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COMPUTING THE EHRHART QUASI-POLYNOMIAL OF A RATIONAL SIMPLEX

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ABSTRACT. We present a polynomial time algorithm to compute any fixed number of the highest coefficients of the Ehrhart quasi-polynomial of a rational simplex. Previously such algorithms were known for integer simplices and for rational polytopes of a fixed dimension. The algorithm is based on the formula relating the kth coefficient of the Ehrhart quasi-polynomial of a rational polytope to volumes of sections of the polytope by affine lattice subspaces parallel to k-dimensional faces of the polytope. We discuss possible extensions and open questions.

1. INTRODUCTION AND MAIN RESULTS

Let $P \subset \mathbb{R}^d$ be a rational polytope, that is, the convex hull of a finite set of points with rational coordinates. Let $t \in \mathbb{N}$ be a positive integer such that the vertices of the dilated polytope

$$tP = \Big\{tx : \ x \in P\Big\}$$

are integer vectors. As is known (see, for example, Section 4.6 of [27]), there exist functions $e_i(P; \cdot) : \mathbb{N} \longrightarrow \mathbb{Q}$, $i = 0, \ldots, d$, such that

$$e_i(P; n+t) = e_i(P; n) \quad \text{for all } n \in \mathbb{N}$$

and

$$|nP \cap \mathbb{Z}^d| = \sum_{i=0}^{d} e_i(P; n) n^i \quad \text{for all } n \in \mathbb{N}.$$

The function on the right-hand side is called the *Ehrhart quasi-polynomial* of P. It is clear that if $\dim P = d$, then $e_d(P; n) = \operatorname{vol} P$. In this paper, we are interested in the computational complexity of the coefficients $e_i(P; n)$.

If the dimension d is fixed in advance, the values of $e_i(P; n)$ for any given P, n, and i can be computed in polynomial time by interpolation, as implied by a polynomial time algorithm to count integer points in a polyhedron of a fixed dimension [4], [6].

If the dimension d is allowed to vary, it is an NP-hard problem to check whether $P \cap \mathbb{Z}^d \neq \emptyset$, let alone to count integer points in P. This is true even when P is a rational simplex, as exemplified by the knapsack problem; see, for example, Section 16.6 of [25]. If the polytope P is integral, then the coefficients $e_i(P; n) = e_i(P)$

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do not depend on n. In that case, for any k fixed in advance, computation of the Ehrhart coefficient $e_{d-k}(P)$ reduces in polynomial time to computation of the volumes of the $(d-k)$-dimensional faces of P [5]. The algorithm is based on efficient formulas relating $e_{d-k}(P)$, volumes of the $(d-k)$-dimensional faces, and cones of feasible directions at those faces; see [22], [6], and [23]. In particular, if $P = \Delta$ is an integer simplex, there is a polynomial time algorithm for computing $e_{d-k}(\Delta)$ as long as k is fixed in advance.

In this paper, we extend the last result to *rational* simplices (a d-dimensional rational simplex is the convex hull in $\mathbb{R}^d$ of $(d+1)$ affinely independent points with rational coordinates).

- Let us fix an integer $k \geq 0$. The paper presents a polynomial time algorithm, which, given an integer $d \geq k$, a rational simplex $\Delta \subset \mathbb{R}^d$, and a positive integer n, computes the value of $e_{d-k}(\Delta; n)$.

We present the algorithm in Section 7 and discuss its possible extensions in Section 8.

This is in contrast to the case of an integral polytope, for a general rational polytope P computation of $e_i(P; n)$ cannot be reduced to computation of the volumes of faces and some functionals of the "angles" (cones of feasible direction) at the faces. A general result of McMullen [19] (see also [21] and [20]) asserts that the contribution of the i-dimensional face F of a rational polytope P to the coefficient $e_i(P; n)$ is a function of the volume of F, the cone of feasible directions of P at F, *and* the translation class of the affine hull $\operatorname{aff}(F)$ of F modulo $\mathbb{Z}^d$.

Our algorithm is based on a new structural result, Theorem 1.1 below, relating the coefficient $e_{d-k}(P; n)$ to volumes of sections of P by affine lattice subspaces parallel to faces F of P with $\dim F \geq d-k$. Theorem 1.1 may be of interest in its own right.

1.1. **Valuations and polytopes.** Let V be a d-dimensional real vector space and let $\Lambda \subset V$ be a lattice, that is, a discrete additive subgroup which spans V. A polytope $P \subset V$ is called a Λ-*polytope* or a *lattice polytope* if the vertices of P belong to Λ. A polytope $P \subset V$ is called Λ-*rational* or just *rational* if tP is a lattice polytope for some positive integer t.

For a set $A \subset V$, let $[A] : V \longrightarrow \mathbb{R}$ be the indicator of A:

$$[A](x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

A complex-valued function ν on rational polytopes $P \subset V$ is called a *valuation* if it preserves linear relations among indicators of rational polytopes:

$$\sum_{i \in I} \alpha_i [P_i] = 0 \Longrightarrow \sum_{i \in I} \alpha_i \nu(P_i) = 0,$$

where $P_i \subset V$ is a finite family of rational polytopes and α_i are rational numbers. We consider only Λ-*valuations* or *lattice valuations* ν that satisfy

$$\nu(P+u) = \nu(P) \quad \text{for all } u \in \Lambda;$$

see [21] and [20].

A general result of McMullen [19] states that if ν is a lattice valuation, $P \subset V$ is a rational polytope, and $t \in \mathbb{N}$ is a number such that tP is a lattice polytope, then there exist functions $\nu_i(P;\cdot) : \mathbb{N} \longrightarrow \mathbb{C}$, $i = 0, \dots, d$, such that

$$\nu(nP) = \sum_{i=0}^{d} \nu_i(P;n)n^i \quad \text{for all } n \in \mathbb{N}$$

and

$$\nu_i(P; n+t) = \nu_i(P;n) \quad \text{for all } n \in \mathbb{N}.$$

Clearly, if we compute $\nu(mP)$ for $m = n, n+t, \dots, n+td$, we can obtain $\nu_i(P;n)$ by interpolation.

We are interested in the counting valuation E, where $V = \mathbb{R}^d$, $\Lambda = \mathbb{Z}^d$, and

$$E(P) = |P \cap \mathbb{Z}^d|$$

is the number of lattice points in P.

The idea of the algorithm is to replace valuation E by some other valuation, so that the coefficients $e_d(P;n), \dots, e_{d-k}(P;n)$ remain intact, but the new valuation can be computed in polynomial time on any given rational simplex Δ, so that the desired coefficient $e_{d-k}(\Delta;n)$ can be obtained by interpolation.

1.2. **Valuations E_L.** Let $L \subset \mathbb{R}^d$ be a lattice subspace, that is, a subspace spanned by the points $L \cap \mathbb{Z}^d$. Suppose that $\dim L = k$ and let $pr : \mathbb{R}^d \longrightarrow L$ be the orthogonal projection onto L. Let $P \subset \mathbb{R}^d$ be a rational polytope, let $Q = pr(P)$, $Q \subset L$, be its projection, and let $\Lambda = pr\left(\mathbb{Z}^d\right)$. Since L is a lattice subspace, $\Lambda \subset L$ is a lattice.

