GLY Conjecture on Upper Estimate of Integral Points in Real Tetrahedra with Application in Singularity Theory

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Abstract

Let $\Delta(a_1, a_2, \cdots, a_n)$ be an n-dimensional real tetrahedron with vertices at $(a_1, 0, \cdots, 0), (0, a_2, \cdots, 0), \cdots, (0, 0, \cdots, a_n)$. Let $P_{(a_1,a_2,\cdots,a_n)}$ be the number of positive integral points lying in $\Delta(a_1, a_2, \cdots, a_n)$. In this paper we prove that $n!P_{(a_1,a_2,\cdots,a_n)} \leq (a_1-1)(a_2-1)\cdots(a_n-1)$. As a consequence we have proved the Durfee conjecture for isolated weighted homogeneous singularities: $n!p_g \leq \mu$ where p_g and μ are the geometric genus and Milnor number of the singularity.

1 Introduction

Let $\Delta(a_1, a_2, \cdots, a_n)$ be an n-dimensional tetrahedron with nonintegral points described by

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1, \quad x_1 \geq 0, x_2 \geq 0 \cdots, x_n \geq 0 \tag{1.1}$$

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where $a_1 \geq a_2 \geq \cdots \geq a_u \geq 1$ are positive real numbers. Let

$$Q_{(a_1,a_2,\cdots,a_n)} = \#\left\{(x_1,x_2,\cdots,x_n) \in \mathbb{Z}^n_{\geq 0} : \sum_{i=1}^n \frac{x_i}{a_i} \leq 1\right\} \tag{1.2}$$

$$P_{(a_1,a_2,\cdots,a_n)} = \#\left\{(x_1,x_2,\cdots,x_n) \in \mathbb{Z}^n_{+} : \sum_{i=1}^n \frac{x_i}{a_i} \leq 1\right\} \tag{1.3}$$

If we let $a = \dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_n}$, then (1.2) and (1.3) are related by the following formula

$$P_{(a_1,a_2,\cdots,a_n)} = Q_{(a_1(1+a),a_2(1+a),\cdots,a_n(1+a))} \tag{1.4}$$

The computation of $Q_{(a_1,a_2,\cdots,a_n)}$ has received attention from many distinguished mathematicians. Hardy and Littlewood wrote two papers for $n = 2$ [Ha-Li 2] [Ha-Li 3] which have applications on some problems of Diophantine approximation [Ha-Li 1]. D. C. Spencer [Sp 1] [Sp 2] followed up the efforts of Hardy and Littlewood and wrote two papers on this subject. However, their results are asymptotic in nature and are not useful in the applications described below. In recent years, there are tremendous activities in finding the exact formula for $Q_{(a_1,a_2,\cdots,a_n)}$ or $P_{(a_1,a_2,\cdots,a_n)}$ where $a_1, a_2, \cdots, a_n$ are integers, see [Mo], [Mor], [Po], [Ca-Sh], [Br-Ve], [Di-Ro], [Ka-Kh]. The exact formula for integral tetrahedron is nice, but it has no application in the following problems in number theory and singularities theory. Since exact formula involves the generalized Dedekind sum, it is difficult to tell how large $P_{(a_1,a_2,\cdots,a_n)}$ is from the exact formula. Therefore one would like to get an upper estimate of $P_{(a_1,a_2,\cdots,a_n)}$ in terms of a polynomial in $a_1, a_2, \cdots, a_n$. Such an upper polynomial estimate of $P_{(a_1,a_2,\cdots,a_n)}$ is important because it would have applications in number theory and singularities theory.

According to Granville [Gr], upper polynomial estimate of $P_{(a_1,a_2,\cdots,a_n)}$ is an extremely important question in number theory; it would have many applications to current problems in analytic number theory, primality testing and factoring. Given a set $\mathcal{P}$ of primes $p_1 < p_2 < \cdots < p_n < y$, number theorists want to count the number of integers $m \leq x$ where $m = p_1^{l_1} p_2^{l_2} \cdots p_n^{l_n}$, $x = y^u$ for all $u \geq 2$. This is equivalent to counting the number of $(l_1, l_2, \cdots, l_n) \in \mathbb{Z}^n_{\geq 0}$ such that $l_1 p_1 + l_2 p_2 + \cdots + l_n p_n \leq \log x$, which is also equivalent to counting the number of $(l_1, l_2, \cdots, l_n) \in \mathbb{Z}^n_{\geq 0}$ such that

$$\frac{l_1}{a_1} + \frac{l_2}{a_2} + \cdots + \frac{l_n}{a_n} \leq 1, \quad \text{where} \quad a_i = \frac{\log x}{\log p_i} = \frac{\log y}{\log p_i} \tag{1.5}$$

