1$, and for max $\alpha < 1$.

If the algorithm is randomized, it must give a solution whose expected cost is within α of optimal.

Why...?

One possible answer to "how do we approach NP-hard optimization problems?"

There are two possible answers:

1. Use algorithms that are not provably efficient.
   - Branch-and-bound, branch-and-cut
2. Use algorithms that are not provably optimal.
   - Local search (including simulated annealing, tabu search, etc.)
   - Approximation algorithms

Approximation algorithms have been around since Vizing (1964) and Graham (1966), but in the past decade there has been a great deal of research, and two different strands have converged.

- **Complexity** Complexity theorists have developed powerful tools for showing that no α-approximation algorithms can exist unless $P = NP$.

- **Algorithms** Algorithm designers have developed techniques that apply to a wide range of problems.

Example 1: TSP

Christofides (1976) gives a $\frac{3}{2}$-approximation algorithm for the Metric TSP (i.e. edge costs obey the triangle inequality).

Theorem: (Arora, Lund, Motwani, Sudan, Szegedy 1992): There exists some $\rho > 1$ such that there is no ρ-approximation algorithm for the Metric TSP unless P=NP.

Arora (1996) and Mitchell (1996) give an approximation scheme for the Euclidean TSP (i.e. for any $\epsilon > 0$, there exists a $(1 + \epsilon)$-approximation algorithm.)

Example 2: Set cover

Johnson (1972), Chvátal (1979) give a $\ln n$-approximation algorithm for the Set Cover problem.

Theorem (Lund, Yannakakis; Feige 1996): There exists no $c \ln n$-approximation algorithm for Set Cover for $c < 1$ unless $NP \subseteq DTIME(n^{\log^k n})$.

Example 3: MAX CUT

Goemans, W '94 give a .878-approximation algorithm for MAX CUT (finding the largest cut in a graph).

Håstad '97 shows that there can be no .941-approximation algorithm for MAX CUT unless $P = NP$.

Example 4: MAX 3SAT

Johnson (1972) gives a $\frac{7}{8}$-approximation algorithm for satisfying the maximum number of clauses of a 3SAT formula (MAX E3SAT) when each clause has exactly three literals.

Theorem (Håstad 1997) There exists no $(\frac{7}{8} + \epsilon)$-approximation algorithm for MAX E3SAT unless $P = NP$.

Karloff and Zwick (1997) give a $(\frac{7}{8} - \epsilon)$-approximation algorithm for MAX 3SAT (when clauses may have 1-3 literals).

Two techniques

I. The Primal-Dual Method

II. Semidefinite Programming

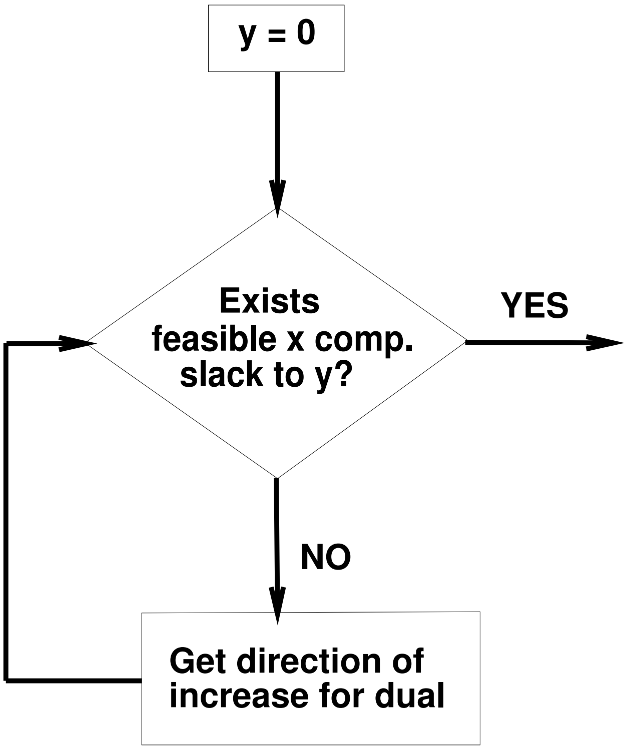
The Primal-Dual Method

$$
\begin{array}{lll}
\text{Min} & \displaystyle\sum_i c_i x_i & \\
& \displaystyle\sum_i a_{ij} x_i \geq b_j & \forall j \\
& x_i \in \{0, 1\} & \forall i
\end{array}
$$

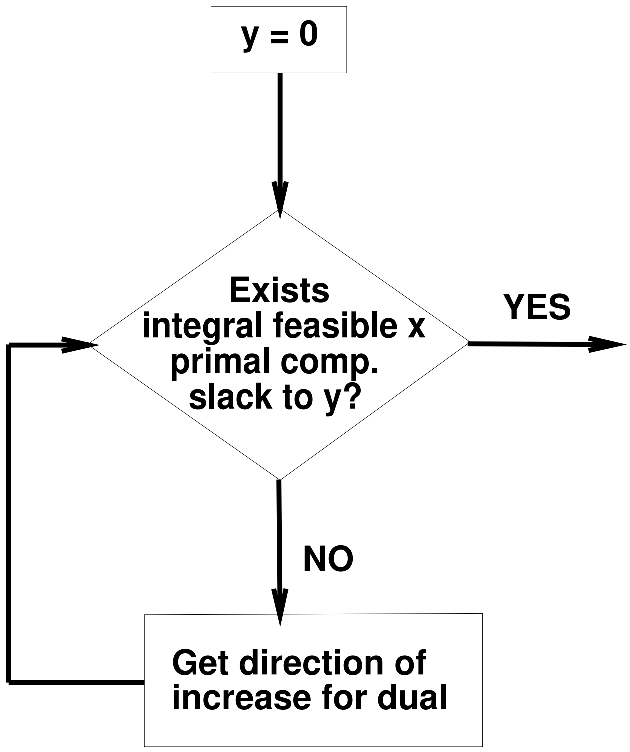
$$
\begin{array}{lll}
\text{Max} & \displaystyle\sum_j b_j y_j & \\
& \displaystyle\sum_j a_{ij} y_j \leq c_i & \forall i \\
& y_j \geq 0 & \forall j
\end{array}
$$

The primal-dual method

The standard primal-dual method for solving combinatorial optimization problems.



The Main Idea for primal-dual approximation algorithms: relax dual complementary slackness conditions (e.g. $y_j > 0 \Rightarrow \sum_i a_{ij} x_i = b_j$).



Choosing the right direction for dual increase allows for proof that the integer solution found is close to the dual solution constructed.

See Goemans & W '97, Bertsimas & Teo '95, Teo '97 for nice surveys.