Let $L^\perp$ be the orthogonal complement of L. Then $L^\perp \subset \mathbb{R}^d$ is a lattice subspace. We introduce the volume form vol_{d-k} on $L^\perp$ which differs from the volume form inherited from $\mathbb{R}^d$ by a scaling factor chosen so that the determinant of the lattice $\mathbb{Z}^d \cap L^\perp$ is 1. Consequently, the same volume form vol_{d-k} is carried by all translations $x + L^\perp$, $x \in \mathbb{R}^d$.

We consider the following quantity

$$E_L(P) = \sum_{m \in \Lambda} \mathrm{vol}_{d-k}\left(P \cap \left(m + L^\perp\right)\right) = \sum_{m \in Q \cap \Lambda} \mathrm{vol}_{d-k}\left(P \cap \left(m + L^\perp\right)\right)$$

(clearly, for $m \notin Q$ the corresponding terms are 0).

In words, we take all lattice translates of $L^\perp$, select those that intersect P, and add the volumes of the intersections.

Clearly, E_L is a lattice valuation, so

$$E_L(nP) = \sum_{i=0}^{d} e_i(P, L; n)n^i$$

for some periodic functions $e_i(P, L; \cdot)$. If tP is an integer polytope for some $t \in \mathbb{N}$, then

$$e_i(P, L; n+t) = e_i(P, L; n) \quad \text{for all } n \in \mathbb{N}$$

and $i = 0, \dots, d$.

Note that if $L = \{0\}$, then $E_L(P) = \mathrm{vol}\, P$ and if $L = \mathbb{R}^d$, then $E_L(P) = |P \cap \mathbb{Z}^d|$, so the valuations E_L interpolate between the volume and the number of lattice points as $\dim L$ grows.

We prove that $e_{d-k}(P;n)$ can be represented as a linear combination of $e_{d-k}(P, L; n)$ for some lattice subspaces L with $\dim L \leq k$.

Theorem 1.1. *Let us fix an integer $k \geq 0$. Let $P \subset \mathbb{R}^d$ be a full-dimensional rational polytope and let t be a positive integer such that tP is an integer polytope. For a $(d-k)$-dimensional face F of P let $\operatorname{lin}(F) \subset \mathbb{R}^d$ be the $(d-k)$-dimensional subspace parallel to the affine hull $\operatorname{aff}(F)$ of F and let $L^F = (\operatorname{lin} F)^{\perp}$ be its orthogonal complement, so $L^F \subset \mathbb{R}^d$ is a k-dimensional lattice subspace.*

Let $\mathcal{L}$ be a finite collection of lattice subspaces which contains the subspaces L^F for all $(d-k)$-dimensional faces F of P and is closed under intersections. For $L \in \mathcal{L}$ let $\mu(L)$ be integer numbers such that the identity

$$\left[\bigcup_{L \in \mathcal{L}} L\right] = \sum_{L \in \mathcal{L}} \mu(L)[L]$$

holds for the indicator functions of the subspaces from $\mathcal{L}$.

Let us define

$$\nu(nP) = \sum_{L \in \mathcal{L}} \mu(L) E_L(nP) \quad \text{for } n \in \mathbb{N}.$$

Then there exist functions $\nu_i(P; \cdot) : \mathbb{N} \longrightarrow \mathbb{Q}$, $i = 0, \ldots, d$, such that

(1)

$$\nu(nP) = \sum_{i=0}^{d} \nu_i(P; n) n^i \quad \text{for all } n \in \mathbb{N},$$

(2)

$$\nu_i(P; n+t) = \nu_i(P; n) \quad \text{for all } n \in \mathbb{N},$$

and

(3)

$$e_{d-i}(P; n) = \nu_{d-i}(P; n) \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad i = 0, \ldots, k.$$

We prove Theorem 1.1 in Section 4 after some preparations in Sections 2 and 3.

Remark 1.2. Valuation E clearly does not depend on the choice of the scalar product in $\mathbb{R}^d$. One can observe that valuation ν of Theorem 1.1 admits a dual description which does not depend on the scalar product. Instead of $\mathcal{L}$, we consider the set $\mathcal{L}^{\vee}$ of subspaces containing the subspaces $\operatorname{lin}(F)$ and closed under taking sums of subspaces, and for $L \in \mathcal{L}^{\vee}$ we define $E_L^{\vee}(\cdot)$ as the sum of the volumes of sections of the polytope by the lattice affine subspaces parallel to L. Then

$$\nu = \sum_{L \in \mathcal{L}^{\vee}} \mu^{\vee}(L) E_L^{\vee},$$

where $\mu^{\vee}$ are some integers computed from the set $\mathcal{L}^{\vee}$, partially ordered by inclusion.

However, using the explicit scalar product turns out to be more convenient.

The advantage of working with valuations E_L is that they are more amenable to computations.

• Let us fix an integer $k \geq 0$. We present a polynomial time algorithm, which, given an integer $d \geq k$, a d-dimensional rational simplex $\Delta \subset \mathbb{R}^d$, and a lattice subspace $L \subset \mathbb{R}^d$ such that $\dim L \leq k$, computes $E_L(\Delta)$.

We present the algorithm in Section 6 after some preparations in Section 5.

1.3. **The main ingredient of the algorithm to compute $e_{d-k}(\Delta; n)$.** Theorem 1.1 allows us to reduce the computation of $e_{d-k}(\Delta; n)$ to that of $E_L(\Delta)$, where $L \subset \mathbb{R}^d$ is a lattice subspace and $\dim L \leq k$. Let us choose a particular lattice subspace L with $\dim L = j \leq k$.

If $P = \Delta$ is a simplex, then the description of the orthogonal projection $Q = pr(\Delta)$ of Δ onto L can be computed in polynomial time. Moreover, one can compute in polynomial time a decomposition of Q into a union of non-intersecting polyhedral pieces Q_i, such that $\mathrm{vol}_{d-j}\left(pr^{-1}(x)\right)$ is a polynomial on each piece Q_i. Thus computing of $E_L(\Delta)$ reduces to computing of the sum

$$\sum_{m \in Q_i \cap \Lambda} \phi(m),$$

where ϕ is a polynomial with $\deg \phi = d - j$, $Q_i \subset L$ is a polytope with $\dim Q_i = j \leq k$, and $\Lambda \subset L$ is a lattice. The sum is computed by applying the technique of "short rational functions" for lattice points in polytopes of a fixed dimension; cf. [7], [6], and [12].

The algorithm for computing the sum of a polynomial over integer points in a polytope is discussed in Section 5.

2. The Fourier expansions of E and E_L

Let V be a d-dimensional real vector space with the scalar product $\langle \cdot, \cdot \rangle$ and the corresponding Euclidean norm $\| \cdot \|$. Let $\Lambda \subset V$ be a lattice and let $\Lambda^* \subset V$ be the *dual* or the *reciprocal* lattice

$$\Lambda^* = \Big\{ x \in V : \ \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in \Lambda \Big\}.$$

For $\tau > 0$, we introduce the *theta function*

$$\begin{aligned}\theta_\Lambda(x, \tau) =& \tau^{d/2} \sum_{m \in \Lambda} \exp\left\{-\pi\tau \|x - m\|^2\right\} \\ =& (\det \Lambda)^{-1} \sum_{l \in \Lambda^*} \exp\left\{-\pi \|l\|^2/\tau + 2\pi i \langle l, x \rangle\right\}, \quad \text{where } x \in V.\end{aligned}$$

The last inequality is the reciprocity relation for theta series (essentially, the Poisson summation formula); see, for example, Section 69 of [9].

