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The Multi-Dimensional Version of $\int_a^b x^p dx$

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1. INTRODUCTION. Besides its own interest, integration of polynomials over simple sets such as simplices has important applications. In particular, in most finite element integration methods ([**7**, p. 90, p. 175]), the domain of integration is decomposed into elementary cells and the function is approximated by a polynomial on each cell. The simplex-like elements (triangles, tetrahedrons,...) are among the most popular type of cells.

In this paper we obtain a new *exact* integration formula for a q-homogeneous polynomial that is not an approximate quadrature formula ([**1**], [**2**], [**6**]) but rather is the multi-dimensional version of the one-dimensional classical formula

$$\int_a^b x^q\,dx = \frac{b^{q+1}-a^{q+1}}{1+q} = \frac{b-a}{1+q}[a^q + a^{q-1}b + \cdots + ab^{q-1} + b^q]. \tag{1.1}$$

Among its nice features, the multi-dimensional analogue of (1.1) has a simple form, is coordinate-free, and uses information at the vertices only. In addition, various simplifications are possible to yield even simpler alternative formulae [**4**].

Since every polynomial can be represented as a sum of homogeneous polynomials, one can easily apply our results to integrate an arbitrary polynomial over a simplex.

Another interesting feature of this formula is that it could be used efficiently in a finite-element method using simplices. For example, while building the elementary simplices Δ of the mesh, it is easy to associate once and for all with each Δ (via this formula), a matrix Q^*_Δ, so that integrating a quadratic functional $x'Qx$ reduces to computing trace(QQ^*_Δ), which requires only n^2 *scalar* multiplications. This may be particularly useful when one has to integrate various quadratic functionals on the same mesh. A similar argument is also valid for arbitrary q-homogeneous functionals.

2. MAIN RESULT. Let $\Delta_n \subset \mathbb{R}^n$ be an n-dimensional (non-degenerate) simplex, that is, $x \in \Delta_n$ if and only if x is a convex combination $\sum_0^n \lambda_i x_i$ (with $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$) of $n+1$ points $x_0, x_1, \ldots, x_n$ such that the vectors $(x_i - x_0)$, $i = 1, 2, \ldots, n$, are linearly independent. We let x' denote the transpose of a vector x.

Let $p(x) : \mathbb{R}^n \to \mathbb{R}$ be a real (positively) *r-homogeneous* polynomial, i.e., $p(\lambda x) = \lambda^r p(x)$ for all $\lambda > 0$, $x \in \mathbb{R}^n$, and some integer $r \geq 0$.

We are interested in computing

$$\int_{\Delta_n} p(x)\,dx. \tag{2.1}$$

We first introduce some notation. With every symmetric multilinear form H : $(\mathbb{R}^n)^q \to \mathbb{R}$, given by

$$(x_1, \ldots, x_q) \mapsto H(x_1, \ldots, x_q), \qquad x_1, \ldots, x_q \in \mathbb{R}^n, \tag{2.2}$$

one may associate a q-homogeneous polynomial $x \mapsto f(x) := H(x, x, \ldots, x)$ and conversely, using a polarization formula, with every q-homogeneous polynomial $f : \mathbb{R}^n \to \mathbb{R}$, one may associate a symmetric q-linear form $H : (\mathbb{R}^n)^q \to \mathbb{R}$. Therefore, we now consider the integration of a q-linear form H over the simplex Δ_n.

Theorem 2.1. *Let $x_0, x_1, \ldots, x_n$ be the vertices of an n-dimensional simplex Δ_n. Then, for a symmetric q-linear form $H : (\mathbb{R}^n)^q \to \mathbb{R}$, one has*

$$\int_{\Delta_n} H(x, x, \ldots, x)\, dx = \frac{\mathrm{vol}(\Delta_n)}{\binom{n+q}{q}} \left[\sum_{0 \le i_1 \le i_2, \ldots, \le i_q \le n} H(x_{i_1}, x_{i_2}, \ldots, x_{i_q}) \right]. \tag{2.3}$$

Proof. We use the well-known formula for integrating a homogeneous polynomial on the *canonical* simplex

$$\int_{\Omega_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n}\, dx = \frac{\alpha_1! \ldots \alpha_n!}{(n + \sum_i \alpha_i)!}, \tag{2.4}$$

where $\Omega_n := \{x \in \mathbb{R}^n | \sum_{i=1}^n x_i \le 1;\ x_i \ge 0,\ i = 1, 2, \ldots, n\}$ ([**2**], [**5**]). Of course, (2.4) is valid only for the canonical simplex Ω_n. The key idea is to use properties of a symmetric q-linear form.

