# On the maximal width of empty lattice simplices

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We construct $d$-dimensional empty lattice simplices of arbitrarily high volume from $(d-1)$-dimensional ones, while preserving the lattice width. In particular, we give an example of infinitely many empty 4-simplices of width 2.

## 1 Introduction

A $k$-dimensional *lattice simplex* $\sigma \subseteq \mathbb{R}^d$ is the convex hull of $k+1$ affinely independent integer points. General *lattice polytopes* are obtained by taking convex hulls of arbitrary finite subsets of $\mathbb{Z}^d$. A lattice simplex or polytope is called *empty* if it intersects the lattice $\mathbb{Z}^d$ only in its vertices. (Such polytopes are studied also under the names *elementary* and *lattice-free*.)

In dimensions $d > 3$ not all empty lattice simplices have lattice width 1 (that is, they are not all enclosed by two adjacent lattice hyperplanes). However, the famous "flatness theorem" of Khinchine (see [KL, BLPS]) implies that the maximal width of an empty lattice simplex is bounded by a constant $w(d)$ in each fixed dimension $d$. Additionally, Bárány conjectured (personal communication) that in each dimension the volume of an empty lattice simplex of width greater than 1 is bounded (equivalently, there are only finitely many combinatorial types, up to unimodular equivalence). In this paper, we disprove Bárány's conjecture. Even stronger, we show that for every "almost empty" lattice simplex of dimension $d$, there are infinitely many empty lattice simplices of the same width in dimension $d+1$. In particular, this produces an infinite sequence of lattice simplices of width 2 in dimension 4. We also propose a modified finiteness conjecture and present some (computational) evidence for it.

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In the following we consider lattice simplices only up to unimodular transformations. Thus examples of lattice simplices are considered to be different if they cannot be related by a lattice-preserving affine map. The *determinant* of a $d$-dimensional lattice simplex $\sigma = \operatorname{conv}\{a_0, a_1, \ldots, a_d\} \subset \mathbb{R}^d$ is given by $\det(\sigma) := |\det(a_1 - a_0, \ldots, a_d - a_0)|$. The volume of $\sigma$ is then $\frac{1}{d!}\det(\sigma)$.

Let $K \subseteq \mathbb{R}^d$ be any full-dimensional lattice polytope (or even a general full-dimensional convex body). For a linear form $\ell \in (\mathbb{R}^d)^*$ define the *width of $K$ with respect to $\ell$* as

$$\operatorname{width}_\ell(K) \quad := \quad \max \ell(K) - \min \ell(K).$$

Given $K$, the assignment $\ell \longmapsto \operatorname{width}_\ell(K)$ defines a norm on $(\mathbb{R}^d)^*$. The *(lattice) width* of $K$ is

$$\operatorname{width}(K) \quad := \quad \min\{\operatorname{width}_\ell(K) \; : \; \ell \in (\mathbb{Z}^d)^* \backslash \{\mathbf{0}\}\},$$

where $(\mathbb{Z}^d)^*$ denotes the dual lattice.

The maximal width of empty lattice simplices in dimension $d$ is thus encoded in the *maximal width function*:

$$\begin{aligned} w : \mathbb{N} &\longrightarrow \mathbb{N} \\ d &\longmapsto \max\{\operatorname{width}(\sigma) : \sigma \text{ is a } d\text{-dimensional empty lattice simplex}\}. \end{aligned}$$

Here are the main facts that are known about this function:

- $w(2) = 1$ (trivial)
- $w(3) = 1$
  (This is White's Theorem [Wh] [MS] [Sc] [Se].)
- $w(4) \geq 4$
  (We have found that the simplex spanned by $(6, 14, 17, 65)^t$ together with the four unit vectors in $\mathbb{R}^4$ is the smallest example of width 4; it seems to be the only one, up to unimodular equivalence. In particular, we believe that $w(4) = 4$.)
- $w(d) \leq M d \log d$ for some $M$
  (This bound is a recent result of Banaszczyk, Litvak, Pajor & Szarek [BLPS])
- $w(d) \geq d - 2$ for all $d \geq 1$, and $w(d) \geq d - 1$ for even $d$.
  (This was proved, by giving explicit examples

  $$S(d) \quad := \quad \operatorname{conv}\{(d-1)e_1, e_1 + (d-1)e_2, \ldots, e_{d-1} + (d-1)e_d, e_d\},$$

  by Sebő and Bárány [Se]; Kantor [K] had the first linear lower bound.)
- $w(d) \leq w(d+1)$ for all $d$.
  (See Corollary 2 below. One should assume that strict inequality holds for $d \geq 3$, but this is not proved.)