For a polytope P, let $\operatorname{int} P$ denote the relative interior of P and let $\partial P = P \setminus \operatorname{int} P$ be the boundary of P.

Lemma 2.1. *Let $P \subset V$ be a full-dimensional polytope such that $\partial P \cap \Lambda = \emptyset$. Then*

$$\begin{aligned}|P \cap \Lambda| =& \lim_{\tau \longrightarrow +\infty} \int_P \theta_\Lambda(x, \tau) \ dx \\ =& (\det \Lambda)^{-1} \lim_{\tau \longrightarrow +\infty} \sum_{l \in \Lambda^*} \exp\left\{-\pi \|l\|^2/\tau\right\} \int_P \exp\{2\pi i \langle l, x \rangle\} \ dx.\end{aligned}$$

Proof. As is known (cf., for example, Section B.5 of [17]), as $\tau \longrightarrow +\infty$, the function $\theta_\Lambda(x, \tau)$ converges in the sense of distributions to the sum of the delta-functions concentrated at the points $m \in \Lambda$. Therefore, for every smooth function $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ with a compact support, we have

$$\lim_{\tau \longrightarrow +\infty} \int_{\mathbb{R}^d} \phi(x)\theta_\Lambda(x, \tau) \ dx = \sum_{m \in \Lambda} \phi(m). \tag{2.1}$$

Since $\partial P \cap \Lambda = \emptyset$, we can replace ϕ by the indicator function $[P]$ in (2.1). $\square$

Remark 2.2. If $\partial P \cap \Lambda \neq \emptyset$, the limit still exists but then it counts every lattice point $m \in \partial P$ with the weight equal to the "solid angle" of m at P, since every term $\exp\left\{-\pi\tau\|x-m\|^2\right\}$ is spherically symmetric about m. This connection between the solid angle valuation and the theta function was described by the author in the unpublished paper [2] (the paper is very different from paper [5] which has the same title) and independently discovered by Diaz and Robins [13]. Diaz and Robins used a similar approach based on Fourier analysis to express coefficients of the Ehrhart polynomial of an integer polytope in terms of cotangent sums [14]. Banaszczyk [1] obtained asymptotically optimal bounds in transference theorems for lattices by using a similar approach with theta functions, with the polytope P replaced by a Euclidean ball.

The formula of Lemma 2.1 can be considered as the Fourier expansion of the counting valuation.

We need a similar result for valuation E_L defined in Section 1.2.

Lemma 2.3. *Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope and let $L \subset \mathbb{R}^d$ be a lattice subspace with* $\dim L = k$. *Let $pr : \mathbb{R}^d \longrightarrow L$ be the orthogonal projection onto L, let $Q = pr(P)$, and let $\Lambda = pr(\mathbb{Z}^d)$, so $\Lambda \subset L$ is a lattice in L. Suppose that $\partial Q \cap \Lambda = \emptyset$.*

Then

$$E_L(P) = \lim_{\tau \longrightarrow +\infty} \sum_{l \in L \cap \mathbb{Z}^d} \exp\left\{-\pi\|l\|^2/\tau\right\} \int_P \exp\{2\pi i\langle l, x\rangle\}\ dx.$$

Proof. We observe that $L \cap \mathbb{Z}^d = \Lambda^*$. For a vector $x \in \mathbb{R}^d$, let x_L be the orthogonal projection of x onto L. Applying the reciprocity relation for theta functions in L, we write

$$\begin{aligned}
&\sum_{l \in L \cap \mathbb{Z}^d} \exp\left\{-\pi\|l\|^2/\tau + 2\pi i\langle l, x\rangle\right\} \\
=&\sum_{l \in L \cap \mathbb{Z}^d} \exp\left\{-\pi\|l\|^2/\tau + 2\pi i\langle l, x_L\rangle\right\} \\
=&(\det\Lambda)\tau^{k/2} \sum_{m \in \Lambda} \exp\left\{-\pi\tau\|x_L - m\|^2\right\}.
\end{aligned}$$

As is known (cf., for example, Section B.5 of [17]), as $\tau \longrightarrow +\infty$, the function

$$g_\tau(x) = \tau^{k/2} \sum_{m \in \Lambda} \exp\left\{-\pi\tau\|x_L - m\|^2\right\}$$

converges in the sense of distributions to the sum of the delta-functions concentrated on the subspaces $m + L^\perp$ (this is the set of points where $x_L = m$) for $m \in \Lambda$.

Therefore, for every smooth function $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ with a compact support, we have

$$\lim_{\tau \longrightarrow +\infty} \int_{\mathbb{R}^d} \phi(x) g_\tau(x)\ dx = \sum_{m \in \Lambda} \int_{m+L^\perp} \phi(x)\ d_{L^\perp}x, \tag{2.2}$$

where $d_{L^\perp}x$ is the Lebesgue measure on $m + L^\perp$ induced from $\mathbb{R}^d$.

Since $\partial Q \cap \Lambda = \emptyset$, each subspace $m + L^\perp$ for $m \in \Lambda$ either intersects the interior of P or is at least some distance $\epsilon = \epsilon(P, L) > 0$ away from P. Hence we may replace ϕ by the indicator $[P]$ in (2.2).

Recall from Section 1.2 that measuring volumes in $m + L^\perp$, we scale the volume form in $L^\perp$ induced from $\mathbb{R}^d$ so that the determinant of the lattice $L^\perp \cap \mathbb{Z}^d$ is 1. One can observe that $\det \Lambda$ provides the required normalization factor, so

$$(\det \Lambda) \int_{m+L^\perp} [P](x)\ d_{L^\perp}(x) = \mathrm{vol}_{d-k}\left(P \cap \left(m + L^\perp\right)\right).$$

The proof now follows. □

Remark 2.4. If $\partial Q \cap \Lambda \neq \emptyset$, the limit still exists, but then for $m \in \partial Q \cap \Lambda$ the volume $\mathrm{vol}_{d-k}\left(P \cap \left(m + L^\perp\right)\right)$ is counted with the weight defined as follows: we find the minimal (under inclusion) face F of P such that $m + L^\perp$ is contained in $\mathrm{aff}(F)$ and the weight is equal to the solid angle of P at F.

3. Exponential valuations

Let V be a d-dimensional Euclidean space, let $\Lambda \subset V$ be a lattice, and let Λ^* be the reciprocal lattice. Let us choose a vector $l \in \Lambda^*$ and let us consider the integral

$$\Phi_l(P) = \int_P \exp\{2\pi i \langle l, x\rangle\}\ dx,$$

where dx is the Lebesgue measure in V. Note that for $l = 0$ we have $\Phi_l(P) = \Phi_0(P) = \mathrm{vol}\, P$. We have

$$\Phi_l(P + a) = \exp\left\{2\pi i \langle l, a\rangle\right\} \Phi_l(P) \quad \text{for all } a \in V.$$

It follows that Φ_l is a Λ-valuation on rational polytopes $P \subset V$.

If $l \neq 0$, then the following lemma (essentially, Stokes' formula) shows that Φ_l can be expressed as a linear combination of exponential valuations on the facets of P. The proof can be found, for example, in [3].