$\underline{\alpha = 1}$

Min-cost spanning tree	Kruskal '56
Shortest s-t path	Dijkstra '59
Min-cost branching	Edmonds '67

$\underline{\alpha = 2}$

Vertex cover	Bar-Yehuda, Even '81
Steiner tree	various
Generalized Steiner tree	Agrawal, Klein, Ravi '95
Matching (w/ triangle inequality)	Goemans, W '95
T-joins	⋮
Non-fixed point-to-point connection	
Partition into trees/paths/cycles of at least k vertices	
Location/design problems	
Prize-collecting TSP	
⋮	
Multicuts in trees	Garg, Vazirani, Yannakakis '93
Feedback vertex sets in undirected graphs	Becker, Geiger '94 Bafna, Berman, Fujito '95

Other α

Survivable network design	W, Goemans, Mihail, Vazirani '93
	Goemans, Goldberg, Plotkin, Shmoys, Tardos, W '94
Intersecting contra-polymatroids	Bertsimas, Teo '95
Feedback vertex sets in planar graphs	Goemans, W '96
Steiner trees in special graphs	Rajagopalan, Vazirani '98
Uncapacitated facility location	Jain, Vazirani '99
k-median	Jain, Vazirani '99
	Guha, Charikar '99

Experimental studies

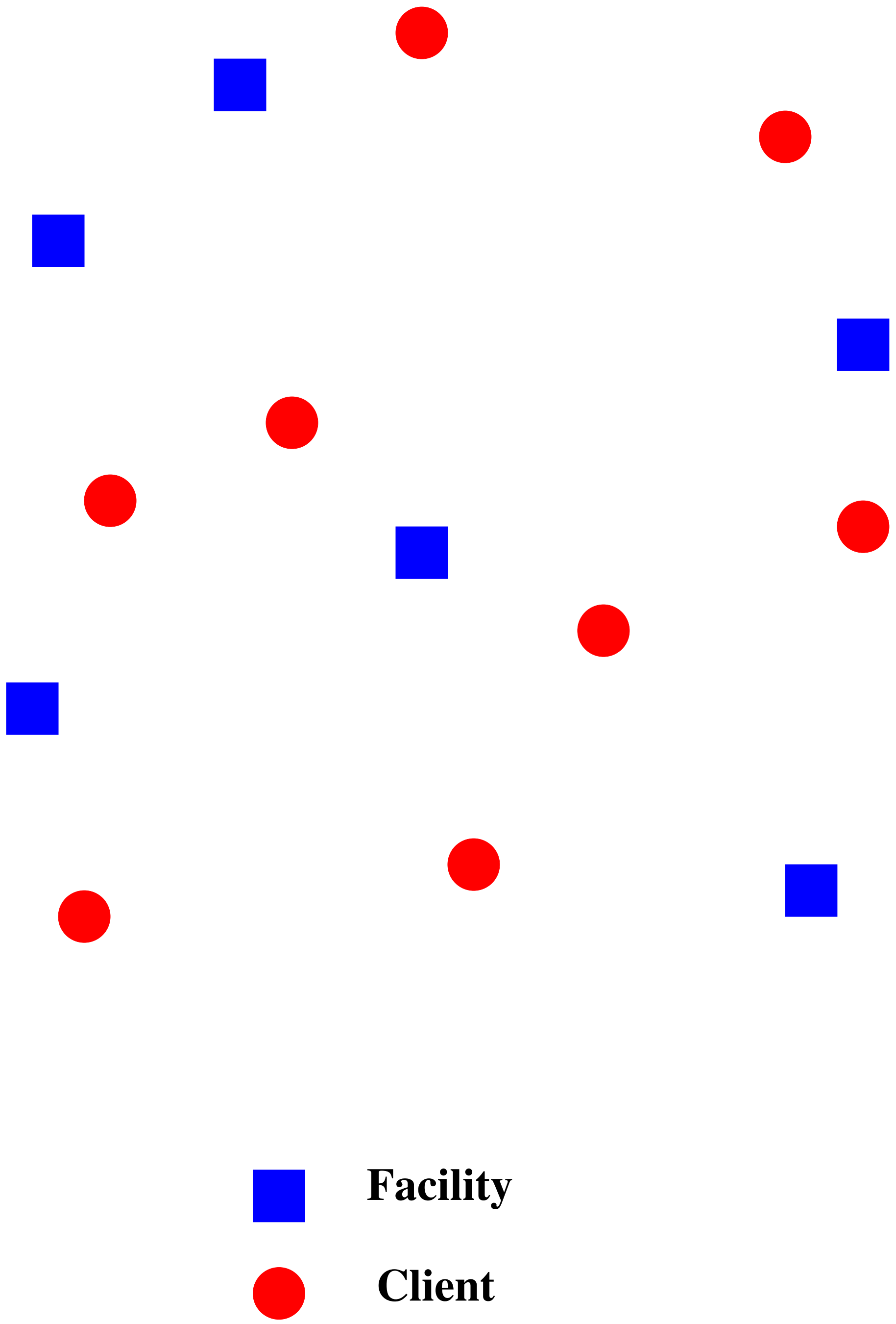
Euclidean matching	Goemans, W '94
Steiner trees	Hall '95
Generalized Steiner trees	Hu & Wein
Network design problems	Mihail, Mostrel, Dean, Shallcross '96
Feedback vertex set	Becker & Geiger '96
Prize-collecting Steiner tree	Johnson, Minkoff, Phillips '99

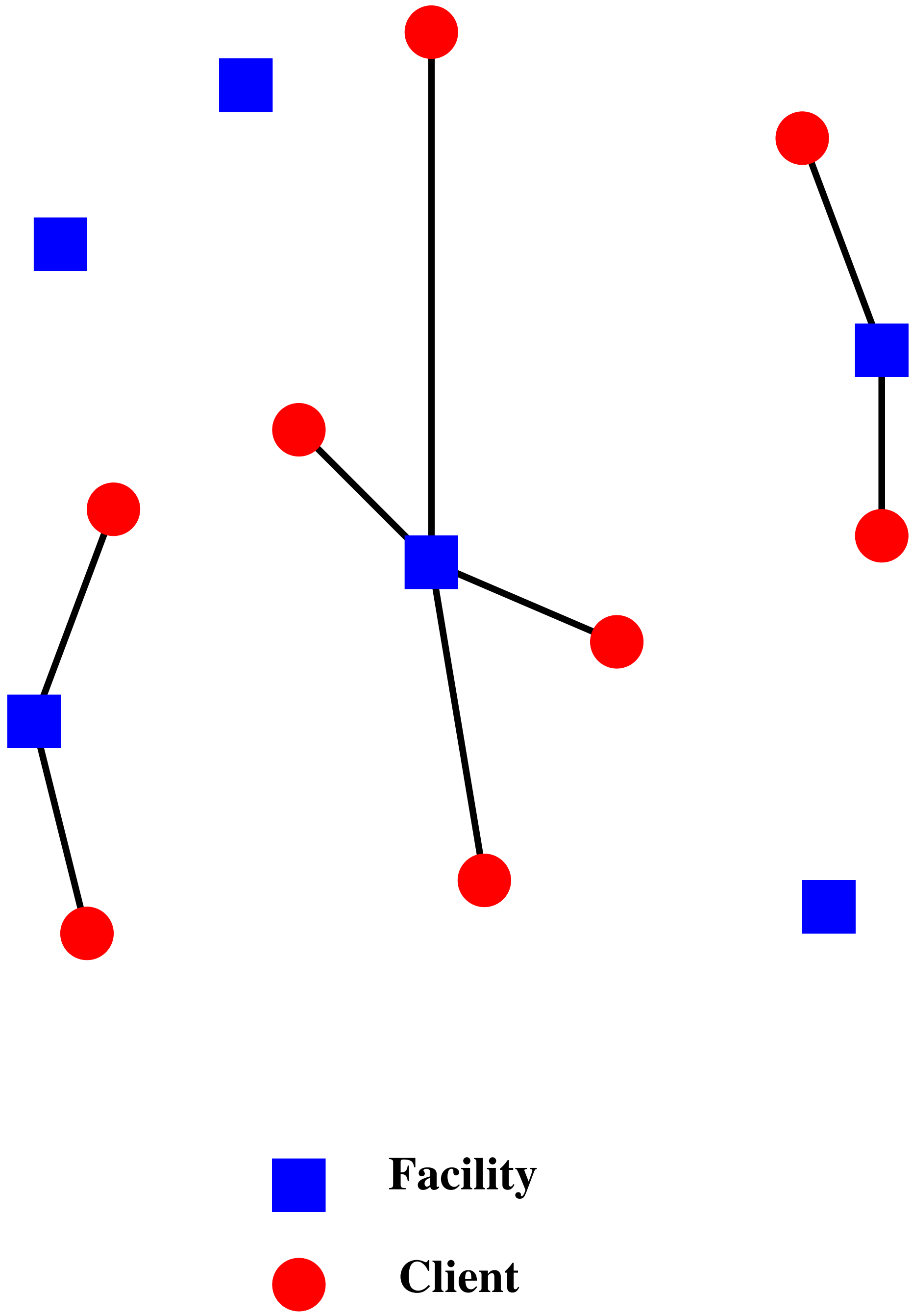
Example: Uncapacitated Facility Location Problem

- **Input:**
  - A set of locations V.
  - Facilities $F \subset V$, and cost f_i for $i \in F$.
  - Clients $D = V - F$, and cost c_{ij} for assigning client $j \in D$ to facility $i \in F$.
  - Edge costs obey the triangle inequality.