## 2 Results

Our computer search for 4-dimensional empty lattice simplices yielded many simplices of width 2, a bounded number of width 3 and exactly one of width 4. The same results were obtained independently by Fermigier and Kantor. The criteria, methods and results for our search are described in more detail in Section 4.

The examples of lattice simplices of width greater than 1 together with the following proposition show that Bárány's conjecture does not hold in dimension $d \geq 5$.

**Proposition 1** *Every empty $(d-1)$-dimensional lattice simplex $\sigma \subset \mathbb{R}^d$ is a facet of infinitely many empty $d$-dimensional lattice simplices $\widetilde{\sigma} \subset \mathbb{R}^d$ that have at least the same width,* $\mathrm{width}(\widetilde{\sigma}) \geq \mathrm{width}(\sigma)$.

**Corollary 2** *The maximal width function is monotone: $w(d) \geq w(d-1)$.*

**Corollary 3** *For all $d \geq 3$, there are infinitely many equivalence classes of $d$-dimensional empty lattice simplices of width $w(d-1)$.*

For a sharper analysis of the situation we introduce the following concept.

**Definition 4** A lattice simplex without interior lattice points and with at least one empty facet is called *almost empty*. Let $\overline{w}(d)$ be the maximal width function for almost empty simplices.

The following result establishes that $w(d)$ is finite for all $d$.

**Proposition 5** *For any $(d-1)$-dimensional almost empty simplex there are infinitely many $d$-dimensional empty simplices of the same width. In particular, $\overline{w}(d-1) \leq w(d)$ for all $d$.*

The proofs of the propositions are the subject of Section 3. As a special case of Proposition 5, we get the following infinite family of 4-dimensional empty lattice simplices of width $2 > w(3)$, which disproves Bárány's conjecture in dimension 4. For this, we use the notation

$$\sigma[v] \quad := \quad \mathrm{conv}\,\{e_1, e_2, \ldots, e_d, v\},$$

to denote the convex hull of the standard unit vectors together with one additional vector $v \in \mathbb{Z}^d$. We always assume that $\sum_{i=1}^{d} v_i = D+1 > 1$, where $D$ is the determinant of $\sigma[v]$.

**Proposition 6** *For every $D \geq 8$, the 4-simplex $\sigma[(2,2,3,D-6)^t]$, the convex hull of the columns of*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & D-6 \end{pmatrix},$$

*has width 2. It is empty if and only if* $\gcd(D,6) = 1$.

# 3 Proving the Propositions

**Proof of Proposition 1.** We generalize Reeve's construction [R] of arbitrarily large empty tetrahedra, $R(r) = \mathrm{conv}\{\mathbf{0}, e_1, e_2, e_1 + e_2 + re_3\} \subseteq \mathbb{R}^3$.

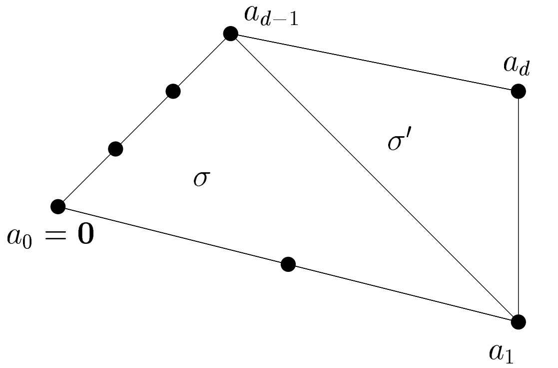
Suppose that $\sigma = \mathrm{conv}\{a_0, \ldots, a_{d-1}\} \subseteq \mathbb{R}^{d-1}$ is an empty simplex with $a_0 = \mathbf{0}$. Every point $p$ in the cone $C$ spanned by $\sigma$ has a unique representation of the form $p = \sum_{i=1}^{d-1} \lambda_i a_i$ with $\lambda_i \geq 0$; we define the *height* of $p$ in $C$ as $ht(p) := \sum_{i=1}^{d-1} \lambda_i$.

