Monomial ideals and the Scarf complex for coherent systems in reliability theory

Beatrice Giglio and Henry P. Wynn

Department of Statistics, University of Warwick, Coventry CV4 7AL (UK)

Abstract

The study of network reliability has received increasing attention in the recent decades because of its applications to computer networks and communications systems. In a monograph (*Network Reliability and Algebraic Structures*, 1991) Shier points out that one of the most interesting aspects of this subject is the "variety of discrete, combinatorial, and algebraic mathematics that can be found lurking just underneath this practical veneer. [...] The new approaches developed in the context of network reliability can have ramifications beyond that particular venue". In the present paper, the authors investigate a strong link between reliability theory and certain algebraic structures which we believe will eventually unify much of the work in the most recent literature. The two particular objects covered in this paper are the Scarf complex and the finely graded Hilbert Series.

Keywords: Network Reliability, Coherent Systems, Monomial Ideals, Scarf Complex.

1 Introduction

The study of network reliability has received increasing attention in the recent decades because of its applications to computer networks and communications systems. In a monograph ([16]) Shier points out that one of the most interesting aspects of this subject is the "variety of discrete, combinatorial, and algebraic mathematics that can be found lurking just underneath this practical veneer. [...] The new approaches developed in the context of network reliability can have ramifications beyond that particular venue".

In Naiman and Wynn ([14]) the authors established a connection between reliability structures and some special topological objects, called abstract tubes. This study lead to a second paper (Giglio, Naiman and Wynn, [9]) where the interactions with computational algebra and Gröbner Bases were investigated. This was particularly benefited by the work of Diaconis and Sturmfels that initiated the use of algebraic techniques in probability and statistics ([6]). A section on reliability was also included in a monograph ([15], Section 4.4) devoted to establish or deepen the numerous links existing between algebra and statistics. The work of Dohmen (see, among others, [4], [5] and [7]) concentrates more on the combinatorial approach but many fruitful and interesting synergies with the abstract tube theory and the algebraic approach are foreseen.

In the present paper, the authors investigate a strong link between reliability theory and certain algebraic structures which we believe will eventually unify much of the work just mentioned. The two particular objects covered in this paper are the Scarf complex and the finely graded Hilbert Series. These can expose more efficient computations and identities, similar to those in [14] and [9].

The main point of contact between the disciplines of algebra and reliability theory is an echelon grid of points with positive and integer coordinates. Such a grid (or lattice) can identify a coherent reliability system but also plays a specific role as an algebraic object.

A reliability system of d components is a system whose failure or non failure state is determined by the state of each of its components. We deal here with the case where each component can assume a discrete and possibly infinite number of states $\{0, 1, 2, 3, \ldots\}$ corresponding to increasing levels of efficiency. Section 4.4 deals with the case of a continuous distribution on the component performance and more formal definitions about reliability systems are given and discussed in Section 4. A state of the given system can thus be described by a non negative integer vector of length d. The

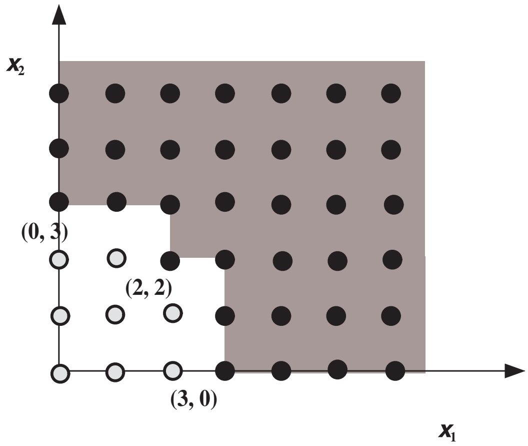

Figure 1: Monomials in $M = \langle x_1^3, x_1^2x_2^2, x_2^3 \rangle$ (black dots) and monomials NOT in M (hollow dots)

state space $\mathcal{D}$ ($\mathcal{D} = \mathbb{N}^d$ in this general setting) can therefore be decomposed into the set of configurations that correspond to failure of the system and the set of configurations that correspond to non failure. We indicate these two sets with $\mathcal{F}$ and $\bar{\mathcal{F}}$ respectively. Obviously it is $\mathcal{D} = \mathcal{F} \cup \bar{\mathcal{F}}$ (that is, $\mathcal{F}$ is the complement of $\bar{\mathcal{F}}$ in $\mathcal{D}$).

Coherent systems are systems for which improving the state of a single component cannot lead the system from an operating to a failure state (or equivalently degrading a component state cannot bring the system from failure to non failure). Thus, in the integer representation a coherent system is such that if a state point α belongs to the failure set $\mathcal{F}$ then any other point with component state levels worse (=less) than α must belong to $\mathcal{F}$. Similarly the non failure set, being the complement of $\mathcal{F}$, must have a similar property: if α is a non failure point then any point with coordinates greater or equal to α's coordinates must still be a non failure point. This is equivalent to say that the integer representation of the failure set must have an echelon structure, that is the grid representing $\mathcal{F}$ (or equivalently $\bar{\mathcal{F}}$) must have no "holes".

For example Figure 1 represents a two component coherent system where the hollow dots correspond to failure states and the filled dots correspond to operating (non failure) configurations.

The link with algebra stems from the fact that we can represent a monomial $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\dots x_d^{\alpha_d}$ with the point $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ given by its exponent vector. Then, for example, the filled dots in Figure 1 represent the monomials belonging to a certain monomial ideal M. The echelon property translates into the algebra language as "order ideal" property: if the monomial x^α belongs to a monomial ideal then so do all the monomials x^β such that x^α divides x^β, that is such that $\alpha \preceq \beta$: $\alpha_i \leq \beta_i,\ i = 1, \dots, d$. The corner points of the set $\mathcal{F}$ (those labeled with their coordinates in Figure 1) are the minimal generators of this monomial ideal but it is interesting to point out that these points play also a specific role as reliability theory objects. They are in fact what we call *minimal non failure points* in Section 4.

The main interesting observation, which we will develop in the following sections, is that there are some points on the boundary between the failure set and the non failure set that are fundamental to the identification of the points in $\mathcal{F}$ and in $\bar{\mathcal{F}}$. We will see that these special points form a simplicial complex, called *Scarf Complex*, which then plays a key role in both algebraic geometry and reliability theory.

Sections 2 and 3 give an introduction to understand the algebraic setting and Section 4 establishes and formalises the connection with reliability. For the introductory material in algebra we have mainly used [3], [12] and [10] as references.

2 Monomial ideals

2.1 Some basics

Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x] = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n indeterminates. A monomial in $\mathbb{K}[x]$ is a product $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}$ for a vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ of nonnegative integers.