- **Output:** A set of facilities F' and an assignment of clients to facilities in F' that minimizes total cost (Facility + Assignment).

- NP-hard (Cornuéjols, Nemhauser, and Wolsey '90).


Facility
Client


Facility
Client

An integer programming formulation

A formulation of the problem (Balinski 1966):

$$
\begin{array}{lll}
\text{Min} & \displaystyle\sum_{i \in F} f_i y_i + \sum_{i \in F, j \in D} c_{ij} x_{ij} & \\
& \displaystyle\sum_i x_{ij} = 1 & \forall j \in D, \\
& x_{ij} \leq y_i & \forall i \in F, j \in D, \\
& y_i \in \{0,1\} & \forall i \in F, \\
& x_{ij} \in \{0,1\} & \forall i \in F, j \in D,
\end{array}
$$

where

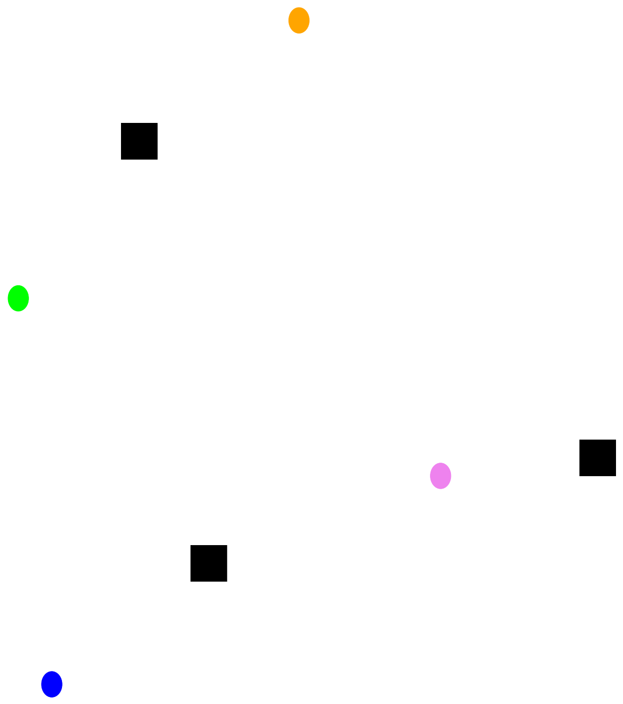
- $y_i = 1$ iff facility i is opened;
- $x_{ij} = 1$ iff client j is assigned to facility i.

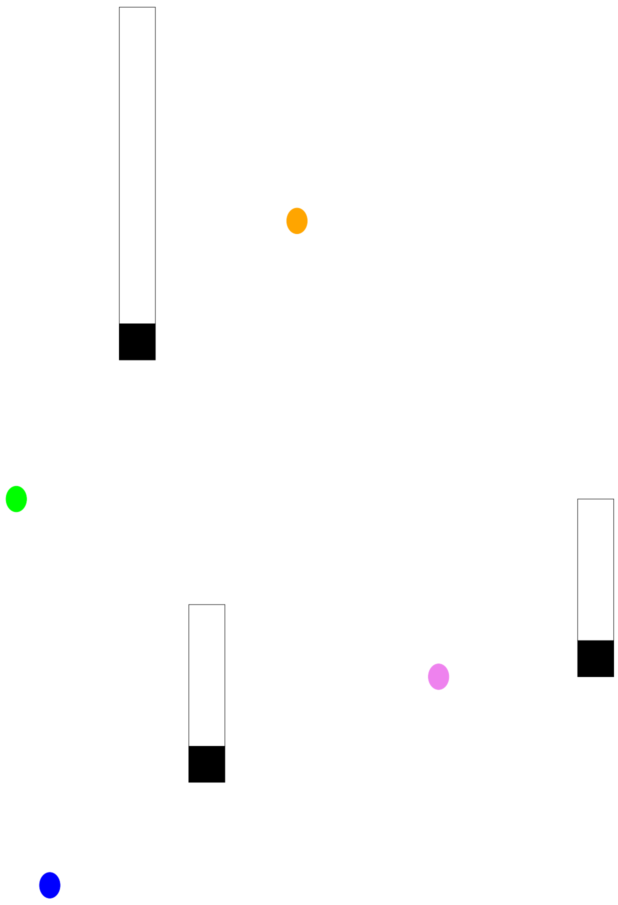
Dual and Algorithm

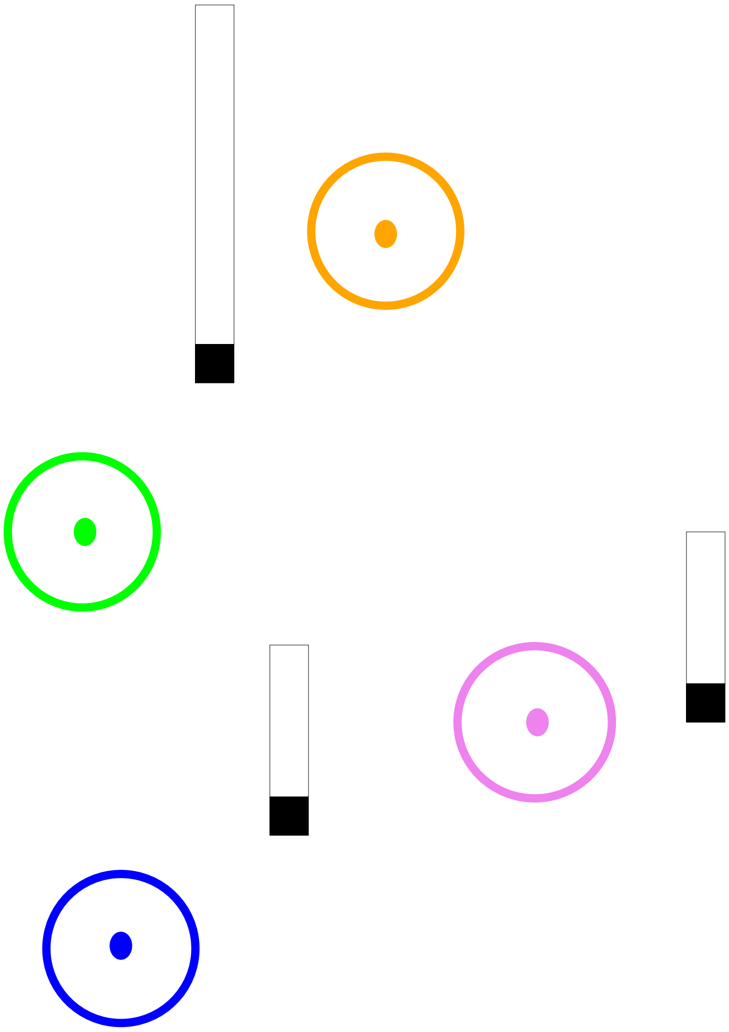
$$
(D) \qquad
\begin{array}{lll}
\text{Max} & \displaystyle\sum_{j \in D} v_j & \\
& \displaystyle\sum_{j \in D} w_{ij} \leq f_i & \forall i \in F, \\
& v_j - w_{ij} \leq c_{ij} & \forall i \in F, j \in D, \\
& w_{ij} \geq 0 & \forall i \in F, j \in D.
\end{array}
$$

