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## INTEGRATION ON A CONVEX POLYTOPE

JEAN B. LASSERRE

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**ABSTRACT.** We present an exact formula for integrating a (positively) homogeneous function  $f$  on a convex polytope  $\Omega \subset R^n$ . We show that it suffices to integrate the function on the  $(n-1)$ -dimensional faces of  $\Omega$ , thus reducing the computational burden. Further properties are derived when  $f$  has continuous higher order derivatives. This result can be used to integrate a continuous function after approximation via a polynomial.

### 1. INTRODUCTION

We consider the integration of a continuous (positively) homogeneous function  $f : R^n \rightarrow R$  on a convex polytope  $\Omega \subset R^n$ . We prove that if  $f$  is continuously differentiable, it suffices to integrate the function on the  $(n-1)$ -dimensional faces of  $\Omega$ . As a continuous function on a compact set in  $R^n$  can be uniformly approximated by a polynomial (a sum of homogeneous functions), this result provides an alternative method for integrating continuous functions on a convex polytope.

A similar result also holds for an exponential  $e^{\langle c, x \rangle}$ . In fact, it has even been shown in [1], [2] that it suffices to evaluate that function at the *vertices* of  $\Omega$ . This result was then used for computing the volume and counting integral points in  $\Omega$ .

When  $f$  is twice continuously differentiable, one may proceed further, and we show that it suffices to integrate  $f$  on the  $(n-2)$ -dimensional faces and its derivatives on the  $(n-1)$ -dimensional faces. One may iterate the process when  $f$  has higher order continuous derivatives, etc.

### 2. INTEGRATION OF A HOMOGENEOUS FUNCTION

Let  $A$  be an  $(m, n)$ -real matrix,  $f : R^n \rightarrow R$  a real continuous (positively) *homogeneous* function of degree  $q$ , i.e.  $f(\lambda x) = \lambda^q f(x)$  for all  $\lambda > 0$ ,  $x \in R^n$ . For a (positively) homogeneous function of degree  $q$  that is continuously differentiable, Euler's formula holds (cf. [5]), i.e.:

$$(2.1) \quad qf(x) = \langle \nabla f(x), x \rangle \quad \text{for all } x.$$

Let

$$(2.2) \quad h(b) := \int_{\Omega} f(x) dx \quad \text{with } \Omega := \{x \in R^n \mid Ax \leq b\}.$$

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We assume that  $\Omega$  is a convex **polytope**. The following fact is straightforward:

**Proposition 2.1.** *If  $f$  is (positively) homogeneous of degree  $q$ , then  $h$  is (positively) homogeneous of degree  $n + q$ .*

*Proof.* We have

$$h(\lambda b) := \int_{Ax \leq \lambda b} f(x) dx = \int_{A(x/\lambda) \leq b} \lambda^q f(x/\lambda) \lambda^n d(x/\lambda) = \lambda^{n+q} \int_{\Omega} f(x) dx,$$

which yields the desired result.  $\square$

Let  $\Omega_i := \{x \in R^n \mid Ax \leq b, A_i^T x = b_i\}$ , i.e.  $\Omega_i$  is the  $(n-1)$ -dimensional face of  $\Omega$  determined by the hyperplane  $A_i^T x = b_i$ , where  $A_i^T$  is the  $i$ th row of the matrix  $A$ . Let  $\mathcal{H}_i$  denote the  $(n-1)$ -dimensional affine variety that contains  $\Omega_i$ . The algebraic distance from the point  $x_0$  to  $\mathcal{H}_i$  is denoted  $d(x_0, \mathcal{H}_i)$ , and  $d(x_0, \mathcal{H}_i) = (b_i - A_i^T x_0) / \|A_i\|$  (with  $\|\cdot\|$  the usual Euclidean norm). Let  $\mu$  be the Lebesgue measure on  $\mathcal{H}_i$ . The  $n$ -dimensional (resp.  $(n-1)$ -dimensional) volume of  $\Omega$  (resp.  $\Omega_i$ ) is denoted by  $\mathcal{V}_n(\Omega)$  (resp.  $\mathcal{V}_{n-1}(\Omega_i)$ ).