Lemma 3.1. *Let $P \subset V$ be a full-dimensional polytope. For a facet Γ of P, let $d_\Gamma x$ be the Lebesgue measure on $\mathrm{aff}(\Gamma)$, and let p_Γ be the unit outer normal to Γ. Then, for every $l \in V \setminus 0$, we have*

$$\int_P \exp\left\{2\pi i \langle l, x\rangle\right\}\ dx = \sum_\Gamma \frac{\langle l, p_\Gamma\rangle}{2\pi i \|l\|^2} \int_\Gamma \exp\left\{2\pi i \langle l, x\rangle\right\}\ d_\Gamma x,$$

where the sum is taken over all facets Γ of P.

Let $F \subset P$ be an i-dimensional face of P. Recall that by $\mathrm{lin}(F)$ we denote the i-dimensional subspace of $\mathbb{R}^d$ that is parallel to the affine hull $\mathrm{aff}(F)$ of F. We need the following result.

Theorem 3.2. *Let $P \subset V$ be a rational full-dimensional polytope and let t be a positive integer such that tP is a lattice polytope. Let $\epsilon \geq 0$ be a rational number and let $a \in V$ be a vector. Let us choose $l \in \Lambda^*$. Then there exist functions $f_i(P, \epsilon, a, l; \cdot) : \mathbb{N} \longrightarrow \mathbb{C}$, $i = 0, \ldots, d$, such that*

(1)

$$\Phi_l\big((n+\epsilon)P + a\big) = \sum_{i=0}^{d} f_i(P, \epsilon, a, l; n) n^i \quad \textit{for all } n \in \mathbb{N}$$

and

(2)

$$f_i(P, \epsilon, a, l; n+t) = f_i(P, \epsilon, a, l; n) \quad \textit{for all } n \in \mathbb{N}$$

and $i = 0, \ldots, d$.

Suppose that $f_{d-k}(P, \epsilon, a, l; n) \neq 0$ *for some* n. *Then there exists a* $(d-k)$*-dimensional face* F *of* P *such that* l *is orthogonal to* $\operatorname{lin}(F)$.

Proof. Since

$$\Phi_l(P+a) = \exp\{2\pi i \langle l, a\rangle\} \Phi_l(P),$$

without loss of generality we assume that $a = 0$. We will denote $f_i(P, \epsilon, 0, l; n)$ just by $f_i(P, \epsilon, l; n)$.

We proceed by induction on d. For $d = 0$ the statement of the theorem obviously holds. Suppose that $d \geq 1$. If $l = 0$, then $\Phi_l\big((n+\epsilon)P\big) = (n+\epsilon)^d \operatorname{vol} P$ and the statement holds as well.

Suppose that $l \neq 0$. For a facet Γ of P, let $\Lambda_\Gamma = \Lambda \cap \operatorname{lin}(\Gamma)$ and let l_Γ be the orthogonal projection of l onto $\operatorname{lin}(\Gamma)$. Thus Λ_Γ is a lattice in the $(d-1)$-dimensional Euclidean space $\operatorname{lin}(\Gamma)$ and $l_\Gamma \in \Lambda_\Gamma^*$, so we can define valuations Φ_{l_Γ} on $\operatorname{lin}(\Gamma)$. Since tP is a lattice polytope, for every facet Γ there is a vector $u_\Gamma \in V$ such that

$$\operatorname{lin}(\Gamma) = \operatorname{aff}(t\Gamma) - tu_\Gamma \quad \text{and} \quad tu_\Gamma \in \Lambda.$$

Let $\Gamma' = \Gamma - u_\Gamma$, so $\Gamma' \subset \operatorname{lin}(\Gamma)$ is a Λ_Γ-rational $(d-1)$-dimensional polytope such that $t\Gamma'$ is a Λ_Γ-polytope. We have

$$(n+\epsilon)\Gamma = (n+\epsilon)\Gamma' + (n+\epsilon)u_\Gamma.$$

Applying Lemma 3.1 to $(n+\epsilon)P$, we get

$$\Phi_l\big((n+\epsilon)P\big) = \sum_\Gamma \psi(\Gamma, l; n) \Phi_{l_\Gamma}\big((n+\epsilon)\Gamma'\big),$$

where

$$\psi(\Gamma, l; n) = \frac{\langle l, p_\Gamma\rangle}{2\pi i \|l\|^2} \exp\big\{2\pi i (n+\epsilon)\langle l, u_\Gamma\rangle\big\}$$

and the sum is taken over all facets Γ of P.

Since $tu_\Gamma \in \Lambda$ and $l \in \Lambda^*$, we have

$$\psi(\Gamma, l; n+t) = \psi(\Gamma, l; n) \quad \text{for all } n \in \mathbb{N}.$$

Hence, applying the induction hypothesis, we may write

$$f_i(P, \epsilon, l; n) = \sum_\Gamma \psi(\Gamma, l; n) f_i(\Gamma', \epsilon, l_\Gamma; n) \quad \text{for all } n \in \mathbb{N}$$

and $i = 0, \ldots, d-1$ and $f_d(P, \epsilon, l; n) \equiv 0$. Hence (1)–(2) follows by the induction hypothesis.

If $f_{d-k}(P, \epsilon, l; n) \neq 0$, then there is a facet Γ of P such that $f_{d-k}(\Gamma', \epsilon, l_\Gamma; n) \neq 0$. By the induction hypothesis, there is a face F' of Γ' such that $\dim F' = d-k$, and l_Γ is orthogonal to $\operatorname{lin}(F')$. Then $F = F' + u_\Gamma$ is a $(d-k)$-dimensional face of P, $\operatorname{lin}(F') = \operatorname{lin}(F)$, and l is orthogonal to $\operatorname{lin}(F)$, which completes the proof. $\square$

4. Proof of Theorem 1.1

First, we discuss some ideas relevant to the proof.

4.1. **Shifting a valuation by a polytope.** Let V be a d-dimensional real vector space, let $\Lambda \subset V$ be a lattice, and let ν be a Λ-valuation on rational polytopes. Let us fix a rational polytope $R \subset V$. McMullen [19] observed that the function μ defined by

$$\mu(P) = \nu(P + R)$$

is a Λ-valuation on rational polytopes P. Here "+" stands for the Minkowski sum:

$$P + R = \Big\{ x + y : \ x \in P, \ y \in R \Big\}.$$

This result follows since the transformation $P \longmapsto P + R$ preserves linear dependencies among indicators of polyhedra; cf. [21].

Let t be a positive integer such that tP is a lattice polytope. McMullen [19] deduced that there exist functions $\nu_i(P, R; \cdot) : \mathbb{N} \longrightarrow \mathbb{C}$, $i = 0, \ldots, d$, such that

$$\nu\big(nP + R\big) = \sum_{i=0}^{d} \nu_i(P, R; n) n^i \quad \text{for all } n \in \mathbb{N}$$

and

$$\nu_i(P, R; n + t) = \nu_i(P, R; n) \quad \text{for all } n \in \mathbb{N}.$$

4.2. **Continuity properties of valuations E and E_L.** Let $R \subset \mathbb{R}^d$ be a full-dimensional rational polytope containing the origin in its interior. Then for every polytope $P \subset \mathbb{R}^d$ and every $\epsilon > 0$ we have $P \subset \big(P + \epsilon R\big)$. We observe that

$$\big|(P + \epsilon R) \cap \mathbb{Z}^d\big| = |P \cap \mathbb{Z}^d|,$$

for all sufficiently small $\epsilon > 0$. If P is a rational polytope, the supporting affine hyperplanes of the facets of nP for $n \in \mathbb{N}$ are split among finitely many translation classes modulo $\mathbb{Z}^d$. Therefore, there exists $\delta = \delta(P, R) > 0$ such that

$$\big|(nP + \epsilon R) \cap \mathbb{Z}^d\big| = |nP \cap \mathbb{Z}^d| \quad \text{for all } 0 < \epsilon < \delta \quad \text{and all } n \in \mathbb{N}.$$

We also note that for every rational subspace $L \subset \mathbb{R}^d$, we have

$$\lim_{\epsilon \longrightarrow 0+} E_L\big(P + \epsilon R\big) = E_L(P).$$

We will use the perturbation $P \longmapsto P + \epsilon R$ to push valuations E and E_L into a sufficiently generic position, so that we can apply Lemmas 2.1 and 2.3 without having to deal with various boundary effects. This is somewhat similar in spirit to the idea of [8].