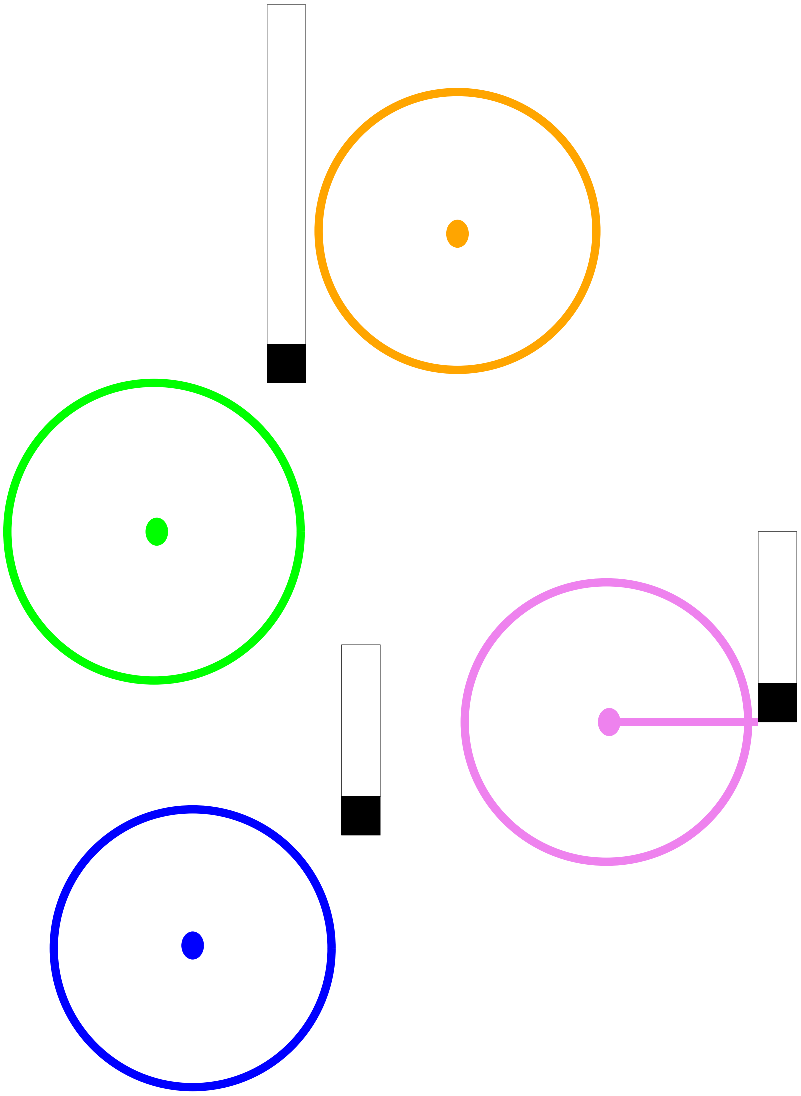
1. Set $v \leftarrow 0$, $w \leftarrow 0$
2. Increase v_j variables uniformly and w_{ij} as necessary, until cannot increase without violating constraints.
3. Open a subset of the facilities i such that $\sum_{j \in D} w_{ij} = f_i$; assign clients to facilities.

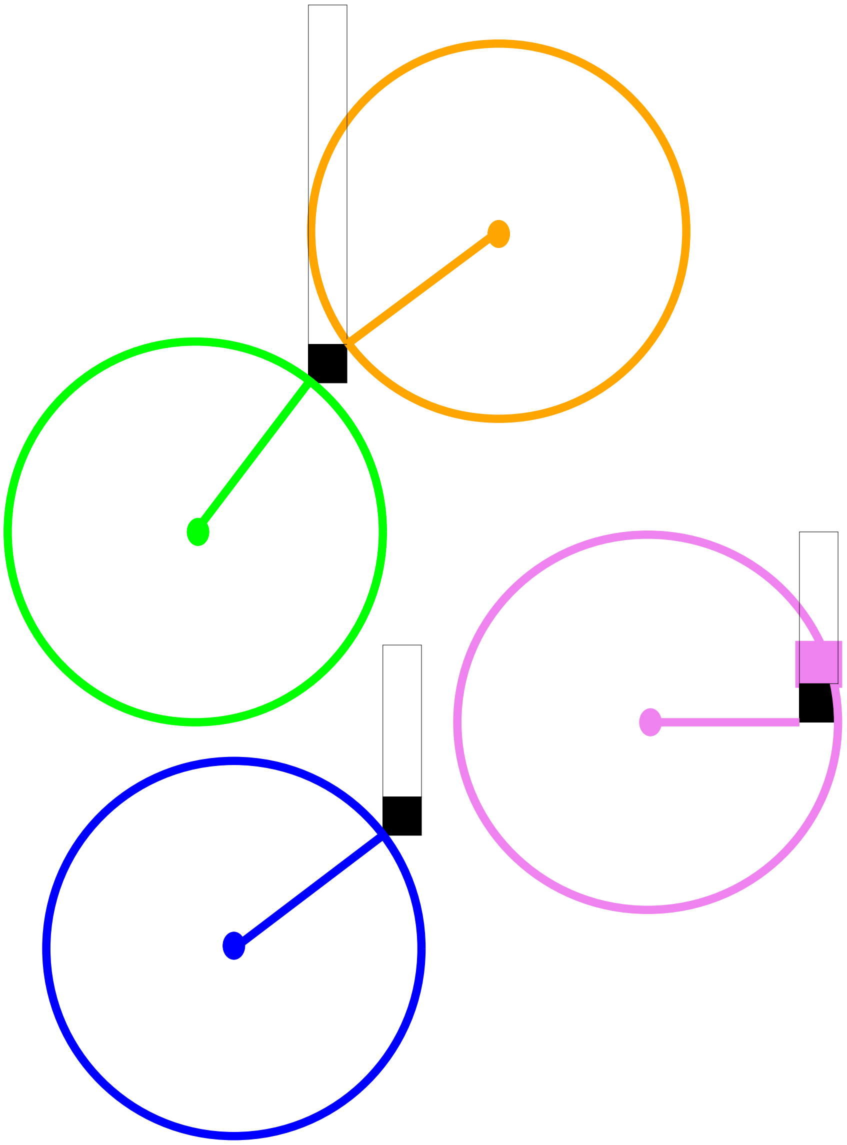
(Jain, Vazirani '99; compare to Erlenkotter '78)