**Lemma 2.2.** *Assume that  $f$  is continuously differentiable,  $\mathcal{V}_n(\Omega) \neq 0$ , and  $\mathcal{V}_{n-1}(\Omega_i) \neq 0$ . Then,  $h$  is continuously differentiable at  $b$  and*

$$(2.3) \quad \frac{\partial h}{\partial b_i} = \frac{1}{\|A_i\|} \int_{\Omega_i} f d\mu,$$

where  $\mu$  is the Lebesgue measure on  $\mathcal{H}_i$ , the  $(n-1)$ -dimensional affine variety that contains  $\Omega_i$ .

*Proof.* The proof is similar to the proof in [4] for the volume of  $\Omega$ , i.e. when  $f(x) \equiv 1$ . For  $\delta b_i > 0$ , let  $\Delta(\delta b_i)$  be the set

$$\Delta(\delta b_i) := \{x \in R^n \mid b_i \leq A_i^T x \leq b_i + \delta b_i, A_j^T x \leq b_j, j \neq i\}.$$

Since  $\mathcal{V}_{n-1}(\Omega_i) \neq 0$ ,  $\Delta(\delta b_i) \neq \emptyset$  for  $\delta b_i$  sufficiently small. Consider the change of variables  $x = x_0 + z A_i / \|A_i\| + \sum_{j=1}^{n-1} y_j v_j$ , where  $A_i^T x_0 = b_i$  and the  $v_j$  form an orthonormal basis of the  $(n-1)$ -dimensional subspace  $A_i^T x = 0$ . Equivalently,  $\Delta(\delta b_i)$  can be written

$$\begin{aligned} 0 \leq z \|A_i\| &\leq \delta b_i, \\ \sum_{k=1}^{n-1} y_k A_j^T v_k &\leq b_j - A_j^T x_0 - z A_j^T A_i / \|A_i\|, j \neq i. \end{aligned}$$

Let

$$s_j := \max[0, \frac{\delta b_i}{\|A_i\|^2} A_j^T A_i], s'_j := \max[0, \frac{-\delta b_i}{\|A_i\|^2} A_j^T A_i], j \neq i,$$

and let  $\Delta^1(\delta b_i)$  and  $\Delta^2(\delta b_i)$  be the domains in  $R^n$ , defined respectively by

$$0 \leq z \leq \frac{\delta b_i}{\|A_i\|}, \sum_{k=1}^{n-1} y_k A_j^T v_k \leq b_j - A_j^T x_0 + s'_j, j \neq i,$$

and

$$0 \leq z \leq \frac{\delta b_i}{\|A_i\|}, \sum_{k=1}^{n-1} y_k A_j^T v_k \leq b_j - A_j^T x_0 - s_j, j \neq i.$$

Obviously,  $\Delta^2(\delta b_i) \subseteq \Delta(\delta b_i) \subseteq \Delta^1(\delta b_i)$ . Define also

$$(2.4) \quad \Delta'^1(\delta b_i) := \{y \in R^{n-1} \mid \sum_{k=1}^{n-1} y_k A_j^T v_k \leq b_j - A_j^T x_0 + s'_j, j \neq i\},$$

and

$$(2.5) \quad \Delta'^2(\delta b_i) := \{y \in R^{n-1} \mid \sum_{k=1}^{n-1} y_k A_j^T v_k \leq b_j - A_j^T x_0 - s_j, j \neq i\}.$$

Note that  $\Delta'^1(0) = \Delta'^2(0) = \Omega_i$ .

Assume first that  $f$  is nonnegative. From  $h(b + \delta b_i e_i) - h(b) = \int_{\Delta(\delta b_i)} f dx$ , we have

$$\begin{aligned} \int_0^{\delta b_i / \|A_i\|} \int_{\Delta'^2(\delta b_i)} f(x_0 + z \frac{A_i}{\|A_i\|} + \sum_k y_k v_k) dy dz &\leq h(b + \delta b_i) - h(b), \\ \int_0^{\delta b_i / \|A_i\|} \int_{\Delta'^1(\delta b_i)} f(x_0 + z \frac{A_i}{\|A_i\|} + \sum_k y_k v_k) dy dz &\geq h(b + \delta b_i) - h(b). \end{aligned}$$