4.3. **Linear identities for quasi-polynomials.** Let us fix positive integers t and d. Suppose that we have a possibly infinite family of quasi-polynomials $p_l : \mathbb{N} \longrightarrow \mathbb{C}$ of the type

$$p_l(n) = \sum_{i=0}^{d} p_i(l; n) n^i \quad \text{for all } n \in \mathbb{N},$$

where functions $p_i(l; \cdot) : \mathbb{N} \longrightarrow \mathbb{C}$, $i = 0, \ldots, d$, satisfy

$$p_i(l; n) = p_i(l; n + t) \quad \text{for all } n \in \mathbb{N}.$$

Suppose further that $p : \mathbb{N} \longrightarrow \mathbb{C}$ is yet another quasi-polynomial

$$p(n) = \sum_{i=0}^{d} p_i(n) n^i \quad \text{where } p_i(n+t) = p_i(n) \quad \text{for all } n \in \mathbb{N}.$$

Finally, suppose that $c_l(\cdot) : \mathbb{R}_+ \longrightarrow \mathbb{C}$ is a family of functions and that

$$p(n) = \lim_{\tau \longrightarrow +\infty} \sum_{l} c_l(\tau) p_l(n) \quad \text{for all } n \in \mathbb{N}$$

and that the series converges absolutely for every $n \in \mathbb{N}$ and every $\tau > 0$.

Then we claim that for $i = 0, \ldots, d$ we have

$$p_i(n) = \lim_{\tau \longrightarrow +\infty} \sum_{l} c_l(\tau) p_i(l; n) \quad \text{for all } n \in \mathbb{N}$$

and that the series converges absolutely for every $n \in \mathbb{N}$ and every $\tau > 0$.

This follows since $p_i(n)$, respectively $p_i(l;n)$, can be expressed as linear combinations of $p(m)$, respectively $p_l(m)$, for $m = n, n+t, \ldots, n+td$ with the coefficients depending on m, n, t, and d only.

Now we are ready to prove Theorem 1.1.

4.4. **Proof of Theorem 1.1.** Let us fix a rational polytope $P \subset \mathbb{R}^d$ as defined in the statement of the theorem. For $L \in \mathcal{L}$ let $P_L \subset L$ be the orthogonal projection of P onto L and let $\Lambda_L \subset L$ be the orthogonal projection of $\mathbb{Z}^d$ onto L.

Let $a \in \operatorname{int} P$ be a rational vector and let

$$R = P - a.$$

Hence R is a rational polytope containing the origin in its interior. Let R_L denote the orthogonal projection of R onto L.

Since P is a rational polytope and $\mathcal{L}$ is a finite set of rational subspaces, there exists $\delta = \delta(P, R) > 0$ such that for all $0 < \epsilon < \delta$ and all $n \in \mathbb{N}$, we have

$$\big(nP + \epsilon R\big) \cap \mathbb{Z}^d = nP \cap \mathbb{Z}^d \quad \text{and} \quad \partial\big(nP + \epsilon R\big) \cap \mathbb{Z}^d = \emptyset \quad \text{for all } n \in \mathbb{N} \tag{4.1}$$

and for all $L \in \mathcal{L}$, we have

$$\begin{gathered} \big(nP_L + \epsilon R_L\big) \cap \Lambda_L = nP_L \cap \Lambda_L \quad \text{and} \\ \partial\big(nP_L + \epsilon R_L\big) \cap \Lambda_L = \emptyset \quad \text{for all } n \in \mathbb{N}; \end{gathered} \tag{4.2}$$

cf. Section 4.2. Let us choose any rational $0 < \epsilon < \delta$.

Because of (4.1), we can write

$$\big|(nP + \epsilon R) \cap \mathbb{Z}^d\big| = \sum_{i=0}^{d} e_i(P; n) n^i \quad \text{for all } n \in \mathbb{N} \tag{4.3}$$

and by Lemma 2.1 we get

$$\big|(nP + \epsilon R) \cap \mathbb{Z}^d\big| = \lim_{\tau \longrightarrow +\infty} \sum_{l \in \mathbb{Z}^d} \exp\{-\pi \|l\|^2/\tau\} \Phi_l\big(nP + \epsilon R\big), \tag{4.4}$$

where Φ_l are the exponential valuations of Section 3.

Since Φ_l is a $\mathbb{Z}^d$-valuation, by Section 4.1 there exist functions $f_i(P, \epsilon, l; \cdot) : \mathbb{N} \longrightarrow \mathbb{C}$, $i = 0, \ldots, d$, such that

$$\Phi_l(nP + \epsilon R) = \sum_{i=0}^{d} f_i(P, \epsilon, l; n) n^i \quad \text{for } n \in \mathbb{N} \tag{4.5}$$

and

$$f_i(P,\epsilon,l;n+t) = f_i(P,\epsilon,l;n) \quad \text{for all } n \in \mathbb{N}. \tag{4.6}$$

Moreover, we can write

$$nP + \epsilon R = nP + \epsilon(P-a) = (n+\epsilon)P - \epsilon a.$$

Therefore, by Theorem 3.2, for $i \le k$ we have $f_{d-i}(P,\epsilon,l;n) = 0$ unless $l \in L^F$ for some face F of P with $\dim F = d-k$. Therefore, combining (4.3)–(4.6) and Section 4.3, we obtain for all $0 \le i \le k$ and all $n \in \mathbb{N}$

$$\begin{aligned} e_{d-i}(P;n) &= \lim_{\tau \longrightarrow +\infty} \sum_{l \in \mathbb{Z}^d} \exp\left\{-\pi \|l\|^2/\tau\right\} f_{d-i}(P,\epsilon,l;n) \\ &= \lim_{\tau \longrightarrow +\infty} \sum_{l \in \bigcup_{L \in \mathcal{L}} (L \cap \mathbb{Z}^d)} \exp\left\{-\pi \|l\|^2/\tau\right\} f_{d-i}(P,\epsilon,l;n), \end{aligned}$$

since vectors $l \in \mathbb{Z}^d$ outside of subspaces $L \in \mathcal{L}$ contribute 0 to the sum. Therefore, for $0 \le i \le k$ and all $n \in \mathbb{N}$

$$e_{d-i}(P;n) = \lim_{\tau \longrightarrow +\infty} \sum_{L \in \mathcal{L}} \mu(L) \sum_{l \in L \cap \mathbb{Z}^d} \exp\left\{-\pi \|l\|^2/\tau\right\} f_{d-i}(P,\epsilon,l;n) \tag{4.7}$$