A monomial ideal M is an ideal of S generated by some monomials x^α, for α belonging to a subset $A \subseteq \mathbb{N}^n$ (possibly infinite). The ideal M consists therefore of all the polynomials of the form $\sum_{\alpha \in A} h_\alpha x^\alpha$, with $h_\alpha \in S$. In this case we write $M = \langle x^\alpha : \alpha \in A \rangle$.

The following lemma allows us to characterise all the monomials that lie in a given monomial ideal.

Lemma 1 *Let $M = \langle x^\alpha : \alpha \in A \rangle$ be a monomial ideal. Then a monomial x^β lies in M if and only if x^β is divisible by x^α for some $\alpha \in A$.*

Observe that the set

$$\alpha + \mathbb{N}^n = \{\alpha + \gamma : \gamma \in \mathbb{N}^n\}$$

consists of the exponents of all monomials divisible by x^α. This observation allows us to draw pictures of the monomials in a given monomial ideal and visualise this set as a union of positive and integer coordinate points. For example the filled dots in Fig 1 represent all the monomials in the ideal $M = \langle x_1^3, x_1^2x_2^2, x_2^3 \rangle$.

A polynomial f is in a monomial ideal $M = \langle x^\alpha | \alpha \in A \rangle$ if and only if each term of f is divisible by one of the given generators x^α. From this it follows that a monomial ideal is uniquely determined by its monomials, that is two monomial ideals are the same if and only if they contain the same monomials. For details and proofs see [3] and [12].

The main result in this section is given by the following theorem (Dickson's Lemma) that shows that each monomial ideal is uniquely and finitely generated.

Theorem 1 *A monomial ideal $M = \langle x^\alpha : \alpha \in A \rangle \subset \mathbb{K}[x_1, \ldots, x_n]$ can be written down in the form $M = \langle x^{\alpha_1}, \ldots, x^{\alpha_s} \rangle$, where $\alpha_i \in A$. In particular, M has a finite basis.*

The polynomial ring $S = \mathbb{K}[x]$ can be seen as a $\mathbb{K}$-vector space and therefore it can be decomposed into the direct sum $S = \bigoplus_{\alpha \in \mathbb{N}^n} S_\alpha$, where S_α is the $\mathbb{K}$-span of the monomial x^α. Since $S_\alpha \cdot S_\beta \subseteq S_{\alpha+\beta}$, we say that S is an $\mathbb{N}^n$-graded $\mathbb{K}$-algebra. More generally, an S-module M is said to be $\mathbb{Z}^n$-graded if $M = \bigoplus_{\beta \in \mathbb{Z}^n} M_\beta$ is a direct sum of $\mathbb{K}$-vector spaces with $S_\alpha \cdot M_\beta \subseteq M_{\alpha+\beta}$. For definitions and properties of modules of polynomial rings see [8].

A monomial ideal M and the corresponding factor ring S/M are both $\mathbb{Z}^n$-graded S-modules: $M = \bigoplus_{x^\alpha \in M} S_\alpha$ and $S/M = \bigoplus_{x^\alpha \notin M} S_\alpha$.

The hollow dots in Figure 1 within the non shaded area form a $\mathbb{K}$-basis for S/M. We show in the next section that the Hilbert Series $H(S/M;x)$ is the sum of all the monomials that are not contained in M.

Given a $\mathbb{Z}^n$-graded module M, the $\mathbb{Z}^n$-graded shift $M[\alpha]$ for $\alpha \in \mathbb{Z}^n$ is the $\mathbb{Z}^n$-graded module defined by $M[\alpha]_\beta = M_{\alpha+\beta}$. In particular, the free S-module of rank one generated in degree α is $S[-\alpha]$. There is an isomorphism between $S[-\alpha]$ and the principal ideal $\langle x^\alpha \rangle \subset S$.

2.2 Hilbert Series and Free Resolutions

Given a $\mathbb{Z}^n$-graded module M_α we consider its dimension as a vector space on $\mathbb{K}$ and we call it $\dim_{\mathbb{K}}(M_\alpha)$. If $\dim_{\mathbb{K}}(M_\alpha)$ is finite for all $\alpha \in \mathbb{Z}^n$ the (finely graded) *Hilbert Series* is defined as the formal power series

$$H(M;x) := H(M;x_1,\ldots,x_n) = \sum_{\alpha\in\mathbb{Z}^n} dim_{\mathbb{K}}(M_\alpha)x^\alpha$$

For example we have that

$$H(S;x) = \sum_{\alpha\in\mathbb{Z}^n} x^\alpha = \prod_{i=1}^{n} \frac{1}{1-x_i}$$

(namely the sum of all monomials in S), and, for $\alpha \in \mathbb{Z}^n$,

$$\begin{aligned} H(S[-\alpha];x) &= \sum_{x^\beta \in \langle x^\alpha \rangle} x^\beta \\ &= x^\alpha(1+x_1+x_1^2+\ldots)\ldots(1+x_n+x_n^2+\ldots) \\ &= x^\alpha \frac{1}{1-x_1}\ldots\frac{1}{1-x_n} \\ &= \frac{x^\alpha}{\prod_{i=1}^n(1-x_i)} \end{aligned} \tag{1}$$

For a monomial ideal M we have that

$$H(S/M;x) = \sum_{\alpha\in\mathbb{N}^n\setminus M} x^\alpha$$

namely the sum of all monomials *not* in M.

A *homological complex* of S-modules is a sequence

$$\ldots \xleftarrow{\phi_{i-1}} F_{i=1} \xleftarrow{\phi_i} F_i \ldots$$

of S-module homomorphisms such that $\phi_{i-1} \circ \phi_i = 0$. A complex is *exact* at the i-th step if it has no homology there, that is if Kernel$(\phi_{i-1}) =$ Image(ϕ_i). The complex is exact if it is exact at the i-th step for all $i \in \mathbb{Z}$.

A *free resolution* of an S-module M is a complex

$$0 \leftarrow F_0 \overset{\phi_1}{\leftarrow} F_1 \ldots F_{t-1} \overset{\phi_t}{\leftarrow} F_t \leftarrow 0$$

of free modules which is exact everywhere except the 0-th step, and such that $M = \text{Coker}(\phi_1) = F_0/\text{Image}(\phi_1)$.

Every S-module has a free resolution, with length $\leq n$. If M is $\mathbb{Z}^n$-graded, then it has a $\mathbb{Z}^n$-graded free resolution. If, in addition, M is finitely generated, there is a $\mathbb{Z}^n$-graded resolution $M \leftarrow \mathbb{F}$ in which all the ranks of the F_i are finite and simultaneously minimised. Such an $\mathbb{F}$ is called a *minimal free resolution* of M and is unique up to non-canonical homomorphism.