$f$  being continuously differentiable, one may write

$$\begin{aligned} &f(x_0 + \sum_k y_k v_k + z \frac{A_i}{\|A_i\|}) \\ &= f(x_0 + \sum_k y_k v_k) + z \langle \nabla f(x_0 + \sum_k y_k v_k + \theta \frac{A_i}{\|A_i\|}), \frac{A_i}{\|A_i\|} \rangle \end{aligned}$$

for some  $0 < \theta < z$ . Therefore,  $\nabla f$  being bounded on a compact set, with a simple continuity argument we get

$$\lim_{\delta b_i \rightarrow 0} \frac{h(b + \delta b_i e_i) - h(b)}{\delta b_i} = \frac{\int_{\Delta'^1(0)} f(x_0 + \sum_k y_k v_k) dy}{\|A_i\|} = \frac{\int_{\Omega_i} f d\mu}{\|A_i\|}.$$

For  $f$  not necessarily nonnegative, simply use the same argument with  $(f + M) - M$ , where  $\sup_{x \in \Omega} |f(x)| \leq M$  (as  $f$  is continuous and  $\Omega$  is compact).

Finally, the same argument also holds if  $\delta b_i < 0$ , and the continuity of the partial derivatives is immediate from (2.3).  $\square$

*Remark 2.3.* We have not used that  $f$  is (positively) homogeneous, so Lemma 2.2 is valid for any continuously differentiable function  $f$ . In addition, note that if  $\Omega_i = \emptyset$ , then  $\partial h(b)/\partial b_i = 0$ , in accordance with  $0 = \int_{\Omega_i} f d\mu$ . Indeed, the constraint  $A_i^T x \leq b_i$  is strictly redundant and remains strictly redundant with a slight perturbation of  $b_i$ .

**Theorem 2.4.** Assume that  $f$  is continuously differentiable,  $\mathcal{V}_n(\Omega) \neq \emptyset$ , and, for all  $i = 1, \dots, m$ ,  $\mathcal{V}_{n-1}(\Omega_i) \neq \emptyset$ . Then

$$(2.6) \quad \int_{\Omega} f(x) dx = \frac{1}{n+q} \sum_{i=1}^m \frac{b_i}{\|A_i\|} \int_{\Omega_i} f d\mu = \sum_{i=1}^m \frac{d(o, \mathcal{H}_i)}{n+q} \int_{\Omega_i} f d\mu,$$

where  $\mu$  is the Lebesgue measure on the  $(n-1)$ -dimensional affine variety  $\mathcal{H}_i$  that contains  $\Omega_i$ .

*Proof.* Since  $h(b)$  is an homogeneous continuously differentiable function at  $b$ , by Euler's formula (2.1), one gets

$$(2.7) \quad (n+q)h(b) = \langle \nabla h(b), b \rangle,$$

which, using Proposition 2.1 and Lemma 2.2 for  $\nabla h(b)$ , yields (2.6).  $\square$

*Remark 2.5.* (a) Formula (2.6) also holds if  $\Omega_i = \emptyset$  for some  $i$ 's. For such  $i$ 's,  $\int_{\Omega_i} f d\mu = 0$ , in accordance with  $\partial h(b)/\partial b_i = 0$  (cf. Remark 2.3).

(b) Note that the proof of Theorem 2.4 only uses Euler's formula. An alternative proof is to use Green's formula, i.e., with notation as in Prop. 2.3, p. 128 in [6],

$$\int_{\Omega} \operatorname{div}(X)f d\omega + \int_{\Omega} Xf d\omega = \int_{\partial\Omega} \langle X, \vec{n} \rangle f d\sigma,$$

where  $\vec{n}$  is the unit outward-pointing normal to  $\partial\Omega$ , and with the vector field  $X := \sum_{i=1}^n x_i \partial/\partial x_i$ .

Hence, the integration of  $f$  on  $\Omega$  reduces to a weighted integration of  $f$  on the  $(n-1)$ -dimensional faces of  $\Omega$  (and in fact, only on those faces that do not contain the origin). A similar formula has already been given for  $f := e^{\langle c, x \rangle}$ , using Stokes' formula (see [1], [2]).