Let $a_d \in \mathrm{int}(C) \cap \mathbb{Z}^{d-1}$ be an integer point in the interior of $C$ with minimal height, that is, so that $\lambda_i > 0$ for all $i$, and such that $\sum_i \lambda_i > 1$ is minimal. Then $\mathrm{conv}\{a_0, \ldots, a_d\} \subseteq \mathbb{R}^{d-1}$, a bipyramid over the facet $\mathrm{conv}\{a_1, \ldots, a_{d-1}\}$ of $\sigma$, is empty. The $d$-dimensional simplex $\widetilde{\sigma} = \mathrm{conv}\{\binom{a_0}{0}, \ldots, \binom{a_{d-1}}{0}, \binom{a_d}{h}\} \subseteq \mathbb{R}^d$ derived from it "by lifting $a_d$ to a new dimension" will satisfy our conditions if $h$ is large enough.

To see this, let $p \in \widetilde{\sigma} \cap \mathbb{Z}^d$. The projection of $p$ to $\mathbb{R}^{d-1}$ is an integral point that lies in the bipyramid, so it must be one of the points $a_0, \ldots, a_d$. But the only points of $\widetilde{\sigma}$ with such a projection are the vertices.

Now any functional $\ell \in (\mathbb{Z}^d)^*$ has the same values on the first $d$ vertices of $\widetilde{\sigma}$ as its restriction $\ell'$ to $\mathbb{R}^{d-1}$ takes on $\sigma$, so $\mathrm{width}_\ell(\widetilde{\sigma}) \geq \mathrm{width}_{\ell'}(\sigma)$. This shows that $\mathrm{width}_\ell(\widetilde{\sigma}) \geq \mathrm{width}(\sigma)$, unless $\ell'$ is zero. In this last case $\ell$ is an integer multiple of the $d^{th}$ coordinate function, which takes the values 0 and $h$ on the vertices of $\widetilde{\sigma}$. Thus we have established that $\mathrm{width}(\widetilde{\sigma}) \geq \min\{h, \mathrm{width}(\sigma)\}$. $\square$

**Proof of Proposition 5.** Suppose that $\sigma = \mathrm{conv}\{a_0, \ldots, a_{d-1}\} \subseteq \mathbb{R}^{d-1}$ with $a_0 = \mathbf{0}$ is an almost empty simplex with empty facet $\mathrm{conv}\{a_1, \ldots, a_{d-1}\}$. Choose $a_d$ as in the previous proof. Then $\sigma' := \mathrm{conv}\{a_1, \ldots, a_{d-1}, a_d\}$ is an empty $(d-1)$-simplex by construction, and $\mathrm{conv}\{a_0, a_1, \ldots, a_{d-1}, a_d\} = \sigma \cup \sigma'$ is a bipyramid with apexes $a_0 = \mathbf{0}$ and $a_d$, such that all the integer points in $\sigma \cup \sigma'$, except for the point $a_d$, are contained in the facets of $\sigma$ that contain the apex $a_0 = \mathbf{0}$.




Now we "lift $a_0$ to the next dimension," obtaining

$$\widetilde{\sigma}(h) \;:=\; \mathrm{conv}\{\begin{pmatrix} a_0 \\ h \end{pmatrix}, \begin{pmatrix} a_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} a_d \\ 0 \end{pmatrix}\}.$$

The following two claims establish that

(1) the simplex $\widetilde{\sigma}(h)$ has width $\mathrm{width}(\widetilde{\sigma}(h)) \geq \mathrm{width}(\sigma)$ if $h$ is large enough, and