Observe that a_i's are real numbers. This is, of course, the problem of computing $Q_{(a_1,a_2,\cdots,a_n)}$. Perhaps the best reference for the application is Carl Pomerance's ICM 1994 lecture at Zürich [Pom 1], and his lecture notes [Pom 2]. The current method in counting $Q_{(a_1,a_2,\cdots,a_n)}$ is the following polynomial estimate (1.7) provided by number theorist. Attach a unit cube to the

right and above each lattice point of $\Delta(a_1, a_2, \cdots, a_n)$. Then

$$\begin{aligned} Q_{(a_1,a_2,\cdots,a_n)} &= \sum \text{volume of the unit cube attached to each lattice point} \\ &\leq \text{volume of} \left\{ (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^n \frac{x_1 - 1}{a_i} \leq 1 \right\} \\ &= \frac{1}{n!}(a_1 a_2 \cdots a_n)(1 + \sum_{i=1}^n \frac{1}{a_i})^n \end{aligned} \tag{1.6}$$

In view of (1.4), (1.6) can be rewritten as

$$P_{(a_1,a_2,\cdots,a_n)} \leq \frac{1}{n!}\, a_1 a_2 \cdots a_n \tag{1.7}$$

The estimate of $P_{(a_1,a_2,\cdots,a_n)}$ given by (1.7) is nice. However, it is not very good particularly when many of the a_i's are small [Gr]. The purpose of this paper is to prove the following theorem

Theorem 1.1 *Let* $P_n = P_{(a_1,a_2\cdots,a_n)} = \#\{(x_1, \cdots, x_n) \in \mathbb{Z}^n_+ : \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1\}$, *where* $a_1 \geq a_2 \geq \cdots \geq a_n \geq 1$ *are real numbers. If* $n \geq 3$, *then*

$$n! P_n \leq (a_1 - 1)(a_2 - 1) \cdots (a_n - 1) \tag{1.8}$$

Equality in (1.8) holds if and only if $a_n = 1$.

Theorem 1.1 was proved for $n = 3$ by Xu and Yau [Xu-Ya 1], for $n = 4$ by Lin and Yau [Li-Ya 1], and for $3 \leq n \leq 6$ by Wang and Yau [Wa-Ya]. It has an interesting application in singularities theory. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a complex analytic function with isolated critical point at the origin. Let M be a resolution of $V = \{(z_1, z_2, \cdots, z_n) \in \mathbb{C}^n : f(z_1, z_2, \cdots, z_n) = 0\}$. The Milnor number of the singularity $(V, 0)$ is

$$\mu = \dim \mathbb{C}\{z_1, z_2, \cdots, z_n\}/(f_{z_1}, f_{z_2}, \cdots, f_{z_n}) \tag{1.9}$$

The geometric genus of the singularity $(V, 0)$ is

$$p_g = \dim H^{n-2}(M, \mathcal{O}) \tag{1.10}$$

Both μ and p_g are important invariants of the singularity $(V, 0)$. As a corollary of Theorem 1.1, we have proved the following Durfee conjecture [Du] made in 1978.

Theorem 1.2 *Let* $(V, 0)$ *be an isolated singularity defined by a weighted homogeneous polynomial* $f(z_1, z_2, \cdots, z_n)$. *Then* $n! p_g \leq \mu$ *and equality holds if and only if* $\mu = 0$.

The connection between Theorem 1.1 and Theorem 1.2 is as follows. A polynomial of $f(z_1, z_2, \cdots, z_n)$ is weighted homogeneous of the type $(w_1, w_2, \cdots, w_n)$, where $w_1, w_2, \cdots, w_n$ are fixed positive rational numbers, if f can be expressed as a linear combination of monomials

$z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n}$ for which $\frac{i_1}{w_1} + \frac{i_2}{w_2} + \cdots + \frac{i_n}{w_n} = 1$. If $f(z_1, z_2, \cdots, z_n)$ is a weighted homogeneous polynomial of type $(a_1, a_2, \cdots, a_n)$ with an isolated singularity at the origin, then Milnor and Orlik [Mi-Or] proved that $M = (a_1 - 1)(a_2 - 1) \cdots (a_n - 1)$. On the other hand, Merle and Teissier [Me-Te] showed that p_g is exactly the number P_n of positive integral points in Theorem 1.1.