On the other hand, because of (4.2), by Lemma 2.3 we get for all $L \in \mathcal{L}$ and all $n \in \mathbb{N}$

$$E_L\big(nP + \epsilon R\big) = \lim_{\tau \longrightarrow +\infty} \sum_{l \in L \cap \mathbb{Z}^d} \exp\left\{-\pi \|l\|^2/\tau\right\} \Phi_l\big(nP + \epsilon R\big). \tag{4.8}$$

Since E_L are $\mathbb{Z}^d$-valuations, by Section 4.1 there exist functions $e_i(P,\epsilon,L;\cdot) : \mathbb{N} \longrightarrow \mathbb{Q}$, $i = 0,\dots,d$, such that

$$E_L\big(nP + \epsilon R\big) = \sum_{i=0}^{d} e_i(P,\epsilon,L;n) n^i \quad \text{for all } n \in \mathbb{N} \tag{4.9}$$

and

$$e_i(P,\epsilon,L;n+t) = e_i(P,\epsilon,L;n) \quad \text{for all } n \in \mathbb{N}. \tag{4.10}$$

Combining (4.5)–(4.6) and (4.8)–(4.10), by Section 4.3 we conclude

$$e_{d-i}(P,\epsilon,L;n) = \lim_{\tau \longrightarrow +\infty} \sum_{l \in L \cap \mathbb{Z}^d} \exp\left\{-\pi \|l\|^2/\tau\right\} f_{d-i}(P,\epsilon,l;n) \quad \text{for all } n \in \mathbb{N}.$$

Therefore, by (4.7), for $0 \le i \le k$ we have

$$e_{d-i}(P;n) = \sum_{L \in \mathcal{L}} \mu(L) e_{d-i}(P,\epsilon,L;n) \quad \text{for all } n \in \mathbb{N}. \tag{4.11}$$

Since E_L is a $\mathbb{Z}^d$-valuation, there exist functions $e_i(P,L;\cdot) : \mathbb{N} \longrightarrow \mathbb{Q}$, $i = 0,\dots,d$, such that

$$E_L(nP) = \sum_{i=0}^{d} e_i(P,L;n) n^i \quad \text{for all } n \in \mathbb{N} \tag{4.12}$$

and

$$e_i(P,L;n+t) = e_i(P,L;n) \quad \text{for all } n \in \mathbb{N}.$$

Let us choose an $m \in \mathbb{N}$. Substituting $n = m, m+t, \dots, m+td$ in (4.12), we obtain $e_i(P,L;m)$ as a linear combination of $E_L(nP)$ with coefficients depending on n, m,

t, and d only. Similarly, substituting $n = m, m+t, \ldots, m+td$ in (4.9), we obtain $e_i(P, \epsilon, L; m)$ as the same linear combination of $E_L\big(nP + \epsilon R\big)$. Since volumes are continuous functions, in view of (4.2) (see also Section 4.2), we get

$$\lim_{\epsilon \longrightarrow 0+} E_L\big(nP + \epsilon R\big) = E_L(nP) \quad \text{for } n = m, m+t, \ldots, m+td.$$

Therefore,

$$\lim_{\epsilon \longrightarrow 0+} e_i(P, \epsilon, L; m) = e_i(P, L; m) \quad \text{for all } m \in \mathbb{N}.$$

Taking the limit as $\epsilon \longrightarrow 0+$ in (4.11), we obtain for $0 \leq i \leq k$

$$e_{d-i}(P; n) = \sum_{L \in \mathcal{L}} \mu(L) e_{d-i}(P, L; n) \quad \text{for all } n \in \mathbb{N}.$$

To complete the proof, we note that

$$\nu_{d-i}(P, L; n) = \sum_{L \in \mathcal{L}} \mu(L) e_{d-i}(P, L; n).$$

5. Summing up a polynomial over integer points in a rational polytope

Let us fix a positive integer k and let us consider the following situation. Let $Q \subset \mathbb{R}^k$ be a rational polytope, let $\operatorname{int} Q$ be the relative interior of Q, and let $f : \mathbb{R}^k \longrightarrow \mathbb{R}$ be a polynomial with rational coefficients. We want to compute the value

$$\sum_{m \in \operatorname{int} Q \cap \mathbb{Z}^k} f(m). \tag{5.1}$$

We claim that as soon as the dimension k of the polytope Q is fixed, there is a polynomial time algorithm to do that. We assume that the polytope Q is given by the list of its vertices and the polynomial f is given by the list of its coefficients.

For an integer point $m = (\mu_1, \ldots, \mu_k)$, let

$$\mathbf{x}^m = x_1^{\mu_1} \cdots x_k^{\mu_k} \quad \text{for } \mathbf{x} = (x_1, \ldots, x_k)$$

be the Laurent monomial in k variables $\mathbf{x} = (x_1, \ldots, x_k)$. We use the following result [6].

5.1. **The short rational function algorithm.** Let us fix k. There is a polynomial time algorithm, which, given a rational polytope $Q \subset \mathbb{R}^k$, computes the generating function (Laurent polynomial)

$$S(Q; \mathbf{x}) = \sum_{m \in \operatorname{int} Q \cap \mathbb{Z}^k} \mathbf{x}^m$$

in the form

$$S(Q; \mathbf{x}) = \sum_{i \in I} \epsilon_i \frac{\mathbf{x}^{a_i}}{(1 - \mathbf{x}^{b_{i1}}) \cdots (1 - \mathbf{x}^{b_{ik}})},$$

where $a_i \in \mathbb{Z}^k$, $b_{ij} \in \mathbb{Z}^k \setminus \{0\}$, and $\epsilon_i \in \mathbb{Q}$. In particular, the number $|I|$ of fractions is bounded by a polynomial in the input size of Q.

Our first step is computing the generating function

$$S(Q, f; \mathbf{x}) = \sum_{m \in \operatorname{int} Q \cap \mathbb{Z}^k} f(m) \mathbf{x}^m.$$

Our approach is similar to that of [12], although we obtain better complexity bounds (our algorithm is polynomial in $\deg f$ whereas the algorithm of [12] is exponential in $\deg f$).

5.2. **The algorithm for computing** $S(Q, f; \mathbf{x})$**.** We observe that

$$S(Q, f; \mathbf{x}) = f\left(x_1 \frac{\partial}{\partial x_1}, \dots, x_k \frac{\partial}{\partial x_k}\right) S(Q; \mathbf{x}).$$

We compute $S(Q; \mathbf{x})$ as in Section 5.1.