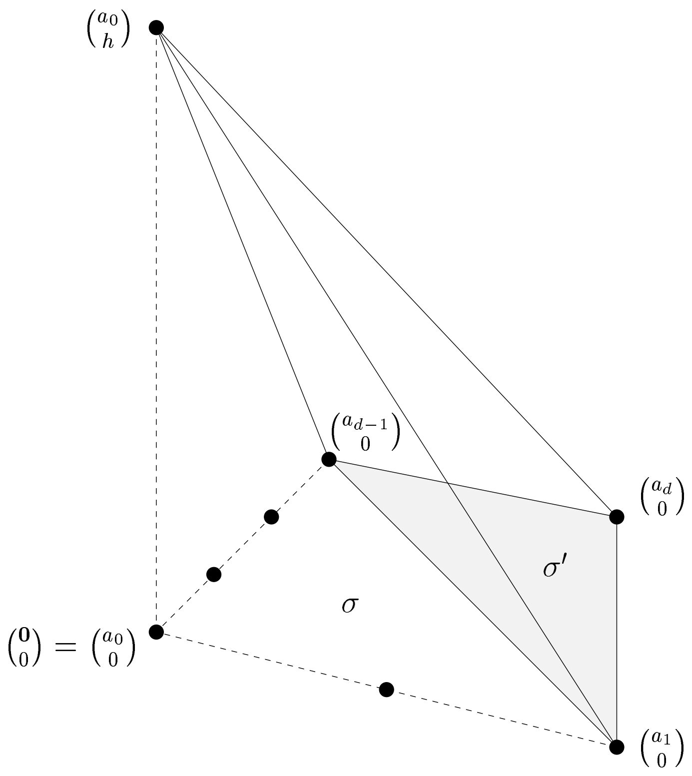
(2) it is empty for infinitely many $h$.

**(1).** *There is a constant $H = H(\sigma, a_d)$ so that* $\mathrm{width}(\widetilde{\sigma}(h)) \geq \mathrm{width}(\sigma)$ *for every $h > H$.*
For $\ell \in (\mathbb{R}^d)^*$ let $\ell'$ denote its restriction to $\mathbb{R}^{d-1}$ and let $l_d$ be the remaining component. Then $\mathrm{width}_\ell(\widetilde{\sigma}(h)) \geq \mathrm{width}_{\ell'}(\sigma')$. Now $\sigma'$ is a full-dimensional lattice simplex (and thus a convex body) in $\mathbb{R}^{d-1}$. Thus the set $\{\ell' \in (\mathbb{R}^{d-1})^* : \mathrm{width}_{\ell'}(\sigma') \leq \mathrm{width}(\sigma)\}$ is bounded. In particular, there is some $M$ such that $\max|\ell'(\sigma')| \leq M$ for every $\ell'$ from this set. From now on, we only consider functionals $\ell$ that satisfy $\mathrm{width}_{\ell'}(\sigma') < \mathrm{width}(\sigma)$ — otherwise the claim is clear anyway.

Now if $l_d = 0$ (and $\ell \neq \mathbf{0}$), then $\mathrm{width}_\ell(\widetilde{\sigma}(h)) = \mathrm{width}_{\ell'}(\sigma \cup \sigma') \geq \mathrm{width}(\sigma)$. But if $l_d \neq 0$ and $h > H := M + \mathrm{width}(\sigma)$, then

$$\left|\ell\left(\binom{a_0}{h} - \binom{a_i}{0}\right)\right| \;=\; |l_d h - \ell'(a_i)| \;\geq\; h - \max|\ell'(\sigma')| \;>\; \mathrm{width}(\sigma).$$

This implies that $\mathrm{width}_\ell(\widetilde{\sigma}(h)) \geq \mathrm{width}(\sigma)$. □



**(2).** *Let $D = \det[a_1, \dots, a_{d-1}]$. If $\gcd(D,h) = 1$, then $\widetilde{\sigma}(h)$ is empty.*

Let $\ell \in (\mathbb{R}^{d-1})^*$ the linear form on $\mathbb{R}^{d-1}$ that takes the value 1 on $a_1, \dots, a_{d-1}$ (and 0 on $a_0 = \mathbf{0}$).

