Local Euler-Maclaurin expansion of Barvinok valuations and Ehrhart coefficients of a rational polytope

Velleda Baldoni, Nicole Berline and Michèle Vergne

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1 Introduction

Let $\mathfrak{p}$ be a rational polytope in $V = \mathbb{R}^d$ and $h(x)$ a polynomial function on V. A classical problem in Integer Programming is to compute the sum of values of $h(x)$ over the set of integral points of $\mathfrak{p}$,

$$S(\mathfrak{p}, h) = \sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} h(x).$$

When $\mathfrak{p}$ is dilated by an integer $n \in \mathbb{N}$, we obtain a function of n which is quasi-polynomial, the so-called Ehrhart quasi-polynomial of the pair $(\mathfrak{p}, h)$

$$S(n\mathfrak{p}, h) = \sum_{m=0}^{d+N} E_m(\mathfrak{p}, h, n) n^m$$

of degree $d + N$ where $N = \deg h$. The coefficients $E_m(\mathfrak{p}, h, n)$ are periodic functions of $n \in \mathbb{N}$, with period the smallest integer q such that $q\mathfrak{p}$ is a lattice polytope.

Replacing $h(x)$ by an exponential, we are led to study the analytic function on V^* defined by

$$S(\mathfrak{p})(\xi) = \sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} e^{\langle \xi, x \rangle}.$$

If $\mathfrak{p}$ is any rational polyhedron, this sum still makes sense as a meromorphic function defined near 0 and the map $\mathfrak{p} \mapsto S(\mathfrak{p})(\xi)$ is a valuation.

In [4], we proved that the meromorphic function $S(\mathfrak{p})(\xi)$ has a local Euler-Maclaurin expansion

$$S(\mathfrak{p})(\xi) = \sum_{\mathfrak{f}} \mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi) \int_{\mathfrak{f}} e^{\langle \xi, x\rangle} dx.$$

The sum is taken over the set of faces $\mathfrak{f}$ of the polyhedron $\mathfrak{p}$. For each face $\mathfrak{f}$, the function $\mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi)$ is holomorphic near 0, and it depends only on the transverse cone $\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$ of $\mathfrak{p}$ along $\mathfrak{f}$. More precisely, once a rational scalar product is chosen on V, we define canonically a map $\mathfrak{a} \mapsto \mu(\mathfrak{a})$ from the set of rational affine cones $\mathfrak{a}$ in quotient spaces of V, with values in the space of functions on V which are analytic near 0, then we prove that these functions satisfy the above formula. The map $\mathfrak{a} \mapsto \mu(\mathfrak{a})$ is invariant under lattice translations, equivariant with respect to lattice preserving isometries, and it is a valuation on the set of affine cones with a fixed vertex ([4], Theorems 17 and 18).

It is easy to see that the Ehrhart quasi-polynomial can be computed in terms of the Taylor coefficients of the functions $\mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi)$. For example, if $\mathfrak{p}$ is a lattice polytope and $h(x) = 1$, we have ([4], Corollary 28)

$$\text{Card}(n\mathfrak{p} \cap \mathbb{Z}^d) = \sum_{\mathfrak{f}} \mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(0) \operatorname{vol}(\mathfrak{f}) n^{\dim \mathfrak{f}}. \tag{1}$$

Using the valuation property of $\mu(\mathfrak{a})$ and Barvinok's decomposition of a cone into unimodular cones, we thus obtained in [4] an algorithm for computing the Ehrhart quasi-polynomial. It has polynomial length with respect to the input $(\mathfrak{p}, h)$, when the dimension d and the degree N are fixed.

The valuation $S(\mathfrak{p}, h)$ has a natural generalization used by Barvinok in [2], the *mixed* valuation $S^L(\mathfrak{p}, h)$, where $L \subseteq V$ is a rational vector subspace. Denote the projected lattice on V/L by Λ_L. For a polytope $\mathfrak{p} \subset V$ and a polynomial $h(x)$

$$S^L(\mathfrak{p}, h) = \sum_{y \in \Lambda_{V/L}} \int_{\mathfrak{p} \cap (y+L)} h(x) dx.$$

In other words, the polytope $\mathfrak{p}$ is sliced along lattice affine subspaces parallel to L and the integrals of h over the slices are added up. For $L = V$, there is only one term and $S^V(\mathfrak{p}, h)$ is just the integral of $h(x)$ over $\mathfrak{p}$, while, for $L = \{0\}$, we recover $S(\mathfrak{p}, h)$, the sum of values of $h(x)$ over the set of integral points of $\mathfrak{p}$.