For instance, if  $P$  (resp.  $Q$ ) is an homogeneous polynomial of degree  $p$  (resp.  $q$ ), then

$$\int_{\Omega} (P+Q)dx = \sum_i d(o, \mathcal{H}_i) \int_{\Omega_i} \left( \frac{P}{n+p} + \frac{Q}{n+q} \right) d\mu.$$

With  $f \equiv 1$ , one retrieves the volume formula given in [4] that is interpreted as a standard result in geometry. Indeed, in (2.6)  $\int_{\Omega_i} f d\mu$  reduces to  $\mathcal{V}_{n-1}(\Omega_i)$ , the  $(n-1)$ -dimensional volume of  $\Omega_i$ , so that  $b_i/(n||A_i||) \times \mathcal{V}_{n-1}(\Omega_i)$  is simply the  $n$ -dimensional version of the standard formula for the area of a triangle (base  $\times$  height/2) and (2.6) reads

$$(2.8) \quad \mathcal{V}_n(\Omega) = n^{-1} \sum_{i=1}^m \frac{b_i}{||A_i||} \mathcal{V}_{n-1}(\Omega_i).$$

In [4], an algorithm based on (2.8) has been proposed, and the interested reader is referred to [3] for a numerical comparison of several algorithms for exact volume computation, including that one.

*Remark 2.6.* In fact Theorem 2.4 is also valid at points  $b$  where  $\mathcal{V}_{n-1}(\Omega_i) = 0$  for some  $i \in I \subset \{1, \dots, m\}$ . Indeed, one may prove that the constraint  $A_i^T x \leq b_i$ ,  $i \in I$ , is redundant and therefore can be removed, i.e.  $\Omega \equiv \{x | A_i^T x \leq b_i, i \notin I\}$ . After having removed all the redundant constraints, (2.6) is valid, with the summation being now over all  $i \notin I$ . But (2.6) is also valid if we maintain those  $i \in I$ , since

$$\mathcal{V}_{n-1}(\Omega_i) = 0 \Rightarrow \mu(\Omega_i) = 0 \Rightarrow \int_{\Omega_i} f d\mu = 0.$$

**2.1. Further results.** We now would like to apply the same technique to  $\int_{\Omega_i} f d\mu$  so as to consider integration on faces of lower dimensions. Indeed, we can do so provided  $f$  has continuous second derivatives.

Let  $b^{(i)}$  be the  $(m-1)$ -vector obtained from  $b$  by deleting its  $i$ th entry, and let  $A^{(i)}$  be the matrix obtained from  $A$  by deleting its  $i$ th row. Let  $\{v_k\}$  be  $n-1$  orthonormal vectors in the vector space associated with  $\mathcal{H}_i$ . For every  $j \neq i$ , let

$B_j$  be the  $(n-1)$ -vector  $\{B_{jk}\}$  with  $B_{jk} := A_j^T v_k$ ,  $k = 1, \dots, n-1$ , and with  $x_0$  arbitrary, define

$$\Gamma_i := \{y \in R^{n-1} \mid B_j^T y \leq b_j - A_j^T x_0, j \neq i\} = \{y \in R^{n-1} \mid By \leq b^i - A^{(i)} x_0\}$$

and

$$(2.9) \quad h(b^i, x_0) := \int_{By \leq b^i - A^{(i)} x_0} f(x_0 + \sum_{k=1}^{n-1} y_k v_k) dy.$$

If  $x_0 \in \mathcal{H}_i$ , then  $\Gamma_i$  is the representation of  $\Omega_i$  in an orthonormal basis of  $\mathcal{H}_i$ , and  $h(b^i, x_0) = \int_{\Omega_i} f d\mu$ , with  $\mu$  the Lebesgue measure on  $\mathcal{H}_i$ . Finally, let

$$\Omega_{ij} := \{x \in \Omega \mid A_i^T x = b_i, A_j^T x = b_j\}$$

be the  $(i, j)$   $((n-2)$ -dimensional) face of  $\Omega$  and  $\mathcal{H}_{ij}$  the  $(n-2)$ -dimensional affine variety that contains  $\Omega_{ij}$ .