Let $\widetilde{x} \in \widetilde{\sigma}(h)$ be an integer point. Its projection $x \in \sigma \cup \sigma'$ to $\mathbb{R}^{d-1}$ has integral coordinates. If $x = a_d$, then $\widetilde{x} = \binom{a_d}{0}$ is a vertex of $\widetilde{\sigma}(h)$. Otherwise $x$ lies in a facet of $\sigma$ that contains $a_0$. We can decompose

$$\widetilde{x} \quad = \quad \begin{pmatrix} x \\ x_d \end{pmatrix} \quad = \quad \frac{x_d}{h}\begin{pmatrix} \mathbf{0} \\ h \end{pmatrix} + \left(1 - \frac{x_d}{h}\right)\begin{pmatrix} v \\ 0 \end{pmatrix}$$

for some $v \in \sigma'$. This yields $x = \left(1 - \frac{x_d}{h}\right) v$. If $x = \mathbf{0}$, then $\widetilde{x} = \binom{a_0}{h}$ is a vertex of $\widetilde{\sigma}(h)$. Otherwise, we find that $x$ lies in a facet of $\sigma$ (and thus of $\sigma \cup \sigma'$) that contains the vertex $a_0 = \mathbf{0}$, while some multiple of $x = (1 - \frac{x_d}{h})v$, namely $v$, lies in $\sigma'$. Thus the geometry of the bipyramid $\sigma \cup \sigma'$ implies that $v \in \sigma \cap \sigma'$, and thus $\ell(v) = 1$. Hence

$$\ell(x) \quad = \quad 1 - \frac{x_d}{h}.$$

On the other hand, by Cramer's rule, $x$ has a unique representation $D \cdot x = \sum_{i=1}^{d-1} \lambda_i a_i$ with $\lambda_i \in \mathbb{Z}$. Thus

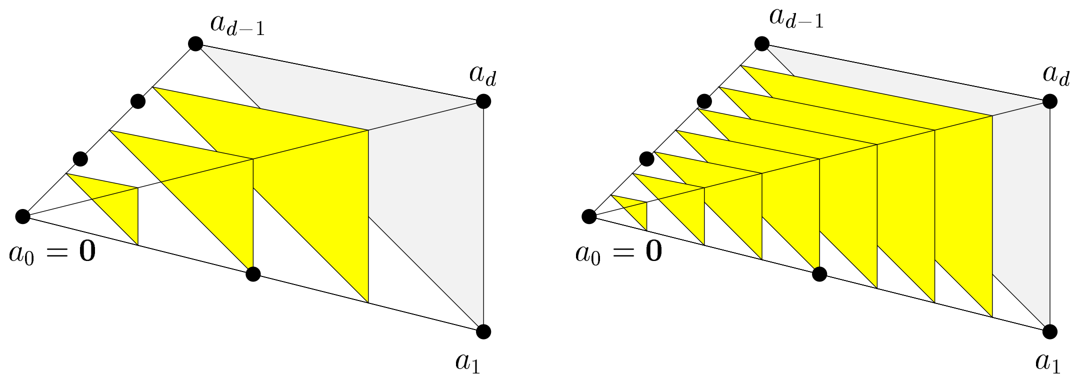
$$D \cdot \ell(x) \quad \in \quad \mathbb{Z}.$$

We conclude that

$$D\frac{x_d}{h} \quad = \quad D - D \cdot \ell(x) \quad \in \quad \mathbb{Z}.$$

Thus if $\gcd(D,h) = 1$, then either $x_d = h$ and thus $\widetilde{x} = \binom{\mathbf{0}}{h}$ is the top vertex of $\widetilde{\sigma}(h)$, or we have $x_d = 0$ and thus $\widetilde{x} \in \sigma' \cap \mathbb{Z}^{d-1}$ is one of the other vertices of $\widetilde{\sigma}(h)$. $\square$

On the other side, if $\sigma$ does have interior lattice points, then $\widetilde{\sigma}(h)$ is an empty simplex only for finitely many values of $h$. To see this, consider the intersections $\widetilde{\sigma}(h) \cap \{\widetilde{x} \in \mathbb{R}^d : x_d = k\}$ for integers $k$. The projection $Z(h)$ of their union to $\mathbb{R}^{d-1}$ is a "forbidden zone" for integer points: if it contains an integer point, then $\widetilde{\sigma}(h)$ is not empty. The following pictures illustrate this for dimension $d = 2+1$ and for the heights 4 and 8.