Theorem 1.1 is the rough upper estimate in GLY conjecture [Li-Ya 3] [Wa-Ya]. The Durfee conjecture in Theorem 1.2 is not sharp although it has been an open question for more than a quarter of century. In 1995, the first author formulated Yau conjecture (see Section 2 below), which is sharper than Durfee conjecture. More importantly, it gives an intrinsic characterization of homogeneous singularities. In order to prove the Yau conjecture, Lin and Yau [Li-Ya 3] [Wa-Ya], independently Granville, formulated a sharp upper estimate GLY conjecture (see Section 2 below). This sharp upper estimate GLY conjecture is only true if a_n is sufficiently large. When we use induction to prove the sharp upper estimate GLY conjecture, we must face the difficulty that we cannot apply the lower dimensional GLY conjecture in every level. Therefore Theorem 1.1 must be used in order to prove the sharp upper estimate GLY conjecture. We hope to address this problem in the future paper.

2 GLY Conjecture on Number of Integral Points and Yau Conjecture in Singularities Theory

The following is a thirty years old problem in singularities theory.

Problem: *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a complex analytic function with isolated critical point at the origin. Find an intrinsic characterization for f to be a homogeneous polynomial.*

In 1971, Saito [Sa] gave an intrinsic characterization for f to be a weighted homogeneous polynomial.

Theorem 2.1 Saito [Sa] *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a complex analytic function with isolated critical point at the origin. Then f is a weighted homogeneous polynomial after biholomorphic change of coordinates if and only if $\mu = \tau$, where $\mu = \dim \mathbb{C}\{z_1, z_2, \cdots, z_n\}/(f_{z_1}, f_{z_2}, \cdots, f_{z_n})$ and $\tau = \dim \mathbb{C}\{z_1, z_2, \cdots, z_n\}/(f, f_{z_1}, f_{z_2}, \cdots, f_{z_n})$.*

In order to characterize homogeneous polynomials with isolated singularity, the first author has made the following conjecture in 1995.

Yau Conjecture: *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin. Let μ, p_g, and ν be the Milnor number, geometric genus and multiplicity of the singularity $V = \{z : f(z) = 0\}$, then*

$$\mu - p(\nu) \geq n! p_g \tag{2.1}$$

where $p(\nu) = (\nu - 1)^n - \nu(\nu - 1) \cdots (\nu - n + 1)$, and equality holds if and only if f is a homogeneous polynomial.

Theorem 2.1 together with Yau conjecture will give an intrinsic characterization for a complex analytic function to be a homogeneous function after a biholomorphic change of variables. Yau conjecture was answered affirmatively by Xu and Yau [Xu-Ya 2] for $n = 3$, and Lin and Yau [Li-Ya 4] for $n = 4$. In order to prove the Yau conjecture above, Lin and Yau [Li-Ya 3] and Granville independently after reading the Lin-Yau's paper [Li-Ya 2] have formulated the following GLY conjecture.

Before we state the GLY conjecture, it is convenient to introduce some notations. Recall the Stirling number of the first kind:

$$s_k^{n-1} = \sum_{1\le i_1<i_2<\cdots<i_k\le n-1} i_1 i_2\cdots i_k,\ S_0^{n-1}=1,\ S_{n-1}^{n-1} = 1\cdot 2\cdots(n-1) \tag{2.2}$$

where $i_1, i_2, \cdots, i_k$ are integers. It has the following property,

$$\begin{aligned} & x(x-1)(x-2)\cdots(x-n+1) \\ = \ & x^n - \left(\sum_{i=1}^n i_1\right) x^{n-1} + (-1)^2 \left(\sum_{1\le i_1<i_2\le n-1} i_1 i_2\right) x^{n-2} \\ & + \cdots + (-1)^k \left(\sum_{1\le i_1<i_2<\cdots<i_k\le n-1} i_1 i_2\cdots i_k\right) x^{n-k} + \cdots + (-1)^{n-1}\left(\prod_{i=1}^{n-1} i\right) x \\ = \ & x^n + (-1)S_1^{n-1}x^{n-1} + (-1)^2 S_2^{n-1}x^{n-2} + \cdots + (-1)^k S_k^{n-1}x^{n-k} + \cdots + (-1)^{n-1}S_{n-1}^{n-1}x \end{aligned}$$