Given a short exact sequence $0 \leftarrow M'' \leftarrow M \leftarrow M' \leftarrow 0$, the rank nullity theorem for linear algebra implies that $\dim_{\mathbb{K}}(M_\alpha) = \dim_{\mathbb{K}}(M''{}_\alpha) + \dim_{\mathbb{K}}(M'_\alpha)$ for all α, and hence $H(M;x) = H(M'';x) + H(M';x)$. More generally, if

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots$$

is a finite sequence such as a free resolution, then

$$H(M;x) = \sum_i (-1)^i H(F_i;x) \tag{2}$$

In particular, if M is finitely generated, the existence of a finite-rank free resolution for M implies that the Hilbert series of M is a rational function of x, because it is an alternating sum of Hilbert series of $S[-\alpha]$ for various α. Moreover the denominator can always be taken to be $\prod_i (1 - x_i)$.

We show, with an example, the connection between minimal resolution and Hilbert Series and why we are particularly interested in it.

Example 1 Consider a monomial ideal in two variables: $M = \langle x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, \ldots, x^{a_r}y^{b_r} \rangle$ in $S = \mathbb{K}[x,y]$. As mentioned above the finely graded Hilbert Series of the factor ring

S/M gives the sum of all the monomials not in M (and for the monomial ideal M it gives the sum of all the monomials in M). A way to obtain the Hilbert Series for S/M is to proceed by inclusion-exclusion, subtracting from the sum of all the monomial in S the list of all the monomials in the quadrants $\langle x^{a_i}y^{b_i}\rangle$ for all $i \in \{1, \dots, r\}$, then adding back the monomials in the two-way "intersections" $\langle \mathrm{lcm}(x^{a_i}y^{b_i}, x^{a_j}y^{b_j})\rangle = \langle x^{a_i}y^{b_i}\rangle \cap \langle x^{a_j}y^{b_j}\rangle$, then removing the three way-intersections $\langle x^{a_i}y^{b_i}\rangle \cap \langle x^{a_j}y^{b_j}\rangle \cap \langle x^{a_k}y^{b_k}\rangle$, and so on. In this way, after r steps, we have counted all the monomials the right number of times. But for a large number of generators this procedure is far from efficient since many terms cancel out. It is easy to see that in this two dimension case it is in fact enough to add back only the principal ideals obtained from couples of adjacent generators.

$\square$

In literature the highly non-minimal free resolution of S/M given by the inclusion-exclusion process is called the Taylor resolution, which we describe in more general terms in Section 3.1. The most compact representation yields instead the minimal resolution. Unfortunately a construction of the minimal free resolution has not been found for all arbitrary monomial ideals. In the literature there are only two general constructions for resolving arbitrary monomial ideals: the Taylor resolution and Lyubeznik's subcomplex. We show in Section 3, based on [2], [11] and [12], a construction of a minimal free resolution for a specific class of monomial ideals, called *generic monomial ideals*. Also a non-minimal free resolution of S/M based on deformation of the exponents is obtained for non generic monomial ideals. Even though this resolution is not minimal it is generally much smaller than Taylor's. We show in Section 4 that in reliability theory we always deal with generic monomial ideals in the most general setting, while in for binary systems (that typically lead to the non generic case) the deformation procedure still gives excellent results.

Notice that the deformation procedure is gratifyingly similar to the perturbation presented in [14] and [9], and the non-minimal resolution obtained via the Scarf complex after deformation leads to the abstract tube formula described in [14] and [9].

3 Monomial resolutions

In this section we first describe the construction of a (highly non minimal) resolution for an arbitrary monomial ideal, called Taylor resolution; then a method to obtain the minimal resolution for generic monomial ideals and finally the deformation procedure to deal with arbitrary monomial ideals. For more details, see [2], [11] and [12].

3.1 Taylor Resolution

For a monomial ideal $M = \langle m_1, \ldots, m_r \rangle$ and a subset $I \subseteq \{1, \ldots, r\}$ we set $m_I = \mathrm{lcm}(m_i | i \in I)$. Let $a_I \in \mathbb{N}^n$ be the exponent vector of m_I and $S(-a_I)$ the free S-module with one generator in multidegree a_I. Let F_s be the $\mathbb{K}$-vector space whose basis elements e_I correspond to the index sets $I \subseteq \{1, \ldots r\}$ of length s. Define the *differential* $\partial_s : F_s \to F_{s-1}$ by

$$\partial_s(e_I) = \sum_{i \in I} \mathrm{sign}(i, I) \frac{m_I}{m_{I \setminus i}} e_{I \setminus i}$$

where $\mathrm{sign}(i, I)$ is $(-1)^{j+1}$ if i is the j-th element in the ordering of I. It is possible to show (see [8]) that

$$0 \leftarrow F_0 \xleftarrow{\partial_1} F_1 \longleftarrow \ldots \longleftarrow F_{r-1} \xleftarrow{\partial_r} F_r \leftarrow 0$$

is a free resolution of S/M of length r, and it is called *Taylor resolution* of S/M. The resolution can be also seen as the $\mathbb{Z}^n$-graded module $\mathbb{F} = \bigoplus_{I \subseteq \{1, \ldots r\}} S(-a_I)$ that has 2^r terms and it is therefore very far from minimal if $r \gg n$.

It is interesting to notice the following way to write the Hilbert Series for a Taylor resolution of a the factor ring S/M:

$$\begin{aligned} H(S/M; x, y) &= H(S; x, y) - H(M; x, y) \\ &= \sum_{\alpha \in \mathbb{N}^d} x^\alpha - \sum_{\substack{x^\alpha \in \langle m_I \rangle \\ I \subseteq \{1, \ldots, d\} \\ |I| = 1}} x^\alpha + \sum_{\substack{x^\alpha \in \langle m_I \rangle \\ I \subseteq \{1, \ldots, d\} \\ |I| = 2}} x^\alpha + \ldots + (-1)^{r+1} \sum_{\substack{x^\alpha \in \langle m_I \rangle \\ I = \{1, \ldots, r\}}} x^\alpha \quad (3) \end{aligned}$$

where $\langle m_I \rangle$ is the principal ideal generated by the monomial m_I. Similarly for the monomial ideal M:

$$H(M;x,y) = \sum_{\substack{x^\alpha \in \langle m_I \rangle \\ I \subseteq \{1,\ldots,r\} \\ |I|=1}} x^\alpha - \sum_{\substack{x^\alpha \in \langle m_I \rangle \\ I \subseteq \{1,\ldots,r\} \\ |I|=2}} x^\alpha + \ldots + (-1)^{r+1} \sum_{\substack{x^\alpha \in \langle m_I \rangle \\ I = \{1,\ldots,r\}}} x^\alpha \tag{4}$$

Notice the analogy with [14] and [9] where the set of monomials belonging to the monomial ideals $\langle m_I \rangle$ are called *orthants*. We see there that the classical inclusion-exclusion formula for orthant arrangements looks very similar to the above expression of the Hilbert Series.