Write $x \in \Delta_n$ as $x = \sum_{i=0}^n \lambda_i x_i$ with $\lambda_i \ge 0$ and $\sum_i \lambda_i = 1$, or equivalently, $x = \sum_{i=1}^n \lambda_i x_i + (1 - \sum_{i=1}^n \lambda_i) x_0$ with $(\lambda_1, \ldots, \lambda_n) \in \Omega_n$, and where

$$\Omega_n := \{\lambda \in \mathbb{R}^n \mid \sum_{i=1}^n \lambda_i \le 1,\ \lambda_i \ge 0,\ i = 1, 2, \ldots, n, \}$$

i.e., Ω_n is just the canonical simplex in $\mathbb{R}^n$.

Therefore, noting that

$$x = \sum_{i=1}^n \lambda_i x_i + \left(1 - \sum_{i=1}^n \lambda_i\right) x_0 = x_0 + \sum_{i=1}^n \lambda_i (x_i - x_0),$$

we have, by a change of variable $x \to \lambda$,

$$\begin{aligned}
\int_{\Delta_n} H(x, x, \ldots, x)\, dx &= \det(x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0) \\
&\times \int_{\Omega_n} H\left(\sum_{i=1}^n \lambda_i x_i + \left(1 - \sum_i \lambda_i\right) x_0, \ldots, \sum_{i=1}^n \lambda_i x_i + \left(1 - \sum_{i=1}^n \lambda_i\right) x_0 \right) d\lambda, \\
&= n!\ \mathrm{vol}(\Delta_n) \\
&\times \int_{\Omega_n} H\left(\sum_{i=1}^n \lambda_i x_i + \left(1 - \sum_{i=1}^n \lambda_i\right) x_0, \ldots, \sum_{i=1}^n \lambda_i x_i + \left(1 - \sum_{i=1}^n \lambda_i\right) x_0 \right) d\lambda.
\end{aligned}$$

Expanding, we get:

$$\begin{aligned}
\int_{\Delta_n} H(x, x, \ldots, x)\, dx &= n!\ \mathrm{vol}(\Delta_n) \\
&\times \sum_{\sum_0^n \alpha_i = q} A(\alpha_0, \ldots, \alpha_n) \int_{\Omega_n} \left(1 - \sum_{i=1}^n \lambda_i\right)^{\alpha_0} \lambda_1^{\alpha_1} \ldots \lambda_n^{\alpha_n}\, d\lambda.
\end{aligned} \tag{2.5}$$

As H is symmetric, in (2.5), we have

$$A(\alpha_0,\ldots,\alpha_n) = \binom{q}{\alpha_0}\binom{q-\alpha_0}{\alpha_1}\cdots\binom{q-\sum_{i=0}^{n-1}\alpha_i}{\alpha_n} H(x_0^{\alpha_0}, x_1^{\alpha_1},\ldots,x_n^{\alpha_n}), \quad (2.6)$$

where $\sum_{i=0}^n \alpha_i = q$ and where the notation $H(x_0^{\alpha_0}, x_1^{\alpha_1},\ldots,x_n^{\alpha_n})$ means that x_0 appears α_0 times, x_1 appears α_1 times, ..., x_n appears α_n times.

Now, with $\sum_{i=0}^n \alpha_i = q$, we have

$$\int_{\Omega_n} \left(1-\sum_{i=1}^n \lambda_i\right)^{\alpha_0} \lambda_1^{\alpha_1}\cdots\lambda_n^{\alpha_n}\, d\lambda = \frac{\alpha_0!\,\alpha_1!\cdots\alpha_n!}{(n+q)!} \quad (2.7)$$

[**2**, (2.2)]. Therefore, using (2.6) and (2.7) in (2.5), and noting that

$$\binom{q}{\alpha_0}\binom{q-\alpha_0}{\alpha_1}\cdots\binom{q-\sum_{i=0}^{n-1}\alpha_i}{\alpha_n} \times \frac{\alpha_0!\,\alpha_1!\cdots\alpha_n!}{(n+q)!} = \frac{q!}{(n+q)!},$$

we get

$$\int_{\Delta_n} H(x,x,\ldots,x)\,dx = \frac{q!\,n!}{(n+q)!}\operatorname{vol}(\Delta_n) \sum_{\sum_0^n \alpha_i = q} H(x_0^{\alpha_0},\ldots,x_n^{\alpha_n})$$

$$= \frac{\operatorname{vol}(\Delta_n)}{\binom{n+q}{q}} \sum_{0\le i_1\cdots\le i_q\le n} H(x_{i_1}, x_{i_2},\ldots,x_{i_q}) \qquad \blacksquare$$

As one may see, the formula (2.3) is extremely simple. Among its nice features:

- it uses only $n+1$ points, the vertices $x_0,\ldots,x_n$ of Δ_n.
- it is coordinate-free, i.e., it is given directly for an arbitrary simplex and not only the canonical simplex.
- all coefficients in the formula are equal, positive, and with ratio to $\operatorname{vol}(\Delta_n)$ bounded as n increases.