In the case $h(x) = 1$, we write $S(\mathfrak{p})$ and $S^L(\mathfrak{p})$ in place of $S(\mathfrak{p}, 1)$ and $S^L(\mathfrak{p}, 1)$.

Using these mixed valuations, Barvinok gave an algorithm which computes the $r+1$ highest degree Ehrhart coefficients of $S(n\mathfrak{p}) = \mathrm{Card}(n\mathfrak{p} \cap \mathbb{Z}^d)$, when $\mathfrak{p}$ is a simplex in $\mathbb{R}^d$. Barvinok's algorithm has polynomial length when d is an input, provided r is fixed. The method consists in reducing the problem to summations over lattice points in dimension $\leq r$.

Barvinok considers particular linear combinations

$$\sum_{L \in \mathcal{L}} \rho(L) S^L(\mathfrak{p}),$$

where $\mathcal{L}$ is a finite set of rational vector subspaces of V which is closed under sum, and the coefficients $\rho(L)$ are integers which satisfy the following relation between characteristic functions:

$$\chi(\cup_{L \in \mathcal{L}} L^\perp) = \sum_{L \in \mathcal{L}} \rho(L) \chi(L^\perp),$$

where $L^\perp \subseteq V^*$ is the orthogonal of L. We call a function $\mathcal{L} \to \mathbb{Z}$ with this property a patchwork function on $\mathcal{L}$.

When $\mathfrak{p}$ is dilated by an integer n, $S^L(n\mathfrak{p})$ is again given by a quasi-polynomial in n, as is a linear combination

$$\sum_{L \in \mathcal{L}} \rho(L) S^L(n\mathfrak{p}) = \sum_{m=0}^{d} \nu_m(\mathfrak{p}, n) n^m.$$

The main theoretical result of [2], Theorem (1.3), is the following: if $\mathcal{L}$ is a family of subspaces which is closed under sum and contains the vector subspace $\mathrm{lin}(\mathfrak{f})$ parallel to $\mathfrak{f}$, for every face $\mathfrak{f}$ of codimension $\leq r$ of $\mathfrak{p}$, and if ρ is a patchwork function on $\mathcal{L}$, then the $r+1$ highest degree coefficients $\nu_m(\mathfrak{p}, n)$, for $m = d, \ldots, d-r$, are equal to the corresponding Ehrhart coefficients of $S(n\mathfrak{p}) = \mathrm{Card}(n\mathfrak{p} \cap \mathbb{Z}^d)$.

In the present article, we introduce the meromorphic functions which extend $S^L(\mathfrak{p})$. For any polyhedron $\mathfrak{p}$,

$$S^L(\mathfrak{p})(\xi) = \sum_{y \in \Lambda_{V/L}} \int_{\mathfrak{p} \cap (y+L)} e^{\langle \xi, x \rangle} dx$$

is defined as a meromorphic function near $\xi = 0$. We show that $S^L(\mathfrak{p})(\xi)$ also enjoys a local Euler-Maclaurin expansion (Theorem 8),

$$S^L(\mathfrak{p})(\xi) = \sum_{\mathfrak{f}} \mu^L(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi) \int_{\mathfrak{f}} e^{\langle \xi, x \rangle} dx.$$

Furthermore, for a linear combination of Barvinok type, if $\mathfrak{f}$ is a face of $\mathfrak{p}$ such that $\mathrm{lin}(\mathfrak{f}) \in \mathcal{L}$, we prove that the $\mathfrak{f}$-term in the Euler-Maclaurin expansions of

$$S^{\mathcal{L},\rho}(\mathfrak{p})(\xi) = \sum_{L \in \mathcal{L}} \rho(L) S^L(\mathfrak{p})(\xi)$$

and of the *usual* valuation $S(\mathfrak{p})(\xi)$ are equal (Theorem 17):

$$\sum_{L \in \mathcal{L}} \rho(L) \mu^L(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi) = \mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi).$$

This is the main result of the present article. From the relation between Ehrhart quasi-polynomials and Euler-Maclaurin expansions, it implies Barvinok's Theorem (1.3).