Let $a_1, a_2, \cdots, a_n$ be positive real numbers. We shall denote

$$\begin{aligned} A_{n-k}^n &= \left(\prod_{i=1}^n a_i\right) \sum_{1\le i_1<i_2<\cdots<i_k\le n} \frac{1}{a_{i_1}a_{i_2}\cdots a_{i_k}} \\ A_n^n &= \prod_{i=1}^n a_i, \quad A_0^n = 1 \end{aligned}$$

Observe that A_k^n is a polynomial in $a_1, a_2, \cdots, a_n$ of degree $n-k$.

Granville-Lin-Yau (GLY) Conjecture ([Li-Ya 3] [Wa-Ya]) *Let* $P_n = P_{(a_1,a_2\cdots,a_n)} = \#\{(x_1,\cdots,x_n)\in\mathbb{Z}_+^n : \frac{x_1}{a_1}+\frac{x_2}{a_2}+\cdots+\frac{x_n}{a_n}\le 1\}$, *where* $a_1\ge a_2\ge\cdots\ge a_n\ge 1$ *are real numbers. If* $n\ge 3$, *then*

(I) *Rough Estimate for all* $a_n\ge 1$*:*

$$n!P_n \le (a_1-1)(a_2-1)\cdots(a_n-1) \tag{2.3}$$

(II) *Sharp Estimate for* a_n *sufficiently large: there exists an integer* $\alpha(n)$ *which depends on* n *such that when* $a_n\ge\alpha(n)$, *then*

$$
\begin{aligned}
n!P_n \;\le\; & \prod_{i=1}^{n} a_i + (-1)\frac{S_1^{n-1}}{n}\left(\prod_{i=1}^{n} a_i\right)\left(\sum_{k=1}^{n}\frac{1}{a_k}\right) + (-1)^2\frac{S_2^{n-1}}{\binom{n-1}{1}}\left(\prod_{i=1}^{n-1} a_i\right)\left(\sum_{k=1}^{n-1}\frac{1}{a_k}\right) \\
& + (-1)^3\frac{S_3^{n-1}}{\binom{n-1}{2}}\left(\prod_{i=1}^{n-1} a_i\right)\left(\sum_{k=1\le i_1<i_2\le n-1}\frac{1}{a_{i_1}a_{i_2}}\right) \\
& + \cdots + (-1)^{k+1}\frac{S_{k+1}^{n-1}}{\binom{n-1}{k}}\left(\prod_{i=1}^{n-1} a_i\right)\left(\sum_{k=1\le i_1<i_2<\cdots<i_k\le n-1}\frac{1}{a_{i_1}a_{i_2}\cdots a_{i_k}}\right) \\
& + \cdots + (-1)^{n-1}\frac{S_{n-1}^{n-1}}{\binom{n-1}{n-2}}(a_1+a_2+\cdots+a_{n-1}) \\
= \;& A_n^n + (-1)\frac{S_1^{n-1}}{n}A_{n-1}^n + (-1)^2\frac{S_2^{n-1}}{\binom{n-1}{1}}A_{n-2}^{n-1} + (-1)^3\frac{S_3^{n-1}}{\binom{n-1}{2}}A_{n-3}^{n-1} \\
& + \cdots + (-1)^{k+1}\frac{S_{k+1}^{n-1}}{\binom{n-1}{k}}A_{n-k-1}^{n-1} + \cdots + (-1)^{n-1}\frac{S_{n-1}^{n-1}}{\binom{n-1}{n-2}}A_1^{n-1}
\end{aligned}
$$

and the equality holds if and only if $a_1 = a_2 = \cdots = a_n =$*integer.*

The GLY conjecture was proved by Xu and Yau for $n = 3$ [Xu-Ya 1] and $n = 4$ [Xu-Ya 3], Lin and Yau for $n = 5$ [Li-Ya 2] and Wang and Yau for $3 \le n \le 6$ [Wa-Ya]. In fact Wang and Yau's method in [Wa-Ya] can be used to prove GLY for any fixed n. It has been checked that GLY conjecture is true for $n \le 10$. However, it takes a long time (several weeks for $n = 10$) for computer to do the computation. The purpose of this paper is to give a proof of the rough estimate of GLY conjecture for all n.