**Theorem 2.7.** *Let  $f$  be twice continuously differentiable. Assume also that for every  $i = 1, \dots, m$ , either  $\Omega_i = \emptyset$  or  $\mathcal{V}_{n-1}(\Omega_i) \neq \emptyset$ , and for every  $j = 1, \dots, m$  with  $j \neq i$ , either  $\Omega_{ij} = \emptyset$  or  $\mathcal{V}_{n-2}(\Omega_{ij}) \neq \emptyset$ . Then:*

(a)  $h(b^i, x_0)$  is positively homogeneous of degree  $n+q-1$ .

(b) With  $x_0 \in \mathcal{H}_i$  fixed, arbitrary,

$$(2.10) \quad \frac{\partial h(b^i, x_0)}{\partial b_j} = \frac{1}{\|B_j\|} \int_{\Omega_{ij}} f d\nu, j \neq i,$$

$$(2.11) \quad \frac{\partial h(b^i, x_0)}{\partial x_{0k}} = \sum_{j \neq i} \frac{-A_{jk}}{\|B_j\|} \int_{\Omega_{ij}} f d\nu + \int_{\Omega_i} \frac{\partial f}{\partial x_k} d\mu,$$

with  $\mu$  (resp.  $\nu$ ) the Lebesgue measure on  $\mathcal{H}_i$  (resp.  $\mathcal{H}_{ij}$ ).

(c) With  $x_0 \in \mathcal{H}_i$  fixed, arbitrary,

$$(2.12) \quad \int_{\Omega_i} f d\mu = \frac{1}{n+q-1} [\sum_{j \neq i} d_i(x_0, \mathcal{H}_{ij}) \int_{\Omega_{ij}} f d\nu + \int_{\Omega_i} \langle \nabla f, x_0 \rangle d\mu],$$

with  $d_i$  the algebraic (Euclidean) distance in  $\mathcal{H}_i$ .

*Proof.* (a) From the definition of  $h(b^i, x_0)$  in (2.9), we get

$$\begin{aligned} h(\lambda b^i, \lambda x_0) &= \int_{By \leq \lambda(b^i - A^{(i)} x_0)} f(\lambda x_0 + \sum_{k=1}^{n-1} y_k v_k) dy \\ &= \int_{B(y/\lambda) \leq b^i - A^{(i)} x_0} \lambda^q f(x_0 + \sum_{k=1}^{n-1} (y_k/\lambda) v_k) \lambda^{n-1} d(y/\lambda) \\ &= \lambda^{n+q-1} \int_{By \leq b^i - A^{(i)} x_0} f(x_0 + \sum_{k=1}^{n-1} y_k v_k) dy \\ &= \lambda^{n+q-1} h(b^i, x_0). \end{aligned}$$

(b) If  $\Omega_i = \emptyset$ , then  $\Omega_{ij} = \emptyset$  as well, and  $\int_{\Omega_{ij}} f d\nu = 0$ . Any slight perturbation of  $b_j$ ,  $j \neq i$ , leaves  $\Omega_i$  empty, so that  $\partial h(b^i, x_0)/\partial b_j = 0$ , and thus (2.10) holds.

Assume now that  $\mathcal{V}_{n-1}(\Omega_i) \neq \emptyset$ . If  $\Omega_{ij} = \emptyset$ , it remains empty for every sufficiently small perturbation of  $b_j$ , and therefore,  $\Omega_i$  remains unchanged. Hence,  $\partial h(b^i, x_0)/\partial b_j = 0$ , in accordance with  $\int_{\Omega_{ij}} f d\nu = 0$ , i.e. (2.10) holds.

Consider now the case where  $\Omega_{ij} \neq \emptyset$  and write  $h(b^i, x_0)$  as  $G(\hat{b}) = \int_{By \leq \hat{b}} g(y) dy$ , with

$$\hat{b} := b^i - A^{(i)} x_0 \text{ and } g(y) := f(x_0 + \sum_{k=1}^{n-1} y_k v_k).$$