Actually, we derive from Theorem 17 another computation of the $r+1$ highest coefficients of the Ehrhart quasi-polynomial for a pair $(\mathfrak{p}, h)$, based on Brion's decomposition of a polytope into cones, in the line of [1] and [6].

For each vertex s of $\mathfrak{p}$, let $\mathfrak{c}_s$ be the cone of feasible directions of $\mathfrak{p}$ at s. Instead of the full family $\mathcal{L}$ generated by taking sums of the subspaces $\mathrm{lin}(\mathfrak{f})$, when $\mathfrak{f}$ runs over the set of faces of codimension $\leq r$ of the polytope $\mathfrak{p}$, we consider, for each vertex s of $\mathfrak{p}$, the family $\mathcal{L}_s$ generated by faces of $\mathfrak{c}_s$ of codimension $\leq r$. The point in taking a family which depends on s lies in the case where $\mathfrak{p}$ is simplicial. Then $\mathcal{L}_s$ consists only of the spaces $\mathrm{lin}(\mathfrak{f})$ where $\mathfrak{f}$ is a face of $\mathfrak{c}_s$ of codimension $\leq r$, as this set is already closed under sum. Moreover the coefficients $\rho(L)$ are just signed binomial coefficients (Lemma 15), and the computation of $S^L(\mathfrak{c}_s)$ is easier when L is parallel to a face of $\mathfrak{c}_s$ (Example 6).

Let us describe our method in the simpler case of a lattice polytope $\mathfrak{p}$ and polynomial $h(x) = 1$. By Brion's theorem, we have

$$S(\mathfrak{p})(\xi) = \sum_{s} e^{\langle \xi, s \rangle} S(\mathfrak{c}_s)(\xi).$$

For each vertex s, let $\rho_s : \mathcal{L}_s \to \mathbb{Z}$ be a patchwork function. We define

$$\mathcal{B}(\mathfrak{p})(\xi) = \sum_{s} e^{\langle \xi, s \rangle} S^{\mathcal{L}_s, \rho_s}(\mathfrak{c}_s)(\xi).$$

For the dilated polytope $n\mathfrak{p}$, we have

$$S(n\mathfrak{p})(\xi) = \sum_s e^{n\langle \xi, s\rangle} S(\mathfrak{c}_s)(\xi) = \sum_{m\geq 0} \frac{n^m}{m!} \sum_s \langle \xi, s\rangle^m \; S(\mathfrak{c}_s)(\xi).$$

Hence, the meromorphic function $\frac{1}{m!}\sum_s \langle \xi, s\rangle^m \; S(\mathfrak{c}_s)(\xi)$ is actually regular at $\xi = 0$ and its value at $\xi = 0$ is the mth Ehrhart coefficient of $\mathfrak{p}$.

We have similarly

$$\mathcal{B}(n\mathfrak{p})(\xi) = \sum_{m\geq 0} \frac{n^m}{m!} \sum_s \langle \xi, s\rangle^m \; S^{\mathcal{L}_s,\rho_s}(\mathfrak{c}_s)(\xi).$$

The meromorphic functions $S^L(\mathfrak{c}_s)(\xi)$ and $S^{\mathcal{L}_s,\rho_s}(\mathfrak{c}_s)(\xi)$ have a special form: they can be written as the quotient of an analytic function by a product of $d' \leq d$ linear forms. Such a function ϕ has an expansion into rational functions $\phi = \sum_{j\geq -d} \phi_{[j]}$ where $\phi_{[j]}$ is homogeneous of total degree j.

Now it follows from our main theorem that, for $m \geq d - r$, we have

$$S(\mathfrak{c}_s)_{[-m]}(\xi) = S^{\mathcal{L}_s,\rho_s}(\mathfrak{c}_s)_{[-m]}(\xi),$$

hence the zero degree terms of $\sum_s \langle \xi, s\rangle^m \; S(\mathfrak{c}_s)(\xi)$ and $\sum_s \langle \xi, s\rangle^m \; S^{\mathcal{L}_s,\rho_s}(\mathfrak{c}_s)(\xi)$ are equal. Therefore the latter is also analytic at $\xi = 0$ and its value at $\xi = 0$ is the mth Ehrhart coefficient of $\mathfrak{p}$. This is the content of Theorem 20.