3 The Main Result

Before proving our main result Theorem 1.1, we need to establish three technical lemmas.

Lemma 3.1 *Given any positive real number* α *with* $0 < \alpha < 1$*, let* $a > 1$ *be any number such that* $\alpha = a - \lfloor a \rfloor$*, where* $\lfloor a \rfloor$ *denotes the greatest positive integer less than or equal to* a*. If* $n \ge 3$*, then*

$$
a - 1 > (n+1)\sum_{k=0}^{\lfloor a \rfloor - 1}\frac{(k+\alpha)^n}{a^n} \tag{3.1}
$$

Proof

We shall prove (3.1) by induction on a. Consider the expression

$$a-1-(n+1)\sum_{k=0}^{\lfloor a\rfloor-1}\frac{(k+\alpha)^n}{a^n} \tag{3.2}$$

Let $a=1+\alpha$, then $k=0$ and (3.2) becomes

$$\begin{aligned} a-1-(n+1)\sum_{k=0}^{\lfloor a\rfloor-1}\frac{(k+\alpha)^n}{a^n} &= 1+\alpha-1-(n+1)\frac{\alpha^n}{(1+\alpha)^n} \\ &= \alpha\left[1-\frac{n\alpha^{n-1}+\alpha^{n-1}}{(1+\alpha)^n}\right] \end{aligned} \tag{3.3}$$

For $n\geq 3$, it can be seen that $(1+\alpha)^n=\alpha^n+n\alpha^{n-1}+\cdots+1>n\alpha^{n-1}+\alpha^{n-1}$ and (3.3) is positive. To finish the proof, we need to prove that if Lemma 3.1 is true for a, then it is also true for $a+1$. By induction hypothesis, we have

$$a-1>(n+1)\sum_{k=0}^{\lfloor a\rfloor-1}\frac{(k+\alpha)^n}{a^n} \tag{3.4}$$

Then

$$\begin{aligned} & (a+1)-1-(n+1)\sum_{k=0}^{\lfloor a+1\rfloor-1}\frac{(k+\alpha)^n}{(a+1)^n} \\ = \;& a-\frac{n+1}{(a+1)^n}\left(\sum_{k=0}^{\lfloor a\rfloor-1}(k+\alpha)^n+a^n\right) \\ > \;& a-\frac{n+1}{(a+1)^n}\left(\frac{a^n(a-1)}{n+1}+a^n\right) \\ = \;& a-\frac{a^n}{(a+1)^n}(a+n) \\ = \;& a\left[1-\frac{a^n+na^{n-1}}{(a+1)^n}\right]>0 \end{aligned} \tag{3.5}$$

The last inequality in (3.5) comes from $(a+1)^n=a^n+na^{n-1}+\cdots+1>a^n+na^{n-1}$. □

Lemma 3.2 *Let $a_{j-1},a_j,\cdots,a_{n+1}$ be real numbers and $\alpha=a_{n+1}-\lfloor a_{n+1}\rfloor$. Assume that $a_{j-1}>1$ and $a_j\geq a_{j+1}\geq\cdots\geq a_n\geq a_{n+1}>1$. If $\frac{a_n}{a_{n+1}}\alpha\geq 1$, and*

$$\begin{aligned} (a_j-1)(a_{j+1}-1)\cdots(a_{n+1}-1) \;>\; & (n+1)\sum_{k=0}^{\lfloor a_{n+1}\rfloor-1}\frac{(k+\alpha)^{j-1}}{a_{n+1}^{j-1}}\left(\frac{a_j}{a_{n+1}}(k+\alpha)-1\right) \\ & \left(\frac{a_{j+1}}{a_{n+1}}(k+\alpha)-1\right)\cdots\left(\frac{a_n}{a_{n+1}}(k+\alpha)-1\right) \end{aligned} \tag{3.6}$$

then

$$(a_{j-1}-1)(a_j-1)\cdots(a_{n+1}-1) > (n+1)\sum_{k=0}^{\lfloor a_{n+1}\rfloor-1}\frac{(k+\alpha)^{j-2}}{a_{n+1}^{j-2}}\left(\frac{a_{j-1}}{a_{n+1}}(k+\alpha)-1\right)\left(\frac{a_j}{a_{n+1}}(k+\alpha)-1\right)\cdots\left(\frac{a_n}{a_{n+1}}(k+\alpha)-1\right) \quad (3.7)$$