As already mentioned, every polynomial $p_n(x) : \mathbb{R}^n \to \mathbb{R}$ of degree q is the sum of at most $q+1$ homogeneous polynomials of degree $0, 1, \ldots, q$. To each one of them corresponds a 0-linear, 1-linear, ..., q-linear form, to which in turn, the formula (2.3) may be applied. Thus, Theorem 2.1 provides a simple way to integrate an arbitrary polynomial on a simplex.

One may see that (2.3) is the n-dimensional counterpart of the one-dimensional formula

$$\int_a^b x^q\,dx = \frac{(b^{q+1}-a^{q+1})}{q+1} = \frac{b-a}{\binom{1+q}{q}}[a^q + a^{q-1}b + \cdots + ab^{q-1} + b^q], \quad (2.8)$$

since $(b-a) = \operatorname{vol}([a,b])$.

Remark 2.2. Consider the integration of a quadratic homogeneous functional $x'Qx$ (with Q an n-by-n symmetric real-valued matrix) on an n-dimensional simplex Δ. The functional $Q \mapsto \int_\Delta x'Qxdx$ may be viewed as a linear form on the $n(n+1)/2$-dimensional Hilbert space of real-valued symmetric matrices, with the Frobenius scalar product $\langle Q, Q^* \rangle = \operatorname{trace}(QQ^*)$. Therefore,

$$\int_\Delta x'Qx\,dx \;=\; \operatorname{vol}(\Delta)\,\langle Q, Q^*_\Delta\rangle, \tag{2.9}$$

for some symmetric matrix Q^*_Δ. The identification of Q^*_Δ is easy from (2.3). For example, with $n = 2$ and $\Delta := \{(x_i, y_i)\}$, $i = 0, 1, 2$,

$$Q^*_\Delta = \frac{1}{6}\begin{bmatrix} \sum_{0\le i\le j\le 2} x_i x_j & \frac{1}{2}\sum_{0\le i\le j\le 2} [x_i y_j + y_i x_j] \\ \frac{1}{2}\sum_{0\le i\le j\le 2} [x_i y_j + y_i x_j] & \sum_{0\le i\le j\le 2} y_i y_j \end{bmatrix}$$

and in the n-dimensional case, for a simplex $\Delta_n := \{x_0, \ldots, x_n\}$,

$$Q^*_\Delta(i, j) = \frac{2}{(n+1)(n+2)} \sum_{0\le k\le l\le n} \frac{1}{2}[x_{ik}x_{jl} + x_{jk}x_{il}].$$

Hence, with an arbitrary n-dimensional simplex Δ, one may associate a symmetric matrix Q^*_Δ so that for every (symmetric) functional $x'Qx$, (2.9) holds. This representation is especially useful when one has to compute (2.3) for several matrices Q. For example, the matrices Q^*_Δ can be precomputed in a finite element method while building a mesh. Then, evaluating (2.3) for a functional $x'Qx$ via (2.9), requires only n^2 *scalar* multiplications, in contrast to evaluating $n(n+1)/2$ terms of the form $x_i'Qx_j$ (each term also requires about n^2 multiplications). Of course, a similar construction holds for q-linear symmetric forms.

REFERENCES

1. I.J. Good and T.N. Tideman, Integration over a simplex, truncated cubes, and Eulerian numbers, *Numer. Math.* **30** (1978) 355–367.
2. A. Grundmann and H.M. Moeller, Invariant integration formulas for the n-simplex by combinatorial methods, *SIAM J. Numer. Anal.* **15** (1978) 282–290.
3. J.B. Lasserre, Integration and homogeneous functions, *Proc. Amer. Math. Soc.* **127** (1999) 813–818.
4. J.B. Lasserre and K.E. Avrachenkov, Integration on a simplex, *Technical report #98334*, Laboratoire d'Analyse et d'Architecture des Systèmes du CNRS, Toulouse, France, 1998.
5. A.H. Stroud, A fifth degree integration formula for the n-simplex, *SIAM J. Numer. Anal.* **6** (1969) 90–98.
6. A.H. Stroud, *Approximate Calculation of Multiple Integrals*, Prentice Hall, Englewood Cliffs, NJ, 1971.
7. O.C. Zienkiewicz and R.L. Taylor, *The Finite Element Method*, 4th ed., vol 1 and 2, McGraw-Hill Book Company, London, 1988.

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