Thus, besides taking care of any polynomial $h(x)$, not only $h(x) = 1$, this method to compute the $r + 1$ highest degree Ehrhart coefficients for the pair $(\mathfrak{p}, h)$ leads to a simpler algorithm than the one proposed in [2]. When $\mathfrak{p}$ is a rational simplex, the contributions of the terms of the form $S^L(\mathfrak{c}_s)(\xi)$ when $L \in \mathcal{L}_s$ are immediately reduced to the computation of a function $S(\mathfrak{a})$ with $\mathfrak{a}$ a simplicial cone of dimension smaller or equal to r.

There is also another possible implementation of an algorithm to compute the $r + 1$ Ehrhart highest degree coefficients for the pair $(\mathfrak{p}, h)$ based on the results of [4]. As seen from Equation (1), this involves the computation of the analytic function $\mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))$, also associated to simplicial cones in dimension smaller or equal to r. We plan to compare the implementation of both methods in the near future.

2 Local Euler-Maclaurin expansion of a mixed valuation S^L

We consider a rational vector space V of dimension d, that is to say a finite dimensional real vector space with a lattice denoted by Λ_V or simply Λ. We will need to consider subspaces and quotient spaces of V, this is why we cannot just let $V = \mathbb{R}^d$ and $\Lambda = \mathbb{Z}^d$. By lattice, we mean a discrete additive subgroup of V which generates V as a vector space. Hence, a lattice is generated by a basis of the vector space V. A basis of V which is a $\mathbb{Z}$-basis of Λ_V is called an integral basis. The elements of Λ are called integral. An element $x \in V$ is called rational if $qx \in \Lambda$ for some integer $q \neq 0$. The space of rational points in V is denoted by $V_{\mathbb{Q}}$. A subspace L of V is called rational if $L \cap \Lambda$ is a lattice in L. If L is a rational subspace, the image of Λ in V/L is a lattice in V/L, so that V/L is a rational vector space. We will call the image of Λ in V/L the projected lattice.

Example 1 *Let $V = \mathbb{R}^2$ with standard lattice $\mathbb{Z}^2$. Let v_1, v_2 be two primitive integral independent vectors. Using an integral basis with first basis vector v_1, a straightforward computation shows that the projected lattice on $\mathbb{R}^2/\mathbb{R}v_1$ is $\mathbb{Z}\frac{\overline{v}_2}{\det(v_1,v_2)}$, where $\overline{v}_2$ is the projection of v_2 on $\mathbb{R}^2/\mathbb{R}v_1$.*

A rational space V, with lattice Λ, has a canonical Lebesgue measure, for which V/Λ has measure 1. An affine subspace L of V is called rational if it is a translate of a rational subspace by a rational element. It is similarly provided with a canonical Lebesgue measure. We will denote this measure by dm_L.

We will denote elements of V by latin letters $x, y, v, \dots$ and elements of V^* by greek letters $\xi, \alpha, \dots$. We denote the duality bracket by $\langle \xi, x \rangle$.

If S is a subset of V, we denote by $< S >$ the affine subspace generated by S. If S consists of rational points, then $< S >$ is rational. Remark that $< S >$ may contain no integral point. We denote by $\operatorname{lin}(S)$ the vector subspace of V parallel to $< S >$.

If S is a subset of V, we denote by $S^\perp$ the subspace of V^* orthogonal to S:

$$S^\perp = \{\xi \in V^* \; ; \; \langle \xi, x \rangle = 0 \text{ for all } x \in S\}.$$

If L is a subspace of V, the dual space $(V/L)^*$ is canonically identified with the subspace $L^\perp \subset V^*$.

A convex rational polyhedron $\mathfrak{p}$ in V (we will simply say polyhedron) is, by definition, the intersection of a finite number of half spaces bounded by rational affine hyperplanes. We say that $\mathfrak{p}$ is solid (in V) if $<\mathfrak{p}>=V$. A polytope $\mathfrak{p}$ is a **compact** polyhedron.