Proof

For fixed $a_j \geq a_{j+1} \geq \cdots \geq a_n \geq a_{n+1} > 1$, let

$$F(a_{j-1}) = (a_{j-1}-1)(a_j-1)\cdots(a_{n+1}-1) - (n+1)\sum_{k=0}^{\lfloor a_{n+1}\rfloor-1}\frac{(k+\alpha)^{j-2}}{a_{n+1}^{j-2}} \times\left(\frac{a_{j-1}}{a_{n+1}}(k+\alpha)-1\right)\left(\frac{a_j}{a_{n+1}}(k+\alpha)-1\right)\cdots\left(\frac{a_n}{a_{n+1}}(k+\alpha)-1\right) \quad (3.8)$$

To prove the lemma, we only need to prove that $F(a_{j-1})$ is a strictly increasing function of a_{j-1} and $F(1) \geq 0$.

Let $a_{j-1} = 1$, it can be seen that $\frac{1}{a_{n+1}}(k+\alpha)-1 < 0$ for $0 \leq k \leq \lfloor a_{n+1}\rfloor - 1$. Furthermore, by the assumption of $a_j \geq a_{j+1} \geq \cdots \geq a_n \geq a_{n+1} > 1$ and $\frac{a_n}{a_{n+1}}\alpha \geq 1$, we have $\frac{a_i}{a_{n+1}}\alpha \geq 1$ for all $j \leq i \leq n$. Therefore

$$\left(\frac{a_j}{a_{n+1}}(k+\alpha)-1\right)\cdots\left(\frac{a_n}{a_{n+1}}(k+\alpha)-1\right) \geq 0 \quad \text{for} \quad 0 \leq k \leq \lfloor a_{n+1}\rfloor - 1 \quad (3.9)$$

Hence we have shown that $F(1) \geq 0$.

$$\begin{aligned}\frac{dF}{da_{j-1}} &= (a_j-1)\cdots(a_{n+1}-1) - (n+1)\sum_{k=0}^{\lfloor a_{n+1}\rfloor-1}\frac{(k+\alpha)^{j-1}}{a_{n+1}^{j-1}} \\ &\quad\times\left(\frac{a_j}{a_{n+1}}(k+\alpha)-1\right)\cdots\left(\frac{a_n}{a_{n+1}}(k+\alpha)-1\right) \quad (3.10)\\ &> 0 \text{ by (3.6).}\end{aligned}$$

Therefore $F(a_{j-1})$ is a strictly increasing function of a_{j-1}. We conclude that $F(a_{j-1}) > 0$ for $a_{j-1} > 1$. □

Lemma 3.3 *Let $a_{j-1}, a_j, \cdots, a_{n+1}$ be real numbers and $\alpha = a_{n+1} - \lfloor a_{n+1}\rfloor$. Assume that $a_{j-1} > 1$ and $a_j \geq a_{j+1} \geq \cdots \geq a_n \geq a_{n+1} > 1$. If $\frac{a_n}{a_{n+1}}\alpha < 1$, and*

$$(a_j-1)(a_{j+1}-1)\cdots(a_{n+1}-1) > (n+1)\sum_{k=1}^{\lfloor a_{n+1}\rfloor-1}\frac{(k+\alpha)^{j-1}}{a_{n+1}^{j-1}}\left(\frac{a_j}{a_{n+1}}(k+\alpha)-1\right)\left(\frac{a_{j+1}}{a_{n+1}}(k+\alpha)-1\right)\cdots\left(\frac{a_n}{a_{n+1}}(k+\alpha)-1\right) \quad (3.11)$$

then

$$(a_{j-1}-1)(a_j-1)\cdots(a_{n+1}-1) > (n+1)\sum_{k=1}^{\lfloor a_{n+1}\rfloor-1}\frac{(k+\alpha)^{j-2}}{a_{n+1}^{j-2}}\left(\frac{a_{j-1}}{a_{n+1}}(k+\alpha)-1\right)\left(\frac{a_j}{a_{n+1}}(k+\alpha)-1\right)\cdots\left(\frac{a_n}{a_{n+1}}(k+\alpha)-1\right) \quad (3.12)$$