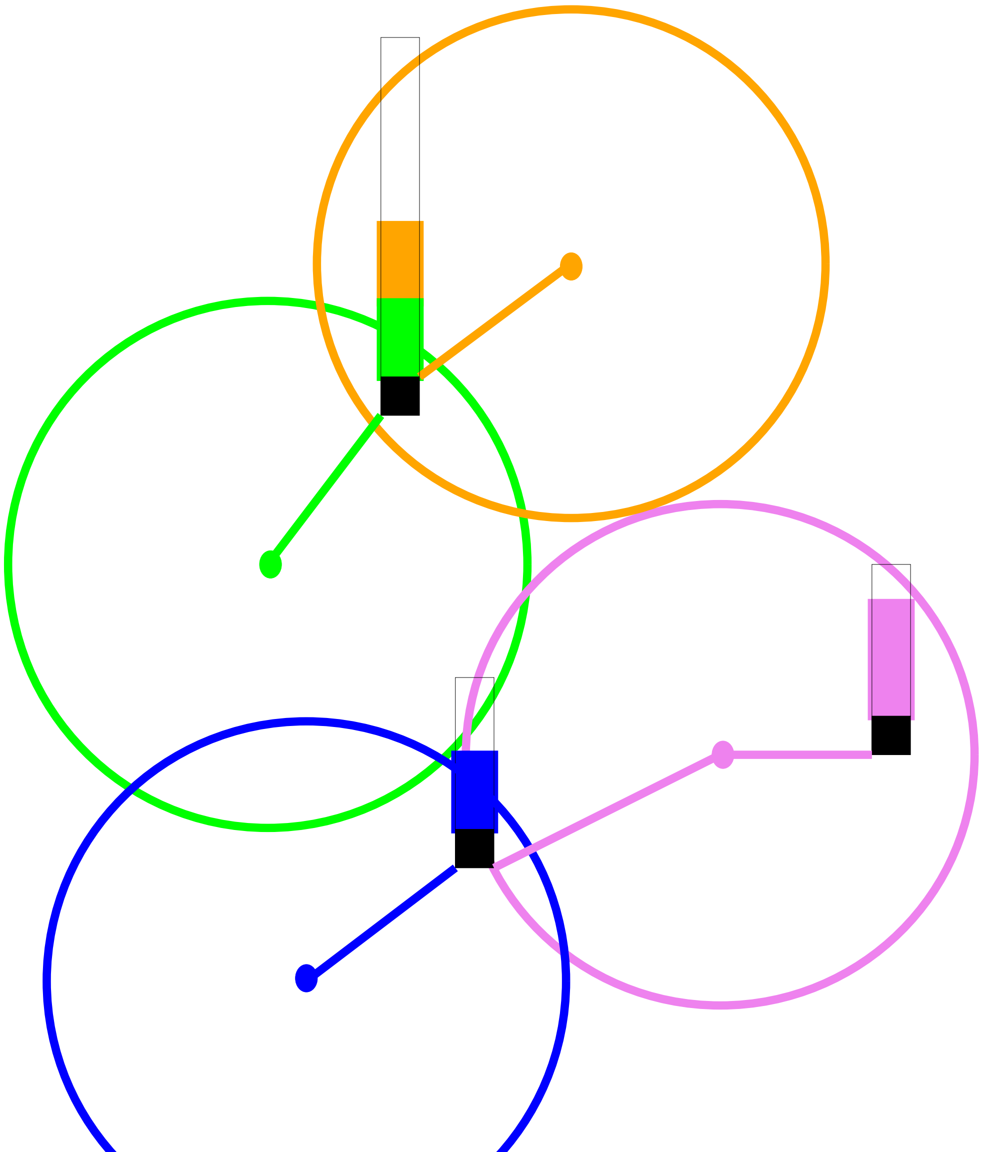


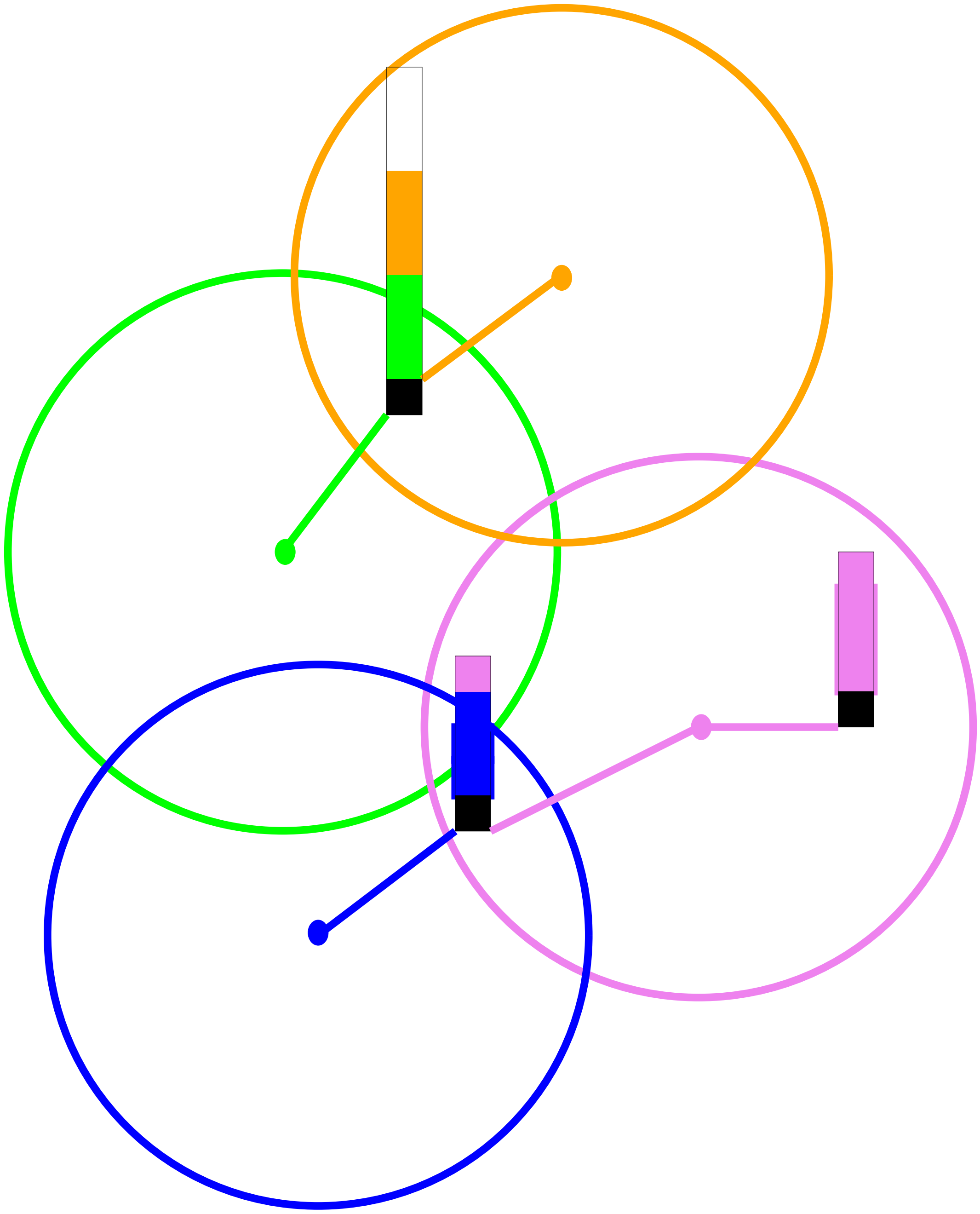


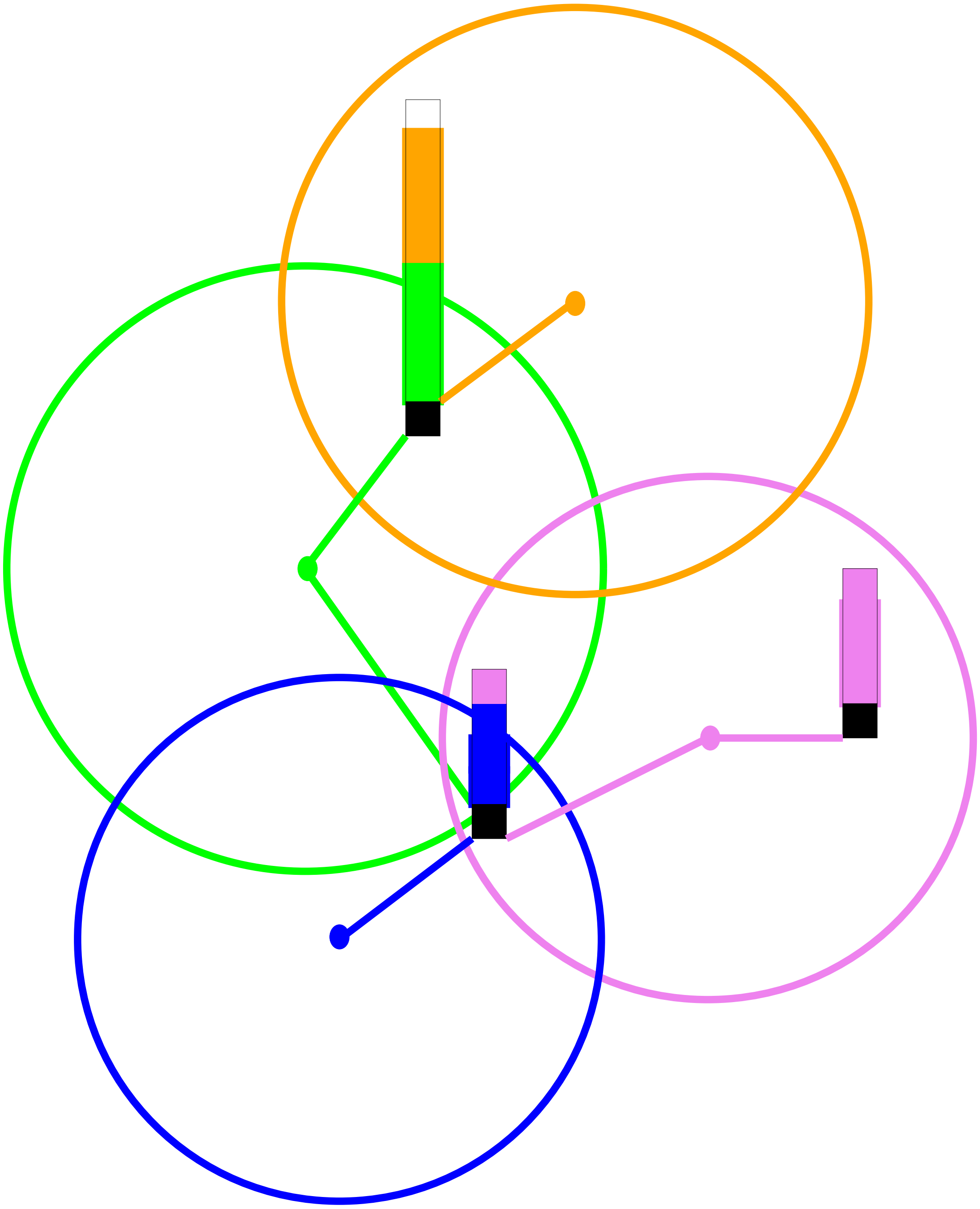


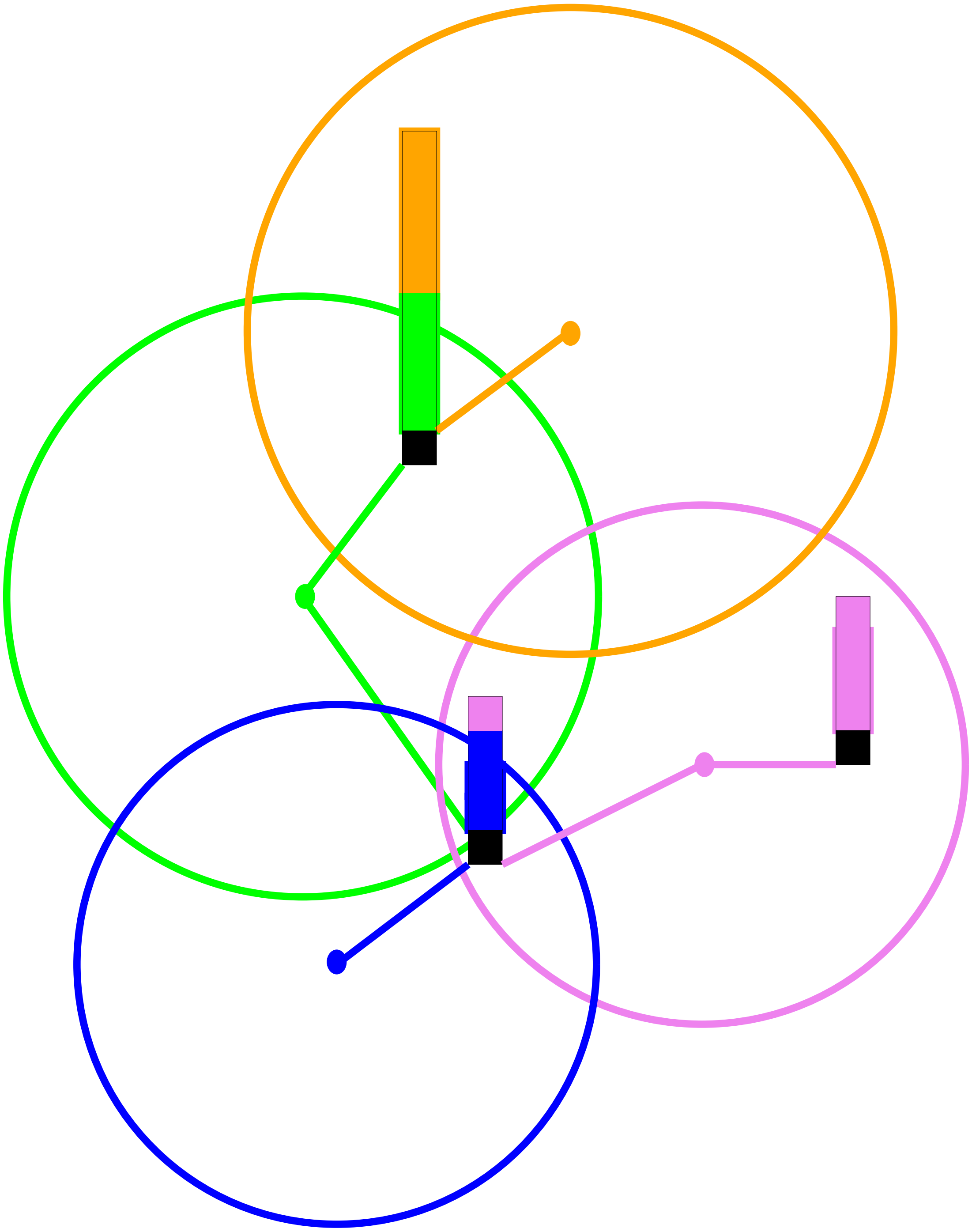












3. Consider facilities i with $\sum_{j \in D} w_{ij} = f_i$ in order inequalities became tight. Open i if no neighboring facility of a neighboring client is open.