We will see later that the classic inclusion-exclusion formula for orthant arrangements has an identical structure. Simply replace any term x^α by the indicator function of the orthants $Q_\alpha = \{\beta \mid \beta \geq \alpha\}$.

We show in the next sections that the Scarf complex allows a more efficient formulation of the above expression and gives an equivalent of the tube formula proposed in [14] and in [9].

Example 2 We construct now the Taylor resolution and the Hilbert Series of S/M where $M = \langle x^3, x^2y^2, y^3 \rangle$ is the monomial ideal given in Section 2 and represented in Figures 1.

The required resolution is given by

$$0 \leftarrow F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} F_2 \xleftarrow{\partial_3} F_3 \leftarrow 0$$

The dimension of each vector space F_s ($s = 0,1,2,3$) in the sequence is given by $\binom{n}{s}$ (that is the number of subsets $I \subseteq \{1,\ldots,n\}$ of length s).

The differentials ∂_s can be expressed via monomial matrices where each column corresponds to an index set I and contains the vector $\partial_s(e_I)$. Therefore we obtain:

$$\partial_1 : \quad \begin{array}{c} \begin{bmatrix} x^3 & x^2y^2 & y^3 \end{bmatrix} \\ \begin{array}{ccc} {\scriptstyle I=\{1\}} & {\scriptstyle I=\{2\}} & {\scriptstyle I=\{3\}} \end{array} \end{array}$$

$$\partial_2 : \quad \begin{array}{c c} \begin{array}{c} x^3 \\ x^2y^2 \\ y^3 \end{array} & \left[\begin{array}{ccc} y^2 & y^3 & 0 \\ -x & 0 & y \\ 0 & -x^3 & -x^2 \end{array} \right] \\ & \begin{array}{ccc} I=\{1,2\} & I=\{1,3\} & I=\{2,3\} \\ m_I = x^3y^2 & m_I = x^3y^3 & m_I = x^2y^3 \end{array} \end{array}$$

$$\partial_3 : \quad \begin{array}{c c} \begin{array}{c} x^3y^2 \\ x^3y^3 \\ x^2y^3 \end{array} & \left[\begin{array}{c} y \\ -1 \\ x \end{array} \right] \\ & \begin{array}{c} I=\{1,2,3\} \\ m_I = x^3y^3 \end{array} \end{array}$$

For example, for $I = \{1, 2\}$ it is

$$\partial_2(e_{12}) = \frac{m_{12}}{m_1} e_1 - \frac{m_{12}}{m_2} e_2 = \left[\begin{array}{c} y^2 \\ -x \\ 0 \end{array} \right]$$

which corresponds to the first column of the matrix representing ∂_2.

Now from Equations (1) and (2) it follows:

$$\begin{aligned} H(S/M; x, y) &= \frac{1 - x^3 - x^2y^2 - y^3 + x^3y^2 + x^3y^3 + x^2y^3 - x^3y^3}{(1-x)(1-y)} \\ &= 1 + x + y + x^2 + xy + y^2 + x^2y + xy^2 \end{aligned}$$

which is exactly the sum of monomials not in M, as expected.

Notice that before cancellation of the term x^3y^3 the numerator of the Hilbert Series for the Taylor resolution contains $2^3 = 8$ terms.

□

3.2 Monomial resolution over labeled simplicial complexes and the Scarf complex

It is possible to define a monomial resolution associated to a simplicial complex labeled by the generators of a given monomial ideal. In this section we show that the resolution of a generic monomial ideal associated with a specific simplicial complex, called the *Scarf complex*, is minimally free.

Definition 1 *Let $V = \{v_1, \ldots, v_r\}$ be a finite set. A* (finite) simplicial complex Δ *on V is a collection of subsets of V such that $I_1 \in \Delta$ whenever $I_1 \subset I_2$ for some $I_2 \in \Delta$, and such that $\{v_i\} \in \Delta$ for $i = 1, \ldots, r$. The elements of Δ are called* faces*, and the* dimension*, $dim(I)$, of a face I is the number $|I| - 1$. The* dimension *of the simplicial complex Δ is $dim(\Delta) = \max\{dim(I) : I \in \Delta\}$.*

Note that the empty set $\emptyset$ is a face of dimension -1 of any non-empty simplicial complex. Faces of dimension 0 and 1 are called *vertices* and *edges* respectively. The maximal faces under inclusion are called *facets* of the simplicial complex.

For a monomial ideal M in $S = \mathbb{K}[x_1, \ldots, x_n]$ we can construct a special simplicial complex whose vertices are labeled with the generators of M. Additionally we label each face I of the simplicial complex by the least common multiple $m_I = lcm(m_i | i \in I)$ of its vertices and we restrict to the faces with unique labels. This leads to the following definition:

Definition 2 *For a monomial ideal $M = \langle m_1, \ldots m_r \rangle$ we call* Scarf complex *the simplicial complex Δ_M consisting of sets of minimal generators with unique labels*

$$\Delta_M = \{I \subseteq \{1, \ldots, r\} \mid m_I \neq m_J \textit{ for all } J \subseteq \{1, \ldots, r\} \textit{ other than } I\}$$

It is not difficult to see that a Scarf complex in $S = \mathbb{K}[x_1, \ldots, x_n]$ is still a simplicial complex and has dimension at most $n - 1$.

For $i \in \mathbb{Z}$, let $F_i(\Delta_M)$ be the set of i-dimensional faces of Δ_M and let $\mathbb{K}^{F_i(\Delta_M)}$ be a $\mathbb{K}$-vector space whose basis elements e_I correspond to the i-faces $I \in F_i(\Delta_M)$. Let $\mathbb{F}_{\Delta_M}$ be the so-called $\mathbb{N}^n$-graded chain complex of Δ_M over S obtained as

$$0 \leftarrow \mathbb{K}^{F_{-1}(\Delta_M)} \xleftarrow{\partial_0} \ldots \leftarrow \mathbb{K}^{F_{i-1}(\Delta_M)} \xleftarrow{\partial_i} \mathbb{K}^{F_i(\Delta_M)} \leftarrow \ldots \xleftarrow{\partial_{n-1}} \mathbb{K}^{F_{n-1}(\Delta_M)} \leftarrow 0$$

with differentials as in Section 3.1

$$\partial_s(e_I) = \sum_{i \in I} \mathrm{sign}(i, I) \frac{m_I}{m_{I \setminus i}} e_{I \setminus i}$$

.

For a special class of monomial ideals, called *generic* the complex defined by the simplicial complex Δ_M is minimally free as Theorem 2 states (for the proof, see [2]).