We can also write  $By \leq \hat{b}$  as

$$B_j^T y \leq \hat{b}_j := b_j - A_j^T x_0 \text{ for all } j \neq i.$$

Applying Lemma 2.2 to  $G$  (in Lemma 2.2, we did not use that  $f$  was positively homogeneous, cf. Remark 2.3), we see that  $G$  is continuously differentiable, and

$$\frac{\partial h(b^i, x_0)}{\partial b_j} = \frac{\partial G(\hat{b})}{\partial \hat{b}_j} = \frac{1}{\|B_i\|} \int_{By \leq \hat{b}, B_j^T y = \hat{b}_j} g d\nu,$$

where  $\nu$  is now the Lebesgue measure on the  $(n-2)$ -dimensional affine variety  $\mathcal{H}_{ij} \subset \mathcal{H}_i$ , that contains the polytope

$$\{y \in R^{n-1} | By \leq \hat{b}, B_j^T y = \hat{b}_j\} = \Omega_{ij}.$$

This yields (2.10). To get (2.11), let  $x_0 := x_0 + \lambda e_k$  with  $e_k$  the  $n$ -vector  $\{\delta_{kj}\}$  (and  $\delta_{kj}$  the Kronecker symbol). Then

$$h(b^i, x_0 + \lambda e_k) = \int_{By \leq b^i - A^{(i)}(x_0 + \lambda e_k)} f(x_0 + \lambda e_k + \sum_{s=1}^{n-1} y_s v_s) dy.$$

Define

$$\Omega_i(\lambda) := \{y \in R^{n-1} | By \leq b^i - A^{(i)} x_0 - \lambda A^{(i)} e_k\} \text{ and } \Omega_i(0) = \Omega_i.$$

Now, writing  $x' := x_0 + \sum_{s=1}^{n-1} y_s v_s$ , with  $f$  twice continuously differentiable, we get

$$f(x_0 + \lambda e_k + \sum_{s=1}^{n-1} y_s v_s) = f(x') + \lambda \frac{\partial f(x')}{\partial x_k} + \lambda^2 \frac{\partial^2 f(x' + \theta e_k)}{\partial x_k^2}$$

for some  $0 < \theta < \lambda$ . Hence,

$$\begin{aligned} \lambda^{-1}(h(b^i, x_0 + \lambda e_k) - h(b^i, x_0)) &= \lambda^{-1} \left[ \int_{\Omega_i(\lambda)} f(x') dy - \int_{\Omega_i} f(x') dy \right] \\ &+ \int_{\Omega_i(\lambda)} \frac{\partial f(x')}{\partial x_k} + \lambda \frac{\partial^2 f(x' + \theta e_k)}{\partial x_k^2} dy. \end{aligned}$$

As  $f$  is twice continuously differentiable,  $(\partial^2 f(x')/\partial x_k^2)$  is bounded on a compact set. In addition, for  $\lambda$  sufficiently small,  $\Omega_i(\lambda)$  is contained in some compact set. Therefore, in the above equation, the term  $\lambda \int_{\Omega_i(\lambda)} (\partial^2 f(x' + \theta e_k)/\partial x_k^2) dy$  vanishes as  $\lambda \rightarrow 0$ .

In addition, by a simple continuity argument,

$$(2.13) \quad \lambda \rightarrow 0 \Rightarrow \int_{\Omega_i(\lambda)} \frac{\partial f(x')}{\partial x_k} dy \rightarrow \int_{\Omega_i(0)} \frac{\partial f(x')}{\partial x_k} dy = \int_{\Omega_i} \frac{\partial f}{\partial x_k} d\mu,$$

with  $\mu$  the Lebesgue measure on  $\mathcal{H}_i$ .

Finally, write

$$g(y) := f(x_0 + \sum_{s=1}^{n-1} y_s v_s) \text{ and } \hat{b}_j(\lambda) := b_j - A_j^T x_0 - \lambda A_{jk}, \quad j \neq i.$$

Denote

$$G(\hat{b}(\lambda)) := \int_{\Omega_i(\lambda)} f(x') dy = \int_{By \leq \hat{b}(\lambda)} g(y) dy.$$