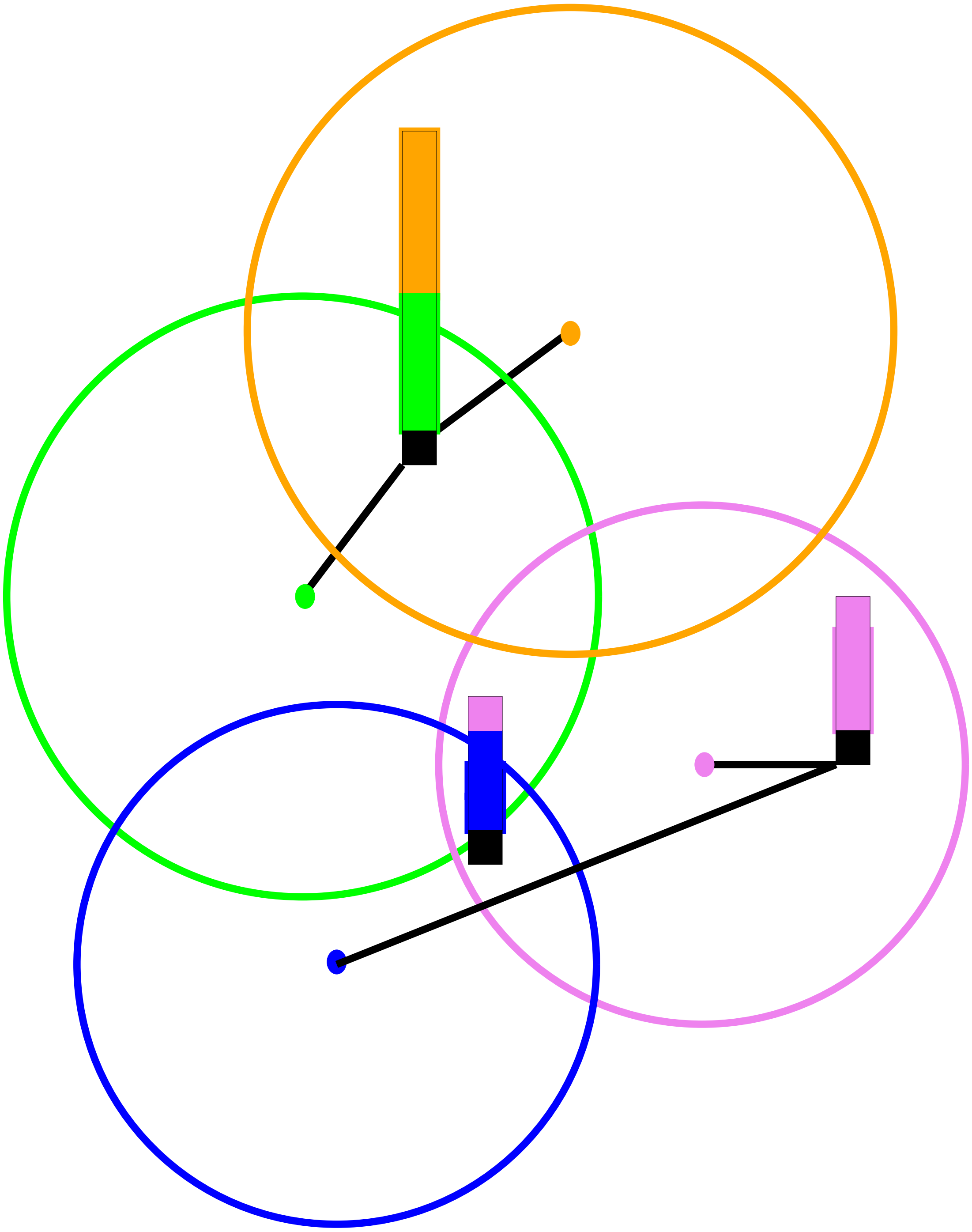
   Assign client j to any open neighbor, or to the first opened neighbor of a neighbor of a neighbor.

Theorem (Jain, Vazirani '99)
For this integer solution $(\bar{x}, \bar{y})$,

$$\begin{aligned} \sum_{i \in F, j \in D} c_{ij}\bar{x}_{ij} + \sum_i f_i \bar{y}_i &\leq 3 \sum_j v_j \\ &\leq 3 \cdot OPT. \end{aligned}$$

Using different technique, get 1.728-approx. alg. (Chudak, Shmoys '98; Guha, Charikar '99)

More complicated applications possible?



Semidefinite programming

A semidefinite program has the form

$$\begin{array}{lll} \text{Max/Min} & C \bullet X & \\ & A_k \bullet X = b_k & \forall k \\ & X \succeq 0 & X \text{ symmetric.} \end{array}$$

where $A \bullet B = \sum_{i,j} a_{ij} b_{ij}$ for $A, B \in \Re^{n \times n}$, and $A \succeq B$ iff $A - B$ is positive semidefinite.

Definition: A matrix $X \in \Re^{n \times n}$ is positive semidefinite iff $a^T X a \geq 0$ for all $a \in \Re^n$.

Useful facts

The following three statements are equivalent for a symmetric matrix X:

1. X is positive semidefinite;
2. X has only non-negative eigenvalues;
3. $X = V^T V$ for some matrix $V \in \Re^{m \times n}$.

Semidefinite programming (SDP) is polynomial-time solvable to within an additive error of ϵ using either:

1. The ellipsoid method: if v is eigenvector of negative eigenvalue, then $v^T X v \geq 0$ is a separating hyperplane. (Grötschel, Lovász, Schrijver '88).
2. Interior-point methods (Nesterov & Nemirovskii; Alizadeh '95)

Using these facts, semidefinite programming is equivalent to a kind of "vector programming".

$$
\begin{array}{lll}
\text{Max/Min} & C \bullet X & \\
 & A_k \bullet X = b_k & \forall k \\
 & X \succeq 0 & X \text{ symm}
\end{array}
$$

is equivalent to

$$
\begin{array}{lll}
\text{Max/Min} & \sum_{i,j} c_{ij}(v_i \cdot v_j) & \\
 & \sum_{i,j} a_{ijk}(v_i \cdot v_j) = b_k & \forall k \\
 & v_i \in \Re^n &
\end{array}
$$

since $X = V^T V$, v_i the ith column of V iff X is symmetric, psd.

The technique

Model problem of interest as a binary quadratic program:

$$\text{Max} \quad \sum_{i,j} a_{ij} x_i x_j$$
$$x \in \{-1, 1\}^n.$$

Relax to a vector program:

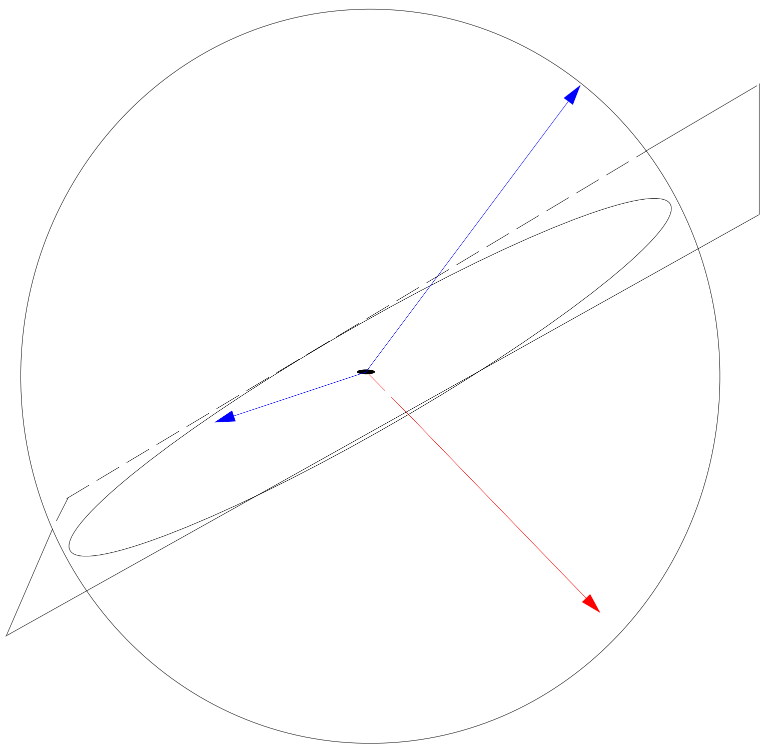
$$\text{Max} \quad \sum_{i,j} a_{ij} (v_i \cdot v_j)$$
$$v_i \cdot v_i = 1$$
$$v_i \in \Re^n$$

Choose random vector r uniformly from unit sphere S^{n-1}. Get solution

$$\bar{x}_i = \begin{cases} 1 & \text{if } v_i \cdot r \geq 0 \\ -1 & \text{if } v_i \cdot r < 0 \end{cases}$$

Geometric interpretation

Vector program is a relaxation of the case where x_i are one-dimensional unit vectors to n dimensions. Choosing random r is the same as choosing a random hyperplane.