One can see that $Z(h)$ contains an inner parallel body of $\sigma$ that grows with $h$, and which completely fills the interior of $\sigma$ for $h \longrightarrow \infty$. So any fixed interior point of $\sigma$ lies in $Z(h)$ if $h$ is large enough.

**Conjecture 7** *For every $d \geq 2$, there are only finitely many equivalence classes of empty $d$-simplices whose width is greater than $\overline{w}(d-1)$, the greatest width that can be achieved in dimension $d-1$ by almost empty simplices.*

# 4 Computer search in dimension 4

The strategy in our search for empty simplices of large width and volume was the following:

1. to enumerate all equivalence classes of (possibly) empty simplices,
2. to check whether or not the simplex is in fact empty, and
3. if so, to calculate the width.

This relied on the following two known results.

**Theorem 8 (Wessels [We])** *Every empty 4-simplex has at least two unimodular facets. In particular, every such simplex is equivalent to a simplex of the form $\sigma[v]$.*

(The analogous statement is false in higher dimensions. For example, the simplex in $\mathbb{R}^5$ given by the columns of

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{pmatrix}$$

is empty, without any unimodular facet [We, p. 21].)

**Theorem 9 (cf. Scarf [Sc])** *A lattice simplex of the form $\sigma[v] = \operatorname{conv}\{e_1, \ldots, e_d, v\}$, of determinant $D := \sum_{i=1}^{d} v_i - 1 > 0$, is empty if and only if*

$$\sum_{i=1}^{d} \left\lceil \frac{kv_i}{D} \right\rceil \quad > \quad k+1 \qquad (*)$$

*holds for all integers $k$ in the range $1 \leq k \leq D/2$.*

**Proof.** Observe that

$$\sum_{i=1}^{d} \left\lceil \frac{kv_i}{D} \right\rceil \quad \geq \quad k+1$$

always holds, since $\sum_{i=1}^{d} \lceil \frac{kv_i}{D} \rceil \geq \sum_{i=1}^{d} \frac{kv_i}{D} = \frac{k}{D} \sum_{i=1}^{d} v_i = k\frac{D+1}{D} > k$.
Also it is readily checked that $\sum_{i=1}^{d} x_i \geq 1$ together with the inequalities

$$\frac{\sum_{i=1}^{d} x_i - 1}{D} v_j \leq x_j < \frac{\sum_{i=1}^{d} x_i - 1}{D} v_j + 1 \qquad (**)$$

describes the set $\sigma \backslash \{e_1, \ldots, e_d\}$: the weak inequalities describe $\sigma$ and the strict ones cut off the vertices $e_i$.
Now if there is a lattice point $x$ in $\sigma \backslash \{e_1, \ldots, e_d\}$, with $k := \sum_{i=1}^{d} x_i - 1$, then by $(**)$ it must have the coordinates

$$x_j \quad = \quad \left\lceil \frac{kv_j}{D} \right\rceil . \qquad (***)$$

Thus $x$ violates $(*)$, but it need not satisfy $k \leq D/2$. However, if $k > D/2$, then we get another lattice point $x' := v + 2(x - v) = 2x - v$ with

$$k' \quad := \quad \sum_{i=1}^{d} x'_i \quad -1 \quad = \quad D + 2(k - D) \quad = \quad 2k - D \quad > \quad 0.$$

Thus from $k' \geq 0$ we get that $x'$ is another lattice point in $\sigma[v]$, which is not a vertex because of $k' > 0$, but whose sum of coordinates is smaller than that of $x$. Iterating this procedure, we finally arrive at a lattice point $x^*$ in $\sigma \backslash \{e_1, \ldots, e_d\}$ which satisfies $k^* \leq D/2$, and which violates the condition $(*)$. This finishes the *only if* part of the proof.
On the other hand, if for some $k$

$$\sum_{i=1}^{d} \left\lceil \frac{kv_i}{D} \right\rceil \quad = \quad k+1,$$

then the vector $x$ given by $(***)$ satisfies $(**)$, and thus provides a lattice point in $\sigma[v]$ which is not a vertex. □

We can further restrict our search for empty simplices, as follows.