Definition 3 *A monomial ideal M is called* generic *if no variable x_i appears with the same nonzero exponents in two distinct minimal generators of M.*

Note that in [11] and [12] the authors prove that it is possible to define genericity in a weaker way and still achieve the same results that we are going to present. They therefore refer to the definition we give above as "strong genericity". For the purpose of this article the stronger version is not only sufficient but also easier and more appropriate to use for applications to reliability. The requirement that the monomial ideal is generic might seem quite strong. In practice though almost all monomial ideals are generic, in the sense that those which fail to be generic lie on finitely many hyperplanes in the space of exponents. From the reliability point of view the requirement is not too strong in the continuous distribution case (see Section 4.4) while in the binary case the deformation procedure can be used to obtain a resolution (typically non minimal).

Theorem 2 *For a generic monomial ideal M the complex $\mathbb{F}_{\Delta_M}$ defined by the Scarf complex Δ_M is the minimal free resolution of S/M over S. The $\mathbb{N}^n$-graded Hilbert Series of S/M (that is the sum of all monomials not in M) is*

$$\frac{\sum_{I\in\Delta_M}(-1)^{|I|}\ m_I}{(1-x_1)\dots(1-x_n)} \tag{5}$$

and there are no cancellations in the alternating sum in the numerator.

Theorem 2 allows us to write the Hilbert Series for the factor ring S/M (and for the monomial ideal M) in a much more parsimonious way than in the Taylor resolution. The best way to see the difference is to compare Equations 3 and 4 with the following expressions for $H(M;x,y)$ and $H(S/M;x,y)$:

$$H(S/M;x,y)=\sum_{\alpha\in\mathbb{N}^d}x^\alpha-\sum_{\substack{x^\alpha\in\langle m_I\rangle\\ I\in\Delta_M\\ |I|=1}}x^\alpha+\sum_{\substack{x^\alpha\in\langle m_I\rangle\\ I\in\Delta_M\\ |I|=2}}x^\alpha+\ldots+(-1)^r\sum_{\substack{x^\alpha\in\langle m_I\rangle\\ I=\{1,\dots,r\}}}x^\alpha \tag{6}$$

$$H(M;x,y) = \sum_{\substack{x^\alpha \in \langle m_I \rangle \\ I \in \Delta_M \\ |I|=1}} x^\alpha - \sum_{\substack{x^\alpha \in \langle m_I \rangle \\ I \in \Delta_M \\ |I|=2}} x^\alpha + \ldots + (-1)^{r+1} \sum_{\substack{x^\alpha \in \langle m_I \rangle \\ I=\{1,\ldots,r\}}} x^\alpha \tag{7}$$

The Taylor resolution can be obtained from the full simplicial complex (with cardinality 2^r) given by the set of *all* subsets of $\{1,\ldots,r\}$ while in the Scarf resolution the summation is done only on the subsets belonging to Δ_M. Let us see now with an example how we can obtain the minimal resolution via the Scarf Complex for the monomial ideal considered in Example 2.

Example 3 The Scarf complex for the monomial ideal $M = \langle x^3, x^2y^2, y^3 \rangle$ can be obtained in the following way:

$I = \{1\} \to m_I = x^3$

$I = \{2\} \to m_I = x^2y^2$

$I = \{3\} \to m_I = y^3$

$I = \{1,2\} \to m_I = x^3y^2$

$I = \{1,3\} \to m_I = x^3y^3$

$I = \{2,3\} \to m_I = x^2y^3$

$I = \{1,2,3\} \to m_I = x^3y^3$

Thus the Scarf Complex Δ_M is given by

$$\Delta_M = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\}\}$$

The index subsets that do not appear in Δ_M are those corresponding to the monomial x^3y^3 and notice that this is exactly the term that cancels in Example 2. It is possible to write functions in CoCoA(or any other computational algebra package) to obtain the Scarf complex easily and efficiently. We do not develop the computational aspects in this paper.

The resolution is given by

$$0 \longleftarrow S \xleftarrow{[x^3,x^2y^2,y^3]} S^3 \xleftarrow{\begin{bmatrix} y^2 & 0 \\ -x & y \\ 0 & -x^2 \end{bmatrix}} S^2 \longleftarrow 0$$

and the Hilbert series (which can be read off directly from the labels of the Scarf Complex) equals

$$H(S/M;x) = \frac{1 - x^3 - x^2y^2 - y^3 + x^3y^2 + x^2y^3}{(1-x)(1-y)}$$

Thus the number of terms in the numerator of the Hilbert Series is six. The gain in comparison with the Taylor resolution, which returned eight terms, is not big for this small example but it becomes extremely relevant in high dimensional problems.

□

3.3 Deformation of exponents

The construction described in this section leads to a (typically non minimal) resolution of an arbitrary monomial ideal via deformation of the exponent vectors of the generators of M. The approach has still the advantage of giving a resolution much smaller and shorter than the Taylor's resolution.

For an arbitrary monomial ideal $M = \langle m_1, \ldots, m_r \rangle$, let $\{\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in}) | 1 \leq i \leq r\}$ be the exponent vectors of the minimal generators of M. Choose vectors $\epsilon_i = (\epsilon_{i1}, \ldots, \epsilon_{in},) \in \mathbb{R}^n$ for $1 \leq i \leq r$ such that, for all i and $s \neq t$, the numbers $a_{is} + \epsilon_{is}$ and $a_{it} + \epsilon_{it}$ are distinct and $a_{is} + \epsilon_{is} < a_{it} + \epsilon_{it}$ implies $a_{is} \leq a_{it}$. Each vector ϵ_i defines a monomial $x_{\epsilon_i} = x_1^{\epsilon_{i1}} \ldots x_n^{\epsilon_{in}}$ with real exponents. We define the generic monomial ideal M_ϵ in a polynomial ring with real exponents as follows:

$$M_\epsilon = \langle m_1 x^{\epsilon_1}, m_2 x^{\epsilon_2}, \ldots, m_r x^{\epsilon_r} \rangle$$

We call M_ϵ a generic deformation of M. Let Δ_{M_ϵ} be the Scarf complex of M_ϵ. We now label the vertex of Δ_{M_ϵ} corresponding to $m_i x^{\epsilon_i}$ with the original monomial m_i. Let

$\mathcal{F}_\epsilon$ be the complex of S-modules defined by this labeling of Δ_{M_ϵ}. Then the following result can be obtained (see proof in [2]):

Theorem 3 *The complex $\mathbb{F}_\epsilon$ is a free resolution of S/M over S.*

There is a simple way to deform M. This is the deformation given in [2] and is equivalent to the perturbation proposed in [9]. The method consists in picking an integer $v > r$ and deforming M using $\epsilon_{ij} = i/v$.