Proof

The proof of Lemma 3.3 is almost identical to that of Lemma 3.2. For fixed $a_j \geq a_{j+1} \geq \cdots \geq a_n \geq a_{n+1} > 1$, let

$$G(a_{j-1}) = (a_{j-1}-1)(a_j-1)\cdots(a_{n+1}-1) - (n+1)\sum_{k=1}^{\lfloor a_{n+1}\rfloor-1}\frac{(k+\alpha)^{j-2}}{a_{n+1}^{j-2}} \times\left(\frac{a_{j-1}}{a_{n+1}}(k+\alpha)-1\right)\left(\frac{a_j}{a_{n+1}}(k+\alpha)-1\right)\cdots\left(\frac{a_n}{a_{n+1}}(k+\alpha)-1\right) \quad (3.13)$$

To prove the lemma, we only need to prove that $G(a_{j-1})$ is a strictly increasing function of a_{j-1} and $G(1) \geq 0$.

Let $a_{j-1} = 1$, it can be seen that $\frac{1}{a_{n+1}}(k+\alpha)-1 < 0$ for $1 \leq k \leq \lfloor a_{n+1}\rfloor - 1$. Furthermore, since $k \geq 1$ and $a_j \geq a_{j+1} \geq \cdots \geq a_n \geq a_{n+1}$, we have

$$\left(\frac{a_j}{a_{n+1}}(k+\alpha)-1\right)\cdots\left(\frac{a_n}{a_{n+1}}(k+\alpha)-1\right) \geq 0 \quad \text{for} \quad 1 \leq k \leq \lfloor a_{n+1}\rfloor - 1 \quad (3.14)$$

Therefore $G(1) \geq 0$

$$\frac{dG}{da_{j-1}} = (a_j-1)\cdots(a_{n+1}-1) - (n+1)\sum_{k=1}^{\lfloor a_{n+1}\rfloor-1}\frac{(k+\alpha)^{j-1}}{a_{n+1}^{j-1}} \times\left(\frac{a_j}{a_{n+1}}(k+\alpha)-1\right)\cdots\left(\frac{a_n}{a_{n+1}}(k+\alpha)-1\right) \quad (3.15)$$
$$> 0 \text{ by (3.11).}$$

Therefore $G(a_{j-1})$ is a strictly increasing function of a_{j-1}. We conclude that $G(a_{j-1}) > 0$ for $a_{j-1} > 1$. □

Proof of the Rough Upper Estimate in GLY Conjecture

Now we are ready to prove Theorem 1.1, which is the rough estimate in GLY conjecture. Without loss of generality, we shall assume $a_1 \geq a_2 \geq \cdots \geq a_n > 1$. Recall that Theorem 1.1 is true for $n = 3, 4, 5, 6$ [Xu-Ya 1] [Xu-Ya 3] [Li-Ya 2] [Wa-Ya]. Out goal is to show that if Theorem 1.1 is true for n-dimensional tetrahedrons, then it must also be true for $n+1$ dimensional tetrahedrons.

Let $n \geq 3$ and $P_n = P_{a_1,a_2,\cdots,a_n}$ be the number of positive integral solutions satisfying

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1 \tag{3.16}$$

where $a_1, a_2, \cdots, a_n$ are positive real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n > 1$. By induction hypothesis, we have

$$n! P_n < (a_1 - 1)(a_2 - 1)\cdots(a_n - 1) \tag{3.17}$$

Consider $P_{n+1} = P_{a_1,a_2,\cdots,a_{n+1}}$ of positive integral solutions satisfying