**Lemma 10** *The simplex $\sigma[v]$ is unimodularly equivalent to $\sigma[v + D\epsilon]$, where $\epsilon \in \mathbb{Z}^d$ is any vector with vanishing coordinate sum. In particular, $\sigma[v]$ is equivalent to some $\sigma[v']$ with $\|v'\|_\infty \leq D$.* □

In principle, the width of a lattice simplex can be found by solving an integer program, as demonstrated by the following lemma.

**Lemma 11** *Let $W$ be an upper bound for the width of the simplex $\sigma[v]$. Then* $\mathrm{width}(\sigma[v])$ *is the optimal value of the following minimization problem:*

$$\begin{array}{rclcl} \multicolumn{5}{l}{\textit{minimize}\ \ w\ \ \textit{subject to}} \\ w_0 & \leq & l_i & \leq & w_0 + w \qquad \textit{for}\ \ 1 \leq i \leq d, \\ w_0 & \leq & \sum_{i=1}^d l_i v_i & \leq & w_0 + w, \\ & & \sum_{i=1}^d l_i W^{i-1} & \geq & 1, \end{array}$$

*with integer variables $w$, $w_0$, and $l_i$. (The values of the $l_i$-variables in an optimal solution yield a linear functional that realizes the width.)*

**Proof.** The width is defined to be the minimal solution to the first constraints, excluding the zero solution ($\ell = \mathbf{0}$, $w = w_0 = 0$). This solution is cut off by the last constraint. We have to see that any $\ell \neq \mathbf{0}$ that "realizes" the width of $\sigma[v]$ satisfies this last constraint. By replacing the $l_i$ by their negatives, we can assure the lefthand side of this constraint to be non-negative. If it was zero, the first non-zero $l_i$ would have to be a multiple of $W$ and some other $l_j$ would have the opposite sign, with the effect that $|\ell(e_i) - \ell(e_j)| = |l_i - l_j| > |l_i| \geq W$. □

An integer programming formulation as in Lemma 11 also shows that the width of a general lattice simplex can be computed in polynomial time if the dimension is fixed. (A different IP formulation was provided by Sebő, [Se, Sect. 5].)

Somewhat surprisingly, our computational tests using CPLEX* showed that the integer programs of Lemma 11 can indeed be solved fast and stably. We used this for an enumeration of 4-dimensional empty lattice simplices up to determinant $D = 350$, and also for tests in dimension 5.

However, for larger determinants a less sophisticated criterion proved to be faster. Namely, a simplex $\sigma[v]$ has lattice width greater than $w$ if and only if there is no solution to

$$\begin{array}{rclcl} 0 & \leq & l'_i & \leq & w \qquad \text{for } 1 \leq i \leq d, \\ & & \sum_{i=1}^d l'_i v_i \pmod{D} & \leq & w \\ & & (l'_1, \ldots, l'_d) & \neq & (0, \ldots, 0). \end{array}$$

This is easily derived from the system of Lemma 11 using the substitutions $l'_i := l_i - w_0$. Thus, to test e. g. whether a 4-dimensional simplex has width greater than 2, one simply has to see whether one of $3^4 - 1 = 80$ different 4-tuples $(l'_1, \ldots, l'_4) \in \{0, 1, 2\}^4 \backslash \{\mathbf{0}\}$ satisfies the modular equation

$$\sum_{i=1}^d l'_i v_i \pmod{D} \quad \leq \quad 2.$$

*CPLEX Linear Optimizer 4.0.8 with Mixed Integer & Barrier Solvers; ©CPLEX Optimization, Inc., 1989-1995

The following records our computational results, based on generation and test of all equivalence classes of empty lattice simplices of determinant $D \le 1000$. They provide evidence for $w(4) = 4$ as well as for Conjecture 7.

**Theorem 12** *Among the 4-dimensional empty lattice simplices of determinant $D \le 1000$,*

- *there are no simplices of width $w \ge 5$,*
- *there is a unique equivalence class of simplices of width 4, given by $\sigma[(6, 14, 17, 65)^t]$, whose determinant is $D = 101$,*
- *all simplices of width 3 have determinant $D \le 179$, where the (unique) smallest example, of determinant $D = 41$, is given by $\sigma[(-10, 4, 23, 25)^t]$, and the (unique) example of determinant $D = 179$ is given by $\sigma[(20, 36, 53, 71)^t]$.*

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