4 Algebra and Reliability

4.1 Coherent Systems

We consider here systems as defined in [15] and [1] but with some slight differences in notation.

Definition 4

We call system *a set S of n components and we code their increasing efficiency levels with the integers* $\{0, 1, 2, \ldots, \}$

An outcome *is an integer string of length n describing the state of each component. We call $\mathcal{D}$ the set of all possible outcomes.*

A failure outcome *is an outcome which leads to failure of the system S.*

The failure set *$\mathcal{F}$ is the set of all failure outcomes. The* non-failure set *is the complement of $\mathcal{F}$ in $\mathcal{D}$: $\bar{\mathcal{F}} = \mathcal{D} \setminus \mathcal{F}$.*

In most cases it is natural to assume that replacing a failed component by a functioning one will never lead to a deterioration of the system. Systems for which such an assumption is valid are called *coherent systems.* In this work we consider only this type of systems. For example, if the outcome $(1, 2, 1, 3, 0)$ is a non-failure outcome then also $(1, 2, 2, 4, 0)$ must be an operating configuration. In terms of the usual partial order relation $\preceq$ in $\mathbb{Z}^n$, for a coherent system we write that if $\alpha \in \bar{\mathcal{F}}$ then $\beta \in \bar{\mathcal{F}}$ for all β such that $\alpha \preceq \beta$ (that is $\alpha_i \leq \alpha_i$, $i = 1, \ldots, n$). In the same way, in a failure configuration, replacing an operating component with a failed one cannot improve the system. Therefore, if $(0,1,0,0,0)$ is a failure outcome then also $(0,0,0,0,0)$ must also be. This observation leads to the following definition:

Definition 5

The maximal set of components whose failure ensures the failure of the system is called cut set*. Any configuration "worse" than these still leads the system to failure. We indicate the set of cuts with the symbol $\mathcal{F}^*$ and we call it sometimes* minimal failure set *(even though in our setting it is a maximal set).*

Similarly the extreme points in the non-failure set are called paths *and they represent the minimal configurations in which the system is operating. We indicate the set of cuts with the symbol $\bar{\mathcal{F}}^*$ and we sometimes call it sometimes* minimal path set *or* minimal non failure set*.*

As mentioned in the introduction it is easy to see how the failure set for a coherent system can be represented by an integer grid. Furthermore this grid has an echelon structure since the system is coherent. A similar observation (but with reverted inequalities) can be made on the non failure set.

At this point the parallel between a monomial ideal and non failure set, and between minimal non failure points and minimal generators is established.

Proposition 1 *Given a system S of n components with non failure set $\bar{\mathcal{F}}$ and minimal non failure set $\bar{\mathcal{F}}^*$:*

1. *The minimal non failure points in $\bar{\mathcal{F}}^*$, seen as the exponent vectors of monomials in $\mathbb{K}[x_1, \ldots, x_n]$, are the (exponents of the) minimal generators of a monomial ideal M.*

2. *The points in $\bar{\mathcal{F}}$ represent the (exponents of the) monomials belonging to the monomial ideal M generated by the minimal failure points.*

3. The points in $\mathcal{F}$ represent the (exponents of the) monomials belonging to the factor ring S/M.

Thus, given this setting, we allow ourselves to interchangeably use the expressions "non failure set $\bar{\mathcal{F}}$" and "monomial ideal generated by $\bar{\mathcal{F}}^*$", or "minimal failure points" and "minimal generators", even though, strictly speaking, we are dealing with monomials in one case and their exponents in the other one.

The main interest in reliability theory is to give a measure of the performance of a system S by evaluating the probability of the failure (or non failure) of the system.

Definition 6 *For a system S the reliability function, $r(S)$, is the probability of the non-failure set $\bar{\mathcal{F}}$ while the unreliability, $u(S)$, is the probability of the failure set $\mathcal{F}$. Clearly, we only need to concentrate on one of the two performance measures, being $u(S) = 1 - r(S)$.*

In the literature it is possible to find many techniques to obtain the reliability of a system but the most common problem encountered is the computational effort required to evaluate this probability for large networks.

One of the methods proposed in the literature to calculate the reliability function in the case of a finite state space is due to Moore and Shannon ([13]) and is based on state space enumeration. Thus the reliability function for a network G with non failure set $\bar{\mathcal{F}}$ can be written as

$$r(S) = \sum_{\omega \in \mathcal{D}} I_{\bar{\mathcal{F}}}(\omega) Pr(\omega)$$

where $\mathcal{D}$ is the full set of network states and $I_{\bar{\mathcal{F}}}(\omega)$ takes the value 1 when ω belongs to $\bar{\mathcal{F}}$, 0 otherwise. The approach is impractical because the space $\mathcal{D}$ has cardinality equal to m^n, if n is the number of components and m the number of levels that each component can assume .

The reliability can be also formulated in terms of the minimal non failure points. For a minimal non failure point $\alpha \in \bar{\mathcal{F}}^*$ we indicate with Q_α the event "the system is in one of the states β, with $\beta \succeq \alpha$". This can also be described as the event that the system state is a point in the orthant that has the point α as corner point:

$$Q_\alpha = \{\beta \mid \beta \succeq \alpha\}$$

Notice that the orthants Q_α's correspond exactly to the set of monomials belonging to the principal ideals $\langle x^\alpha \rangle$. Thus the non failure set can be written as a union of the orthants Q_α's:

$$\bar{\mathcal{F}} = \bigcup_{\alpha \in \bar{\mathcal{F}}^*} Q_\alpha$$

The reliability can therefore be obtained as the probability of union of orthants. Unfortunately this union is not disjoint and the classical approach to this problem is to use the inclusion-exclusion formula:

$$\begin{aligned} r(S) &= Prob(\bigcup_{\alpha \in \bar{\mathcal{F}}^*} Q_\alpha) \\ &= \sum_{\alpha \in \bar{\mathcal{F}}^*} Prob(Q_\alpha) - \sum_{\alpha, \alpha' \in \bar{\mathcal{F}}^*} Prob(Q_\alpha \cap Q_{\alpha'}) + \ldots + (-1)^r Prob(Q_\alpha \cap Q_{\alpha'} \cap \ldots) \end{aligned}$$

This approach is computationally very inefficient since many terms in the summation cancel out. The inclusion-exclusion procedure for the calculation of the reliability is the equivalent of the Taylor resolution described in Section 3.1. The solution is therefore to obtain the minimal resolution via the Scarf complex.