Let $a = (\alpha_1, \dots, \alpha_k)$ be an integer vector, let $b_j = (\beta_{j1}, \dots, \beta_{jk})$ be non-zero integer vectors for $j = 1, \dots, k$, and let $\gamma_1, \dots, \gamma_k$ be positive integers. Then

$$\begin{aligned}&\left(x_i \frac{\partial}{\partial x_i}\right) \frac{\mathbf{x}^a}{(1-\mathbf{x}^{b_1})^{\gamma_1} \cdots (1-\mathbf{x}^{b_k})^{\gamma_k}} \\ =&\alpha_i \frac{\mathbf{x}^a}{(1-\mathbf{x}^{b_1})^{\gamma_1} \cdots (1-\mathbf{x}^{b_k})^{\gamma_k}} + \sum_{j=1}^{k} \gamma_j \beta_{ji} \frac{\mathbf{x}^{a+b_j}}{(1-\mathbf{x}^{b_j})^{\gamma_j+1}} \prod_{s \neq j} \frac{1}{(1-\mathbf{x}^{b_s})^{\gamma_s}}.\end{aligned}$$

Consecutively applying the above formula and collecting similar fractions, we compute

$$f\left(x_1 \frac{\partial}{\partial x_1}, \dots, x_k \frac{\partial}{\partial x_k}\right) \frac{\mathbf{x}^a}{(1-\mathbf{x}^{b_1}) \cdots (1-\mathbf{x}^{b_k})}$$

as an expression of the type

$$\sum_j \rho_j \frac{\mathbf{x}^{a_j}}{(1-\mathbf{x}^{b_1})^{\gamma_{j1}} \cdots (1-\mathbf{x}^{b_k})^{\gamma_{jk}}}, \tag{5.2}$$

where $\rho_j \in \mathbb{Q}$, $\gamma_{j1}, \dots, \gamma_{jk}$ are non-negative integers satisfying $\gamma_{j1} + \cdots + \gamma_{jk} \leq k + \deg f$ and a_j are vectors of the type

$$a_j = a + \mu_1 b_1 + \cdots + \mu_k b_k,$$

where μ_i are non-negative integers and $\mu_1 + \cdots + \mu_k \leq \deg f$. The number of terms in (5.2) is bounded by $(\deg f)^{O(k)}$, which shows that for a k fixed in advance, the algorithm runs in polynomial time.

Consequently, $S(Q, f; \mathbf{x})$ is computed in polynomial time.

Formally speaking, to compute the sum (5.1), we have to substitute $x_i = 1$ into the formula for $S(Q, f; \mathbf{x})$. This, however, cannot be done in a straightforward way since $\mathbf{x} = (1, \dots, 1)$ is a pole of every fraction in the expression for $S(Q, f; \mathbf{x})$. Nevertheless, the substitution can be done via efficient computation of the relevant residue of $S(Q, f; \mathbf{x})$ as described in [4] and [7].

5.3. **The algorithm for computing the sum.** The output of Algorithm 5.2 represents $S(Q, f; \mathbf{x})$ in the general form

$$S(Q, f; \mathbf{x}) = \sum_{i \in I} \epsilon_i \frac{\mathbf{x}^{a_i}}{(1-\mathbf{x}^{b_{i1}})^{\gamma_{i1}} \cdots (1-\mathbf{x}^{b_{ik}})^{\gamma_{ik}}},$$

where $\epsilon_i \in \mathbb{Q}$, $a_i \in \mathbb{Z}^k$, $b_{ij} \in \mathbb{Z}^k \setminus \{0\}$, and $\gamma_{ij} \in \mathbb{N}$ are such that $\gamma_{i1} + \cdots + \gamma_{ik} \leq k + \deg f$ for all $i \in I$.

Let us choose a vector $l \in \mathbb{Q}^k$, $l = (\lambda_1, \dots, \lambda_k)$, such that $\langle l, b_{ij} \rangle \neq 0$ for all i, j (such a vector can be computed in polynomial time; cf. [4]). For a complex τ, let

$$\mathbf{x}(\tau) = \left(e^{\tau\lambda_1}, \dots, e^{\tau\lambda_k}\right).$$

We want to compute the limit

$$\lim_{\tau \longrightarrow 0} G(\tau) \quad \text{for } G(\tau) = S\big(Q, f; \mathbf{x}(\tau)\big).$$

In other words, we want to compute the constant term of the Laurent expansion of $G(\tau)$ around $\tau = 0$.

Let us consider a typical fraction

$$\frac{\mathbf{x}^a}{(1-\mathbf{x}^{b_1})^{\gamma_1} \cdots (1-\mathbf{x}^{b_k})^{\gamma_k}}.$$

Substituting $\mathbf{x}(\tau)$, we get the expression

$$\frac{e^{\alpha\tau}}{(1-e^{\tau\beta_1})^{\gamma_1} \cdots (1-e^{\tau\beta_k})^{\gamma_k}}, \tag{5.3}$$

where $\alpha = \langle a, l \rangle$ and $\beta_i = \langle b_i, l \rangle$ for $i = 1, \ldots, k$. The order of the pole at $\tau = 0$ is $D = \gamma_1 + \cdots + \gamma_k \leq k + \deg f$. To compute the constant term of the Laurent expansion of (5.3) at $\tau = 0$, we do the following.

We compute the polynomial

$$q(\tau) = \sum_{i=0}^{D} \frac{\alpha^i}{i!} \tau^i$$

that is the truncation at τ^D of the Taylor series expansion of $e^{\alpha\tau}$. For $i = 1, \ldots, k$ we compute the polynomial $p_i(\tau)$ with $\deg p_i = D$ such that

$$\frac{\tau}{1 - e^{\tau\beta_i}} = p_i(\tau) + \text{ terms of higher order in } \tau$$

at $\tau = 0$. Consecutively multiplying polynomials $\operatorname{mod} \tau^{D+1}$, we compute a polynomial $u(\tau)$ with $\deg u = D$ such that

$$q(\tau) p_1^{\gamma_1}(\tau) \cdots p_k^{\gamma_k}(\tau) \equiv u(\tau) \mod \tau^{D+1}.$$

The coefficient of τ^D in $u(\tau)$ is the desired constant term of the Laurent expansion.

6. Computing $E_L(\Delta)$

Let us fix a positive integer k. Let $\Delta \subset \mathbb{R}^d$ be a rational simplex given by the list of its vertices and let $L \subset \mathbb{R}^d$ be a rational subspace given by its basis and such that $\dim L = k$. In this section, we describe a polynomial time algorithm for computing the value of $E_L(\Delta)$ as defined in Section 1.2.

Let $pr : \mathbb{R}^d \longrightarrow L$ be the orthogonal projection. We compute the vertices of the polytope $Q = pr(\Delta)$ and a basis of the lattice $\Lambda = pr(\mathbb{Z}^d)$. For basic lattice algorithms see [25] and [16].

As is known, as $x \in \Delta$ varies, the function

$$\phi(x) = \operatorname{vol}_{d-k}(P_x) \quad \text{where } P_x = \big(\Delta \cap \big(x + L^\perp\big)\big)$$

is a piecewise polynomial on Q. Our first step consists of computing a decomposition

$$Q = \bigcup_i C_i \tag{6.1}$$

such that $C_i \subset Q$ are rational polytopes (chambers) with pairwise disjoint interiors and polynomials $\phi_i : L \longrightarrow \mathbb{R}$ such that $\phi_i(x) = \phi(x)$ for $x \in C_i$.

We observe that every vertex of P_x is the intersection of $x + L^\perp$ and some k-dimensional face F of Δ.

For every face G of Δ with $\dim G = k-1$ and such that $\operatorname{aff}(G)$ is not parallel to $L^{\perp}$, let us compute

$$A_G = \Big\{x \in L: \ x + L^{\perp} \cap \operatorname{aff}(G) \neq \emptyset\Big\}.$$

Then A_G is an affine hyperplane in L. The number of different hyperplanes A_G is $d^{O(k)}$ and hence they cut Q into at most $d^{O(k^2)}$ polyhedral chambers C_i; cf. Section 6.1 of [18]. As long as x stays within the relative interior of a chamber C_i, the strong combinatorial type of P_x does not change (the facets of P_x move parallel to themselves) and hence the restriction ϕ_i of ϕ onto C_i is a polynomial; cf. Section 5.1 of [24]. Since in the $(d-k)$-dimensional space $x+L^{\perp}$ the polytope P_x is defined by d linear inequalities, ϕ_i can be computed in polynomial time; see [15] and [3].