Applications

Maximum cut	Goemans, W '94
Maximum directed cut	Goemans, W '94
	Feige, Goemans '94
MAX 2SAT	Goemans, W '94
	Feige, Goemans '94
MAX SAT	Goemans, W '94
	Asano, Ono, Hirata '96
	Asano '97
	Zwick '99
	Asano, W '99
Maximum k-cut	Frieze, Jerrum '94
Maximum bisection	Frieze, Jerrum '94
	Ye '99
Minimum k-coloring	Karger, Motwani, Sudan '95
Betweeness	Chor, Sudan '95
Quadratic programming	Nesterov '97
	Ye '97,'98
	Zhang '98

Applications

MAX 3SAT	Trevisan, Sorkin, Sudan, W '96
	Karloff, Zwick '97
MAX 3CSP	Zwick '98
Scheduling	Skutella '98
Linear Eqs mod p	Andersson, Engebretsen, Håstad '98
Minimum Bandwidth	Blum, Konjevod, Ravi, Vempala '98
MAX 4SAT	Halperin, Zwick '99

<u>Experimental Studies</u>

Maximum cut	Goemans, W '95
	Homer, Peinado '95
	Berry, Goldberg '95
	Benson, Ye, Zhang '97
	Burer, Monteiro '98

Quadratic programming

Nesterov '97 gives a $\frac{2}{\pi}$-approximation algorithm to the following quadratic programming problem with $A = (a_{ij})$ is symmetric, psd.

$$\begin{aligned} \text{Max} \quad & \sum_{i,j} a_{ij} x_i x_j \\ & x_i \in \{-1, 1\} \qquad\qquad \forall i \end{aligned}$$

Note that

$$\sum_{i,j} a_{ij} x_i x_j = x^T A x \geq 0.$$

Relaxation

Consider the vector programming relaxation

$$
\begin{array}{lll}
\text{Max} & \sum_{i,j} a_{ij}(v_i \cdot v_j) & \\
 & v_i \cdot v_i = 1 & \forall i, \\
 & v_i \in \Re^n, &
\end{array}
$$

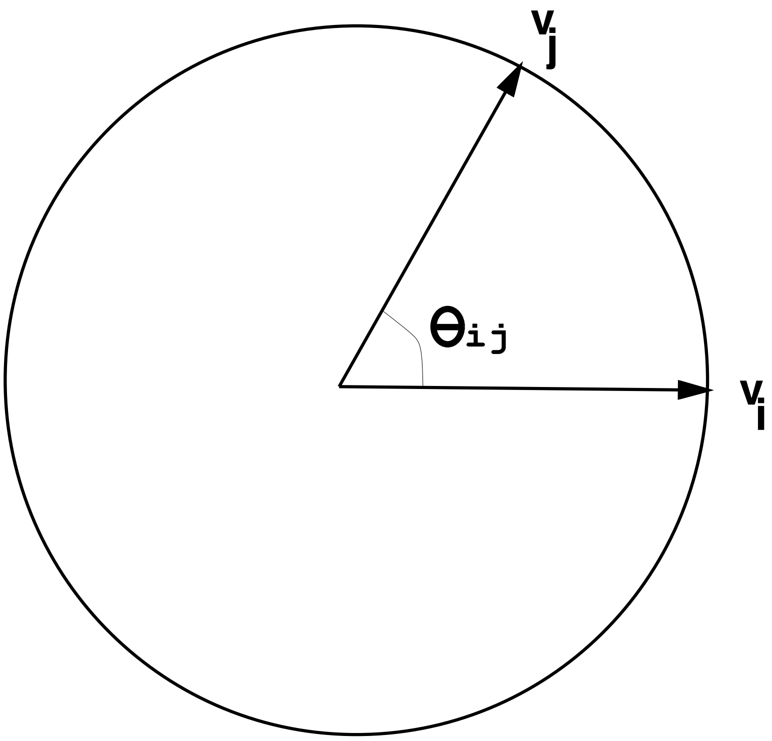
or, equivalently,

$$
\begin{array}{lll}
\text{Max} & A \bullet X & \\
 & x_{ii} = 1 & \forall i, \\
 & X \succeq 0. &
\end{array}
$$

Let X^* and v_i^* denote the optimal solution to the relaxation, and obtain the solution $\bar{x}$ as indicated.

Lemma 1

$$\Pr[\bar{x}_i \bar{x}_j = -1] = \frac{1}{\pi}\theta_{ij} = \frac{1}{\pi}\arccos(v_i^* \cdot v_j^*) = \frac{1}{\pi}\arccos(x_{ij}^*)$$



Analysis

Then

$$
\begin{aligned}
E[\bar{x}_i \bar{x}_j] &= \left[\left(1 - \frac{1}{\pi}\arccos(x^*_{ij})\right) - \frac{1}{\pi}\arccos(x^*_{ij})\right] \\
&= \left[1 - \frac{2}{\pi}\arccos(x^*_{ij})\right] \\
&= \left[1 - \frac{2}{\pi}\left(\frac{\pi}{2} - \arcsin(x^*_{ij})\right)\right] \\
&= \frac{2}{\pi}\arcsin(x^*_{ij}),
\end{aligned}
$$

using the fact that $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$.

Analysis cont.

Let $\arcsin[X] = (\arcsin(x_{ij}))$ and $\bar{X} = (\bar{x}_i \bar{x}_j)$.

The expected value of our solution is

$$E[\sum_{ij} a_{ij} \bar{x}_i \bar{x}_j] = \frac{2}{\pi} \sum_{ij} a_{ij} \arcsin(x^*_{ij}),$$

or

$$E[A \bullet \bar{X}] = \frac{2}{\pi} A \bullet \arcsin[X^*].$$

The value of the SDP relaxation is

$$\sum_{ij} a_{ij} x^*_{ij} = A \bullet X^* \geq OPT.$$

If

$$E[A \bullet \bar{X}] - \frac{2}{\pi} A \bullet X^* \geq 0,$$

then have a $\frac{2}{\pi}$-approximation algorithm.

Analysis cont.

Lemma $\arcsin[X] - X \succeq 0$ if $|x_{ij}| \leq 1$ for all i, j.

Fact If $C \succeq 0$, $D \succeq 0$, then $C \bullet D \geq 0$.

$$\begin{aligned}
E[A \bullet \bar{X}] - \frac{2}{\pi} A \bullet X^* &= \frac{2}{\pi}(A \bullet \arcsin[X^*] - A \bullet X^*) \\
&= \frac{2}{\pi} A \bullet (\arcsin[X^*] - X^*) \\
&\geq 0,
\end{aligned}$$

it follows that

$$E[A \bullet \bar{X}] \geq \frac{2}{\pi} A \bullet X^* \geq \frac{2}{\pi} OPT.$$

Other techniques: LP rounding

Idea: Solve LP relaxation of IP formulation of the problem. Round all decision vars above a certain threshold to the nearest integer. Repeat on remaining subproblem as necessary.
(Hochbaum '82)

Application: Survivable network design.
Jain '98 shows that at least one variable x_e in the LP relaxation has $x_e \geq 1/2$, implies a 2-approximation algorithm.

Other techniques: Randomized rounding

Idea: Solve LP relaxation of IP formulation of the problem. Create integer solution $\bar{x}$ from LP soln. x^* by setting $\bar{x}_i = 1$ with probability x_i^*.
(Raghavan, Thompson '87)

Application: Uncapacitated facility location.
Chudak and Shmoys '98 show that interleaving randomized rounding on facility vars. with a decomposition technique gives a performance guarantee of $1 + \frac{2}{e} \approx 1.736$.

“New” technique: Local search?

Idea: Apply local search techniques.

Application: Uncapacitated/capacitated facility location. Add/drop/swap facilities as it improves the cost of the solution.

- Guha, Charikar '99 show that this gives performance guarantee of 3 for uncapacitated facility location.

- Korupolu, Plaxton, Rajaraman '98 show that this gives performance guarantee of 8 for capacitated facility location (improved to 6 in Chudak, W '99).

Question: What's the lower bound?