4.2 The Method

For a system S with minimal non failure set $\bar{\mathcal{F}}^* = \{\alpha_1, \ldots, \alpha_r\}$ what we actually need in practice to obtain the improved version of the inclusion-exclusion formula is only the Scarf complex Δ for the set of the minimal generators $x^{\alpha_1}, \ldots, x^{\alpha_r}$. The minimal resolution and the Hilbert Series are only the background for our calculations. The Scarf complex tells us in fact directly which least common multiples are needed to be included in the identity. We decompose the complex in terms of its facets and we write

$$\Delta = \{\Delta_0, \Delta_1, \ldots, \Delta_{d-1}\}, \quad \text{with } d \leq n$$

where Δ_s is the set of facets of Δ of dimension $s+1$ (thus Δ_0 is the set of vertices, Δ_1 is the set of edges and so on).

From Equation (7) we obtain the improved version of the inclusion-exclusion formula:

$$
\begin{aligned}
r(S) &= Prob(\bigcup_{\alpha\in\bar{\mathcal{F}}^*} Q_\alpha) \\
&= \sum_{m_I\in\Delta_0} Prob(Q_{\alpha_I}) - \sum_{m_I\in\Delta_1} Prob(Q_{\alpha_I}) + \ldots + (-1)^{d-1} Prob(Q_{\alpha_I})
\end{aligned}
\tag{8}
$$

This formula is the equivalent of the improved inclusion-exclusion formula obtained in [14] (tube identity) for orthant arrangements. The authors prove there that cutting the formula at even and odd level leads to upper and lower bounds for the reliability:

$$
\begin{aligned}
\sum_{m_I\in\Delta_0} Prob(Q_{\alpha_I}) - \sum_{m_I\in\Delta_1} Prob(Q_{\alpha_I}) \leq & r(S) \leq \sum_{m_I\in\Delta_0} Prob(Q_{\alpha_I}) \\
\ldots \leq & r(S) \leq \ldots
\end{aligned}
\tag{9}
$$

It is of some interest that these inequalities have not, to the authors' knowledge, been established in the algebraic literature.

The difficulty with genericity found in the algebraic theory is manifested faithfully in the reliability theory. We observe though that in some cases (as in the continuous case, see Section 4.4) the requirement is not too strong. In other cases such as the binary case (Section 4.5) the deformation procedure can be used. The algebraic theory tells us that this does not guarantee a minimal resolution but in general the improved inclusion-exclusion formula will lead to a much more efficient solution of the problem and also supply sharper bounds.

4.3 The Finite State case

In the situation where each component has only a finite number of states the orthants Q_α are finite. Therefore the non failure set $\bar{\mathcal{F}}$ is simply given by the intersection of the monomials in the monomial ideal generated by the set $\bar{\mathcal{F}}^*$ with the state space $\mathcal{D}$. The formula in Equation (8) is still valid if we take care of intersecting each orthant with the set $\mathcal{D}$.

4.4 The Continuous Distribution Case

The definition of a systems in Section 4 uses discrete states and this has implications for the definition of genericity, also called *general position condition* in [14] and [9].

With care the continuous case can be mapped into the discrete case in such a way that the inclusion-exclusion lemma and associated tube bounds can be inferred back from the discrete to the continuous case.

A simple way to define the mapping is to replace continuous values by their rank along each dimension. In addition one needs to "quantize" the continuous distributions.

Thus let $Z = (Z_1, \ldots, Z_n)$ be an n-vector random variable with distribution function

$$F_Z(z) = Prob(Z \leq z)$$

Define critical points

$$z^{(1)}, \ldots, z^{(m)} \tag{10}$$

and the corresponding orthants

$$Q_{z^{(i)}} = \{z \mid z > z^{(i)}\}$$

Thus the non failure event is given by

$$\bar{\mathcal{F}} = \bigcup_{i=1}^{m} Q_{z^{(i)}}$$

Now consider for a particular dimension i the set of all values on that dimension

$$z_i^{(1)}, \ldots, z_i^{(m)}$$

Assume the continuous genericity (general position) condition:

$$\text{for all } \; j \neq j' \quad \Rightarrow \quad z_i^{(j)} \neq z_i^{(j')} \qquad\qquad (j = 1, \ldots, m) \tag{11}$$

The condition is not too strong in the this case because for continuous variables the critical points (10) can be easily chosen so that condition (11) is verified.

Fix the dimension i and relabel the values so that

$$z_i^{(1)} < z_i^{(2)} < \ldots < z_i^{(m)}$$

Finally, create $m+1$ states for the discrete variable X_i so that

$$X_i \text{ is in state } j \quad \Longleftrightarrow \quad z_i^{(j)} < Z_i \leq z_i^{(j+1)}$$

This induces a distribution on the $(m+1)^n$ grid with probabilities

$$Prob \bigcap_{i=1}^{n} \{z_i^{(j)} < Z_i \leq z_i^{(j+1)}\}$$

which allows the orthant probabilities to be expressed in a straightforward way in terms of the distribution function F:

$$Prob(Q_{z^{(i)}}) = 1 - F(z_1^{(i)}, \ldots, z_n^{(i)})$$

The discrete identity (and bounds) obtained in the previous section can then be used to create an identity and bounds for the union as in the continuous case. The Scarf complex maps the discrete case precisely into the tube simplicial complex.

4.5 The Binary case

In the binary case each component in the system can only assume one of two states: failure (generally coded with 0) and non failure (generally coded with 1).

Example 4 An example of a binary system is a binary network, for which the functionality and the connections among components can be represented via a graph and each component can either fail or be operating . Examples of networks are electrical power systems, where the components (*nodes*) are the physical locations of the users and the connections among them (*links*) are the transmission lines; or computer networks, where the nodes are the single computers and the links are the connection cables. A system failure can occur in different ways: in the example of the power network the system fails when a user (or a group of users) has no power supply, whereas in the computer case the system can be considered to have failed if the communication between two given users is interrupted.

We assume that the links are perfect and not subject to failure, while the nodes represent the components and fail independently and according to a known probability

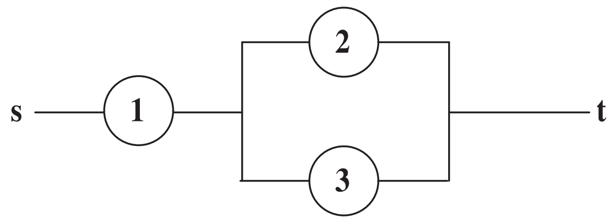

Figure 2: A three component system

distribution. There is a wide variety of cases and situations that can be coded via an appropriate measure of network reliability.

We are interested here only in a specific (although most common) case: the situation where the failure of the system is defined as the failure of communication between two special nodes, called source node s and terminal node t. Thus our network performance measure is the so-called *two-terminal reliability* $r_{st}(G)$, which is the probability that there is an operative path from a source node s to a terminal node t.