The decomposition (6.1) gives rise to the formula

$$[Q] = \sum_j [Q_j],$$

where Q_j are open faces of the chambers C_i (the number of such faces is bounded by a polynomial in d); cf. Section 6.1 of [18]. Hence we have

$$E_L(\Delta) = \sum_j \sum_{m \in Q_j \cap \Lambda} \phi(m).$$

Each inner sum is the sum of a polynomial over lattice points in a polytope of dimension at most k. By a change of the coordinates, it becomes the sum over integer points in a rational polytope and we compute it as described in Section 5.

7. Computing $e_{d-k}(\Delta; n)$

Let us fix an integer $k \geq 0$. We describe our algorithm, which, given a positive integer $d \geq k$, a rational simplex $\Delta \subset \mathbb{R}^d$ (defined, for example, by the list of its vertices), and a positive integer n, computes the number $e_{d-k}(\Delta; n)$.

We use Theorem 1.1.

7.1. Computing the set $\mathcal{L}$ of subspaces. We compute subspaces L and numbers $\mu(L)$ described in Theorem 1.1. Namely, for each $(d-k)$-dimensional face F of Δ, we compute a basis of the subspace $L^F = (\operatorname{lin} F)^{\perp}$. Hence $\dim L^F \leq k$. Clearly, the number of distinct subspaces L^F is $d^{O(k)}$. We let $\mathcal{L}$ be the set consisting of the subspaces L^F and all other subspaces obtained as intersections of L^F. We compute $\mathcal{L}$ in k (or fewer) steps. Initially, we let

$$\mathcal{L} := \Big\{L^F: \ F \text{ is a } (d-k)\text{-dimensional face of } \Delta\Big\}.$$

Then, at every step, we consider the previously constructed subspaces $L \in \mathcal{L}$, consider the pairwise intersections $L \cap L^F$ as F ranges over the $(d-k)$-dimensional faces of Δ, and add the obtained subspace $L \cap L^F$ to the set $\mathcal{L}$ if it is not already there. If no new subspaces are obtained, we stop. Clearly, in the end of this process, we will obtain all subspaces L that are intersections of different L^{F_i}. Since $\dim L^{F_i} = k$, each subspace $L \in \mathcal{L}$ is an intersection of some k subspaces L^{F_i}. Hence the process stops after k steps and the total number $|\mathcal{L}|$ of subspaces is $d^{O(k^2)}$.

Having computed the subspaces $L \in \mathcal{L}$, we compute the numbers $\mu(L)$ as follows.

For each pair of subspaces $L_1, L_2 \in \mathcal{L}$ such that $L_1 \subset L_2$, we compute the number $\mu(L_1, L_2)$ recursively: if $L_1 = L_2$, we let $\mu(L_1, L_2) = 1$. Otherwise, we let

$$\mu(L_1, L_2) = -\sum_{\substack{L \in \mathcal{L} \\ L_1 \subset L \subset L_2 \\ L \neq L_2}} \mu(L_1, L).$$

In the end, for each $L \in \mathcal{L}$, we let

$$\mu(L) = \sum_{\substack{L_1 \in \mathcal{L} \\ L \subset L_1}} \mu(L, L_1).$$

Hence $\mu(L_i, L_j)$ are the values of the Möbius function on the set $\mathcal{L}$ partially ordered by inclusion, so

$$\left[\bigcup_{L \in \mathcal{L}} L \right] = \sum_{L \in \mathcal{L}} \mu(L)[L]$$

follows from the Möbius inversion formula; cf. Section 3.7 of [27].

Now, for each $L \in \mathcal{L}$ and $m = n, n+t, \ldots, n+td$ we compute the values of $E_L(m\Delta)$ as in Section 6, compute

$$\nu(m\Delta) = \sum_{L \in \mathcal{L}} \mu(L) E_L(m\Delta),$$

and find $\nu_{d-k}(\Delta; n) = e_{d-k}(\Delta, n)$ by interpolation.

8. Possible extensions and further questions

8.1. **Computing more general expressions.** Let $P \subset \mathbb{R}^d$ be a rational polytope, let $\alpha \geq 0$ be a rational number, and let $u \in \mathbb{R}^d$ be a rational vector. One can show (cf. Section 4.1) that

$$\left| \big((n+\alpha)P + u \big) \cap \mathbb{Z}^d \right| = \sum_{i=0}^{d} e_i(P, \alpha, u; n) n^i \quad \text{for all } n \in \mathbb{N},$$

where $e_i(P, \alpha, u; \cdot) : \mathbb{N} \longrightarrow \mathbb{Q}$, $i = 0, \ldots, d$, satisfy

$$e_i(P, \alpha, u; n+t) = e_i(P, \alpha, u; n) \quad \text{for all } n \in \mathbb{N},$$

provided $t \in \mathbb{N}$ is a number such that tP is an integer polytope. As long as k is fixed in advance, for given α, u, n, and a rational simplex $\Delta \subset \mathbb{R}^d$, one can compute $e_{d-k}(\Delta, \alpha, u; n)$ in polynomial time. Similarly, Theorem 1.1 and its proof extend to this more general situation in a straightforward way.

8.2. **Computing the generating function.** Let $P \subset \mathbb{R}^d$ be a rational polytope. Then, for every $0 \leq i \leq d$, the series

$$\sum_{n=1}^{+\infty} e_i(P; n) t^n$$

converges to a rational function $f_i(P; t)$ for $|t| < 1$.

It is not clear whether $f_{d-k}(\Delta; t)$ can be efficiently computed as a "closed form expression" in any meaningful sense, although it seems that by adjusting the methods of Sections 5–7, for any given t such that $|t| < 1$ one can compute the value of $f_{d-k}(\Delta; t)$ in polynomial time (again, k is assumed to be fixed in advance).

8.3. **Extensions to other classes of polytopes.** If k is fixed in advance, the coefficient $e_{d-k}(P; n)$ can be computed in polynomial time, if the rational polytope $P \subset \mathbb{R}^d$ is given by the list of its $d+c$ vertices or the list of its $d+c$ inequalities, where c is a constant fixed in advance. A similar result holds for rational parallelepipeds P, that is, for Minkowski sums of d rational intervals that do not lie in the same affine hyperplane in $\mathbb{R}^d$.

8.4. **Possible applications to integer programming and integer point counting.** If $P \subset \mathbb{R}^m$ is a rational polytope given by the list of its defining linear inequalities, the problem of testing whether $P \cap \mathbb{Z}^m = \emptyset$ is a typical problem of integer programming; see [16] and [25]. Moreover, a general construction of "aggregation" (see Section 16.6 of [25] and Section 2.2 of [26]) establishes a bijection between the sets $P \cap \mathbb{Z}^m$ and $\Delta \cap \mathbb{Z}^d$ provided P is defined by $d+1$ linear inequalities. Here $\Delta \subset \mathbb{R}^d$ is a rational simplex whose definition is computable in polynomial time from that of P. It would be interesting to find out whether approximating valuation E by valuation ν of Theorem 1.1 for some $k \ll d$ and applying the algorithm of this paper to compute $\nu(\Delta)$ can be of any practical use to solve higher-dimensional integer programs and integer point counting problems. It could complement existing software packages [11] and [10] based on the "short rational functions" calculus.

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