The set of non negative real numbers is denoted by $\mathbb{R}_+$. A convex rational cone $\mathfrak{c}$ in V is a closed convex cone $\sum_{i=1}^{k}\mathbb{R}_+ v_i$ which is generated by a finite number of elements v_i of $V_{\mathbb{Q}}$. In this article, we simply say cone instead of convex rational cone.

An affine (rational) cone $\mathfrak{a}$ is, by definition, the translate of a cone in V by a rational element $s \in V_{\mathbb{Q}}$. This cone is uniquely defined by $\mathfrak{a}$.

A cone $\mathfrak{c}$ is called simplicial if it is generated by independent elements of $V_{\mathbb{Q}}$. A simplicial cone $\mathfrak{c}$ is called unimodular if it is generated by independent integral vectors $v_1, \dots, v_k$ such that $\{v_1, \dots, v_k\}$ can be completed in an integral basis of V. An affine cone $\mathfrak{a}$ is called simplicial (resp. simplicial unimodular) if it is the translate of a simplicial (resp. simplicial unimodular) cone.

An affine cone $\mathfrak{a}$ is called pointed if it does not contain any straight line.

The set of faces of an affine cone $\mathfrak{a}$ is denoted by $\mathcal{F}(\mathfrak{a})$. If $\mathfrak{a}$ is pointed, then the vertex of $\mathfrak{a}$ is the unique face of dimension 0, while $\mathfrak{a}$ is the unique face of maximal dimension $\dim \mathfrak{a}$.

Let us recall the definition of the *transverse cone* $\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$ of a polyhedron $\mathfrak{p}$ along one of its faces $\mathfrak{f}$. Let x be a point in the relative interior of $\mathfrak{f}$. The cone of feasible directions of $\mathfrak{p}$ at x is the set $\mathfrak{c}(\mathfrak{p}, \mathfrak{f}) := \{v \in V \;; x + \epsilon v \in \mathfrak{p} \text{ for } \epsilon > 0 \text{ small enough}\}$. It does not depend on the choice of x. We denote the projection $V \to V/\operatorname{lin}(\mathfrak{f})$ by $\pi_{\mathfrak{f}}$. Then $\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$ is the image $\pi_{\mathfrak{f}}(\mathfrak{f} + \mathfrak{c}(\mathfrak{p}, \mathfrak{f}))$ of the affine cone $\mathfrak{f} + \mathfrak{c}(\mathfrak{p}, \mathfrak{f})$ in $V/\operatorname{lin}(\mathfrak{f})$. It is a solid pointed **affine** cone in the quotient space $V/\operatorname{lin}(\mathfrak{f})$ with vertex $\pi_{\mathfrak{f}}(<\mathfrak{f}>)$. In particular, if v is a vertex of $\mathfrak{p}$, the transverse cone $\mathfrak{t}(\mathfrak{p}, v)$ coincides with the supporting cone $v + \mathfrak{c}(\mathfrak{p}, v) \subset V$.

If $\mathfrak{a}$ is an affine cone and $\mathfrak{f}$ is a face of $\mathfrak{a}$, then $\mathfrak{c}(\mathfrak{a}, \mathfrak{f}) = \mathfrak{a} + \operatorname{lin}(\mathfrak{f})$ and the transverse cone $\mathfrak{t}(\mathfrak{a}, \mathfrak{f})$ of $\mathfrak{a}$ along $\mathfrak{f}$ is just the projection $\pi_{\mathfrak{f}}(\mathfrak{a})$ of $\mathfrak{a}$ on $V/\operatorname{lin}(\mathfrak{f})$.

Definition 2 *Denote by $\mathcal{H}(V^*)$ the ring of analytic functions around $0 \in V^*$. Denote by $\mathcal{M}(V^*)$ the ring of meromorphic functions defined around $0 \in V^*$ and by $\mathcal{M}_\ell(V^*) \subset \mathcal{M}(V^*)$ the subring consisting of those meromorphic functions $\phi(\xi)$ such that there exists a product of linear forms $D(\xi)$ with*

$$D(\xi)\phi(\xi) \in \mathcal{H}(V^*).$$

A function $\phi(\xi) \in \mathcal{M}_\ell(V^*)$ has a unique expansion into homogeneous rational functions

$$\phi(\xi) = \sum_{m>>-\infty} \phi_{[m]}(\xi)$$

where m is the total degree.