Assume first that  $\mathcal{V}_{n-2}(\Omega_{ij}) \neq \emptyset$ . Again, we can apply Lemma 2.2 to  $G$ , since  $g$  is continuously differentiable and  $\mathcal{V}_{n-2}(\Omega_{ij}) \neq \emptyset$ . Therefore, one gets

$$\frac{\partial G(\hat{b})}{\partial \hat{b}_j} = \frac{1}{\|B_j\|} \int_{By \leq \hat{b}, B_j^T y = \hat{b}_j} g d\nu = \frac{1}{\|B_j\|} \int_{By \leq \hat{b}, B_j^T y = \hat{b}_j} f d\nu,$$

with  $\nu$  the Lebesgue measure on the  $(n-2)$ -dimensional affine variety  $\mathcal{H}_{ij} \subset \mathcal{H}_i$  that contains the convex polytope  $\{y \in R^{n-1} | By \leq \hat{b}, B_j^T y = \hat{b}_j\} = \Omega_{ij}$ . Hence, from

$$\lim_{\lambda \rightarrow 0} \lambda^{-1} \left( \int_{\Omega_i(\lambda)} f(x') dy - \int_{\Omega_i} f(x') dy \right) = \sum_{j \neq i} \frac{\partial G(\hat{b}(0))}{\partial \hat{b}_j} \frac{d\hat{b}_j(0)}{d\lambda}$$

and  $d\hat{b}_j/d\lambda = -A_{jk}$ , one gets

$$(2.14) \quad \lim_{\lambda \rightarrow 0} \lambda^{-1} \left( \int_{\Omega_i(\lambda)} f(x') dy - \int_{\Omega_i} f(x') dy \right) = \sum_{j \neq i} \frac{-A_{jk}}{\|B_j\|} \int_{\Omega_{ij}} f d\nu.$$

If  $\Omega_{ij} = \emptyset$ , then  $\Omega_i(\lambda) = \Omega_i$  for  $\lambda$  sufficiently small, and therefore,

$$\lim_{\lambda \rightarrow 0} \lambda^{-1} \left[ \int_{\Omega_i(\lambda)} g(y) dy - \int_{\Omega_i} g(y) dy \right] = 0,$$

in accordance with  $\int_{\Omega_{ij}} f d\nu = 0$ . Finally, combining (2.13) and (2.14) yields (2.11).

(c) To get (2.12), we just apply Euler's formula (2.1) to  $h(b^i, x_0)$ , which is positively homogeneous of degree  $n+q-1$ , and continuously differentiable. This yields

$$\begin{aligned} \int_{\Omega_i} f d\mu &= h(b^i, x_0) = \frac{1}{n+q-1} \langle \nabla h(b^i, x_0), (b^i, x_0) \rangle \\ &= \frac{1}{n+q-1} [\langle \nabla_{b^i} h(b^i, x_0), b^i \rangle + \langle \nabla_{x_0} h(b^i, x_0), x_0 \rangle]. \end{aligned}$$

Using (2.10)-(2.11) for  $\nabla h(b^i, x_0)$  in the above expression, one gets

$$\int_{\Omega_i} f d\mu = \frac{1}{n+q-1} \left[ \sum_{j \neq i} \frac{b_j - A_j^T x_0}{\|B_j\|} \int_{\Omega_{ij}} f d\nu + \int_{\Omega_i} \langle \nabla f, x_0 \rangle d\mu \right].$$

Noting that  $(b_j - A_j^T x_0)/\|B_j\|$  is just  $d_i(x_0, \mathcal{H}_{ij})$  (the algebraic distance in  $\mathcal{H}_i$  from the origin  $x_0$  to  $\mathcal{H}_{ij}$ ), one gets (2.12).  $\square$

Hence, integrating  $f$  on  $\Omega$  reduces to

- either integrating  $f$  on the  $(n-1)$ -dimensional faces of  $\Omega$  (cf. Theorem 2.4),
- or integrating  $f$  on the  $(n-2)$ -dimensional faces of  $\Omega$  and its derivatives on the  $(n-1)$ -dimensional faces of  $\Omega$  (cf. Theorem 2.7).

Provided  $f$  has continuous partial derivatives of order  $p+1$ , one may iterate the above procedure and show that it suffices to evaluate  $f$  and its first, second, ...,  $p$ th derivatives at the vertices of  $\Omega$ , the  $(1)$ -dimensional faces, etc.