Consider the simple three component network in Figure 2. The outcome set is given by $\{0,1\}^3$ and the failure set contains the following five points:

$$\mathcal{F} = \{(011),(010),(001),(000),(100)\}$$

There are two extreme points in the failure set (cuts):

$$\mathcal{F}^* = \{(0,1,1),(1,0,0)\}$$

In a similar way the non failure set is the listing of the configurations for which the system is operating. In our example we obtain:

$$\bar{\mathcal{F}} = \{(110),(101),(111)\}$$

The path set is, in this case,

$$\bar{\mathcal{F}}^* = \{(110),(101)\}$$

$\square$

As mentioned the immediate problem with using the Scarf resolution for binary systems is that the genericity condition never holds in non trivial cases and the deformation procedure has to be used. The Scarf complex Δ_{M_ϵ} of Section 3.3 has still a meaning as a combinatorial object in our identity since we label each face $I \in \Delta_{M_\epsilon}$ by m_I and not by $\mathrm{lcm}(m_i x^{\epsilon_i},\ i \in I)$.

Notice that in the binary case if we assume independence among the component failure and we indicate with p_i the probability of failure of component i, the probability of a quadrant Q_α is easily obtained as

$$p(Q_\alpha) = \prod_{\alpha_i=1} p_i \tag{12}$$

Also notice that, as pointed out in [9], where the procedure is called perturbation, different deformations give rise to different Scarf complexes.

Example 5 Consider a 5 component binary system whose minimal non failure points are given by

$$\bar{\mathcal{F}}^* = \{(1,0,0,1,0),(0,1,0,0,1),(0,0,1,0,0),(0,1,0,1,0),(1,0,0,0,1)\}$$

The corresponding monomial ideal in $\mathbb{K}[x_1, x_2, x_3, x_4, x_5]$ is

$$M = \langle x_1x_4, x_2x_5, x_3, x_2x_4, x_1x_5 \rangle$$

Labeling the monomials in M from 1 to 5, after deformation (with $\epsilon_i = i/10$), we obtain the following Scarf Complex:

$$\begin{aligned}\Delta_{M_\epsilon} = \{ &\{1,2,3,5\},\{1,2,3,4\},\{2,3,5\},\{1,3,5\},\{1,2,5\},\{2,3,4\},\{1,3,4\},\{1,2,4\},\\ &\{1,2,3\},\{3,5\},\{2,5\},\{1,5\},\{3,4\},\{2,4\},\{1,4\},\{2,3\},\{1,2\},\\ &\{1,3\},\{5\},\{4\},\{2\},\{3\},\{1\}\}\end{aligned}$$

Figure 3 shows the geometric realisation of Δ_{M_ϵ}. Thus the procedure has given only 23 terms compared to the 32 we would obtain using the classical inclusion-exclusion.

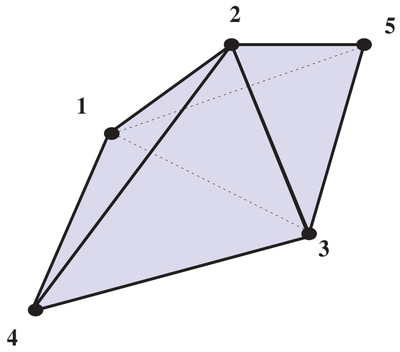

Figure 3: Geometric realisation of the Scarf complex in Example 4

The reliability function is obtained following Equations (8) and (12):

$$
\begin{aligned}
r = {} & p_1p_4 + p_2p_5 + p_3 + p_2p_4 + p_1p_5 \\
& - p_1p_2p_4p_5 - p_2p_3p_5 - p_1p_3p_4 - p_1p_4p_5 - p_1p_2p_5 - p_1p_3p_5 - p_2p_4p_5 - p_2p_3p_4 - p_1p_2p_4 \\
& + p_1p_2p_3p_4p_5 + p_1p_2p_4p_5 + p_1p_2p_3p_5 + p_2p_3p_4p_5 + p_1p_2p_4p_5 + p_1p_2p_3p_4 + p_1p_3p_4p_5 \\
& - p_1p_2p_3p_4p_5 - p_1p_2p_3p_4p_5
\end{aligned}
$$

Notice that there are only two term cancellations (the term $p_1p_2p_3p_4p_5$ and the term $p_1p_2p_4p_5$).

□

References

[1] R. E. Barlow. *Engineering Reliability.* ASA-SIAM series on Statistics and Applied Probability, 1998.

[2] D. Bayer, I. Peeva, and B. Sturmfels. Monomial resolutions. *Mathematical Research Letters*, 5:31–46, 1998.

[3] D. Cox, J. Little, and D. O'Shea. *Ideal, Varieties and Algorithms.* Springer-Verlag, New York, 1997.

[4] K. Dhomen. An improvement of the inclusion-exclusion principle. *Arch. Math*, 72(4):298–303, 1999a.

[5] K. Dhomen. Improved inclusion-exclusion identities and inequalities based on a particular class of abstract tubes. *Electr. J. Probab.*, 4(5):12pp (electronic), 1999b.

[6] P. Diaconis and B. Sturmfels. Algebraic algorithms for sampling from conditional distributions. *Annals of Statistics*, 26:363–97, 1998.

[7] K. Dohmen. *Improved Inclusion-Exclusion Identities and Bonferroni Inequalities with Applications to Reliability Analysis of Coherent Systems.* Habilitationsschrift, University of Humboldt, Berlin, 2000.

[8] D. Eisenbud. *Commutative algebra with a view toward algebraic geometry.* Springer-Verlag, New York, 1995.

[9] B. Giglio, D. Q. Naiman, and H. P. Wynn. Gröbner bases, abstract tubes, and inclusion-exclusion reliability bounds. *IEEE Transactions on Reliability*, 2002. To appear.

[10] M. Kreuzer and L. Robbiano. *Computational Commutative Algebra 1.* Springer-Verlag, 2000.

[11] E. Miller. *Resolutions of monomial ideals.* Phd thesis, University of California at Berkeley, 2000.

[12] E. Miller and D. Perkinson. Eight lectures on monomial ideals. *To appear in Queen's Papers in Pure and Applied Mathematics.*

[13] E. F. Moore and C. E. Shannon. Reliable circuits using less reliable relays. *Journal of the Franklin Institute*, 262:191–208, 1956.

[14] D. Naiman and H. P. Wynn. Improved inclusion-exclusion inequalities for simplex and orthant arrangements. *Journal of Inequalities in Pure and Applied Mathematics*, 2001. In press.

[15] G. Pistone, E. M. Riccomagno, and H. P. Wynn. *Algebraic Statistics.* Chapman and Hall, CRC, 2001.

[16] D. R. Shier. *Network Reliability and Algebraic Structures.* Clarendon Press, Oxford Science Publications, 1991.