If P is a homogeneous polynomial on V^* of degree p, and D a product of r linear forms, then $\frac{P}{D}$ is an element in $\mathcal{M}_\ell(V^*)$ homogeneous of degree $m = p - r$.

Let us recall the definition of the function $I(\mathfrak{p}) \in \mathcal{M}_\ell(V^*)$ associated to a polyhedron $\mathfrak{p}$, (see for instance the survey [3]).

Proposition 3 *There exists a map I which to every polyhedron $\mathfrak{p} \subset V$ associates a meromorphic function with rational coefficients $I(\mathfrak{p}) \in \mathcal{M}_\ell(V^*)$, so that the following properties hold:*

(a) If $\mathfrak{p}$ contains a straight line, then $I(\mathfrak{p})$=0.

(b) If $\xi \in V^*$ is such that $|e^{\langle \xi, x\rangle}|$ is integrable over $\mathfrak{p}$, then

$$I(\mathfrak{p})(\xi) = \int_{\mathfrak{p}} e^{\langle \xi, x\rangle} dm_{<\mathfrak{p}>}(x).$$

(c) For every point $s \in V_{\mathbb{Q}}$, we have

$$I(s + \mathfrak{p})(\xi) = e^{\langle \xi, s\rangle} I(\mathfrak{p})(\xi).$$

(d) The map I is a simple valuation*: if the characteristic functions $\chi(\mathfrak{p}_i)$ of a family of polyhedra $\mathfrak{p}_i$ satisfy a linear relation $\sum_i r_i \chi(\mathfrak{p}_i) = 0$, then the functions $I(\mathfrak{p}_i)$ satisfy the relation*

$$\sum_{\{i, <\mathfrak{p}_i>=V\}} r_i I(\mathfrak{p}_i) = 0.$$

In the following proposition, we define the *mixed valuation* $\mathfrak{p} \mapsto S^L(\mathfrak{p})$ associated to a rational vector subspace $L \subsetneqq V$. To any polyhedron $\mathfrak{p}$, we associate a *meromorphic function* $S^L(\mathfrak{p})(\xi) \in \mathcal{M}(V^*)$. If $\mathfrak{p}$ is compact, this function is actually regular at 0, and its value for $\xi = 0$ is the valuation $E_{L^\perp}(\mathfrak{p})$ considered by Barvinok [2].

We denote by $\Lambda_{V/L}$ the projection on V/L of the lattice Λ.

Proposition 4 *Let $L \subsetneqq V$ be a rational subspace. There exists a map S^L which to every rational polyhedron $\mathfrak{p} \subset V$ associates a meromorphic function with rational coefficients $S^L(\mathfrak{p}) \in \mathcal{M}(V^*)$ so that the following properties hold:*

(a) If $\mathfrak{p}$ contains a line, then $S^L(\mathfrak{p})=0$.

(b)

$$S^L(\mathfrak{p})(\xi) = \sum_{y\in\Lambda_{V/L}} \int_{\mathfrak{p}\cap(y+L)} e^{\langle\xi,x\rangle} dm_L(x), \tag{2}$$

for every $\xi \in V^*$ such that the above sum converges.

(c) For every point $s \in \Lambda$, we have

$$S^L(s+\mathfrak{p})(\xi) = e^{\langle\xi,s\rangle} S^L(\mathfrak{p})(\xi).$$

(d) The map S^L is a valuation: if the characteristic functions $\chi(\mathfrak{p}_i)$ of a family of polyhedra $\mathfrak{p}_i$ satisfy a linear relation $\sum_i r_i\chi(\mathfrak{p}_i) = 0$, then the functions $S^L(\mathfrak{p}_i)$ satisfy the same relation

$$\sum_i r_i S^L(\mathfrak{p}_i) = 0.$$

For $L = \{0\}$, we recover the valuation S given by

$$S(\mathfrak{p})(\xi) = \sum_{x\in\mathfrak{p}\cap\Lambda} e^{\langle\xi,s\rangle},$$

provided this sum is convergent.