For instance, consider the term  $\int_{\Omega_i} \langle \nabla f, x_0 \rangle d\mu$ . Let  $z_0 \in \mathcal{H}_i$  be arbitrary, and with the same notation as in the proof of Theorem 2.7, write

$$g(b^i, z_0) := \int_{\Omega_i} \langle \nabla f, x_0 \rangle d\mu = \int_{By \leq b^i - A^{(i)} z_0} \langle \nabla f(z_0 + \sum_{k=1}^{n-1} y_k v_k), x_0 \rangle dy.$$

Again,  $g$  is (positively) homogeneous of degree  $(n + q - 2)$  since  $\nabla f$  is positively homogeneous of degree  $q - 1$ . Therefore, if  $f$  has continuous third derivatives, proceeding with similar arguments as in the proof of Theorem 2.7, one gets:

$$\int_{\Omega_i} \langle \nabla f, x_0 \rangle d\mu = \frac{1}{n + q - 2} \left[ \sum_{j \neq i} \frac{d_i(z_0, \mathcal{H}_{ij})}{\|B_j\|} \int_{\Omega_{ij}} \langle \nabla f, x_0 \rangle d\nu + \int_{\Omega_i} \langle z_0, (\partial^2 f) x_0 \rangle d\mu \right],$$

with  $\partial^2 f$  the Hessian matrix of  $f$ .

An interesting case is when  $f$  is an homogeneous polynomial of degree  $q$ . Then the  $(q + 1)$ th derivatives vanish, and integrating that polynomial on  $\Omega$  requires only knowledge of the polynomial and all its partial derivatives at the vertices of  $\Omega$ , i.e. at a finite number of points. As a continuous function on a compact set can be approximated by polynomials (a sum of homogeneous polynomials), one may compute a good approximation of the integral by considering only the vertices of  $\Omega$ .

Finally, one may notice that integration on a *nonconvex* polytope reduces to the above case after a partition of the original polytope into convex polytopes.

**2.2. Illustrative example.** In  $R^2$ , consider  $J := \int_{\Omega} xy dx dy$  with

$$\Omega := \{(x, y) \in R^2 \mid x + y \leq 1, x \geq a, y \geq b\},$$

i.e.  $n = q = 2$ . A direct integration yields

$$J = \frac{1}{8}[(1 - b)^4 - a^4] - \frac{1}{3}[(1 - b)^3 - a^3] + \frac{1}{4}(1 - b^2)[(1 - b)^2 - a^2].$$

Now, with  $\Omega_1 := \Omega \cap \{x = a\}$ , we get

$$d(o, \mathcal{H}_1) \int_{\Omega_1} f d\mu = -a \int_b^{1-a} av dv = -a^2[(1 - a)^2 - b^2]/2.$$

With  $\Omega_2 := \Omega \cap \{y = b\}$ , we get

$$d(o, \mathcal{H}_2) \int_{\Omega_2} f d\mu = -b \int_a^{1-b} bv dv = -b^2[(1 - b)^2 - a^2]/2.$$

With  $\Omega_3 := \Omega \cap \{x + y = 1\}$ , we get

$$d(o, \mathcal{H}_3) \int_{\Omega_3} f d\mu = \frac{1}{\sqrt{2}} \int_a^{1-b} \sqrt{2} v(1 - v) dv = \frac{1}{2}[(1 - b)^2 - a^2] - \frac{1}{3}[(1 - b)^3 - a^3]$$

and one may check that

$$J = \frac{1}{4}[-a^2 \int_b^{1-a} v dv - b^2 \int_a^{1-b} v dv + \int_a^{1-b} v(1 - v) dv],$$

i.e.,

$$J = \frac{1}{4} \sum_{i=1}^3 d(o, \mathcal{H}_i) \int_{\Omega_i} f d\mu,$$

or equivalently, (2.6) is satisfied.

Similarly, take  $x_0 := (1 - b, b) \in \mathcal{H}_3$ . Then

$$d_3(x_0, \mathcal{H}_2) = 0, \quad d_3(x_0, \mathcal{H}_1) = \sqrt{2}(1 - a - b), \quad f((a, 1 - a)) = a(1 - a).$$

In addition,

$$\int_{\Omega_3} \langle \nabla f(x), x_0 \rangle \mu(dx) = \sqrt{2} \int_b^{1-a} [v(1 - b) + (1 - v)b] dv,$$

and one may check that

$$\frac{\sqrt{2}}{3} [(1 - a - b)a(1 - a) + \int_b^{1-a} [v(1 - b) + b(1 - v)] dv] = \int_{\Omega_3} f d\mu,$$

i.e. (2.12) is satisfied.

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