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_{n+1}}{a_{n+1}} \leq 1 \tag{3.18}$$

where $a_1, a_2, \cdots, a_{n+1}$ are positive real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_{n+1} > 1$. We can dissect (3.18) into $\lfloor a_{n+1} \rfloor$ numbers of n-dimensional tetrahedrons. Specifically, the n-dimensional tetrahedron at $x_{n+1} = \lfloor a_{n+1} \rfloor - k$, where $\lfloor a_{n+1} - 1 \rfloor \geq k \geq 0$, is

$$\frac{x_1}{\frac{a_1}{a_{n+1}}(k+\alpha)} + \frac{x_2}{\frac{a_2}{a_{n+1}}(k+\alpha)} + \cdots + \frac{x_n}{\frac{a_n}{a_{n+1}}(k+\alpha)} \leq 1 \tag{3.19}$$

where $\alpha = a_{n+1} - \lfloor a_{n+1} \rfloor$. Examine (3.19) closely, we see that if $\dfrac{a_n}{a_{n+1}}\alpha < 1$, then the number of lattice points is zero in the layer $k = 0$ of the $n+1$ dimensional tetrahedron. Therefore k goes from one to $\lfloor a_{n+1} \rfloor - 1$ in this case.

Case 1, $\dfrac{a_n}{a_{n+1}}\alpha \geq 1$

According to the induction hypothesis in (3.17), the number of lattice points in (3.18) has the following upper estimate:

$$n!\ P_{n+1} < \sum_{k=0}^{\lfloor a_{n+1} \rfloor - 1} \left(\frac{a_1}{a_{n+1}}(k+\alpha) - 1\right)\left(\frac{a_2}{a_{n+1}}(k+\alpha) - 1\right)\cdots\left(\frac{a_n}{a_{n+1}}(k+\alpha) - 1\right) \tag{3.20}$$

Our goal in this case is to show that

$$(n+1)\sum_{k=0}^{\lfloor a_{n+1} \rfloor - 1} \left(\frac{a_1}{a_{n+1}}(k+\alpha) - 1\right)\cdots\left(\frac{a_n}{a_{n+1}}(k+\alpha) - 1\right) < (a_1 - 1)\cdots(a_{n+1} - 1) \tag{3.21}$$

From Lemma 3.1, we have

$$(n+1)\sum_{k=0}^{\lfloor a_{n+1} \rfloor - 1} \frac{(k+\alpha)^n}{a_{n+1}^n} < (a_{n+1} - 1) \tag{3.22}$$

We can repeatly reapply Lemma 3.2 to (3.22), then after n times we have (3.21),

$$(a_1 - 1)\cdots(a_{n+1} - 1) > (n+1)\sum_{k=0}^{\lfloor a_{n+1} \rfloor - 1} \left(\frac{a_1}{a_{n+1}}(k+\alpha) - 1\right)\cdots\left(\frac{a_n}{a_{n+1}}(k+\alpha) - 1\right) \tag{3.23}$$

Case 2, $\dfrac{a_n}{a_{n+1}}\alpha < 1$

From (3.17), the number of lattice points in (3.18) has the following estimate:

$$n!\, P_{n+1} < \sum_{k=1}^{\lfloor a_{n+1}\rfloor -1} \left(\frac{a_1}{a_{n+1}}(k+\alpha)-1\right)\left(\frac{a_2}{a_{n+1}}(k+\alpha)-1\right)\cdots\left(\frac{a_n}{a_{n+1}}(k+\alpha)-1\right) \tag{3.24}$$

Therefore we only need to show that if $\dfrac{a_n}{a_{n+1}}\alpha < 1$, then

$$(n+1)\sum_{k=1}^{\lfloor a_{n+1}\rfloor -1} \left(\frac{a_1}{a_{n+1}}(k+\alpha)-1\right)\cdots\left(\frac{a_n}{a_{n+1}}(k+\alpha)-1\right) < (a_1-1)\cdots(a_{n+1}-1) \tag{3.25}$$

From Lemma 3.1, we have

$$\begin{aligned}(a_{n+1}-1) &> (n+1)\sum_{k=0}^{\lfloor a_{n+1}\rfloor -1}\frac{(k+\alpha)^n}{a_{n+1}^n}\\ &> (n+1)\sum_{k=1}^{\lfloor a_{n+1}\rfloor -1}\frac{(k+\alpha)^n}{a_{n+1}^n}\end{aligned} \tag{3.26}$$

We can repeatly reapply Lemma 3.3 to (3.26), then after n times we have (3.25). We have so far shown that if $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n > 1$, then $n!P_n < (a_1-1)(a_2-1)\cdots(a_n-1)$. Notice that if $a_n = 1$, then the number of lattice points in any n-dimensional tetrahedron becomes zero. Hence our upper estimate becomes an equality if and only if $a_n = 1$. □

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