For $L = V$, we have $S^V(\mathfrak{p}) = I(\mathfrak{p})$, if $\mathfrak{p}$ is solid, and $S^V(\mathfrak{p}) = 0$ otherwise.

The proof is entirely analogous to the case $L = \{0\}$, see Theorem 3.1 in [3], and we omit it.

Remark 5 *The function $S^L(\mathfrak{p})$ is actually an element of $\mathcal{M}_\ell(V^*)$, but we do not prove it at this point. Let $\mathfrak{a}$ be an affine cone and $\{v_i\}$ the generators of its edges. It will follow from the Euler-Maclaurin expansion of $S^L(\mathfrak{a})$ (Theorem 8) that $\prod_i\langle\xi, v_i\rangle S^L(\mathfrak{a})(\xi)$ is analytic near zero for any L. It would be interesting to prove it a priori. By Brion's theorem and the valuation property, it follows in particular that $S^L(\mathfrak{p}) \in \mathcal{M}_\ell(V^*)$.*

Example 6 *Let* $\mathfrak{a}$ *be a simplicial affine cone in the space* V*, and assume that* $L = \mathrm{lin}(\mathfrak{f}_1)$ *for some face* $\mathfrak{f}_1$ *of* $\mathfrak{a}$*. In this case,* $S^L(\mathfrak{a})(\xi)$ *decomposes as product of an integral and a discrete sum. For simplicity, assume that* $\mathfrak{a}$ *is solid. Let* $\mathfrak{f}_2$ *be the face of* $\mathfrak{a}$ *such* $V = \mathrm{lin}(\mathfrak{f}_1) \oplus \mathrm{lin}(\mathfrak{f}_2)$*. We write* $x = x_1 + x_2$ *and* $\xi = \xi_1 + \xi_2$ *for the corresponding decompositions of* $x \in V$ *and* $\xi \in V^*$*. Then* $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2$ *where* $\mathfrak{a}_i$ *is a simplicial affine cone in* $\mathrm{lin}(\mathfrak{f}_i)$*. Let us denote by* Λ_2 *the projected lattice in* $V/\mathrm{lin}(\mathfrak{f}_1) \sim \mathrm{lin}(\mathfrak{f}_2)$*. From (2), we obtain immediately*

$$S^L(\mathfrak{a})(\xi_1 + \xi_2) = I(\mathfrak{a}_1)(\xi_1) \sum_{x_2 \in \mathfrak{a}_2 \cap \Lambda_2} e^{\langle \xi_2, x_2 \rangle}. \tag{3}$$

Notice that the lattice Λ_2 *is usually bigger than* $\Lambda \cap \mathrm{lin}(\mathfrak{f}_2)$*.*

Example 7 *Let* $V = \mathbb{R}^2$ *with the standard lattice. We compute* $S^L(\mathfrak{a})$ *when* $\mathfrak{a}$ *is a cone and* L *is a line. Let* $\mathfrak{a} = \mathbb{R}_+ v_1 + \mathbb{R}_+ v_2$*, where* v_1, v_2 *are two linearly independent primitive integral vectors.*

(a) Assume that L *is the line supporting an edge of* $\mathfrak{a}$*, say* $L = \mathbb{R}v_1$*. We identify* V/L *to* $\mathbb{R}v_2$*. The projected lattice is* $\Lambda_2 = \mathbb{Z}\frac{v_2}{\det(v_1,v_2)}$*, (Example 1), hence, by (3) in Example 6, we have*

$$S^L(\mathfrak{a})(\xi) = -\frac{1}{\langle \xi, v_1 \rangle} \frac{1}{1 - e^{\frac{\langle \xi, v_2 \rangle}{|\det(v_1,v_2)|}}}. \tag{4}$$

(b) Assume now that L *is transverse to both edges of* $\mathfrak{a}$*. Assume that* $\det(v_1, v_2) > 0$*. Let* $L = \mathbb{R}u$ *where* u *is a primitive integral vector chosen so that* $\det(u, v_2) > 0$*. Let* $\mathfrak{a}_i = \mathbb{R}_+ u + \mathbb{R}_+ v_i$ *for* $i = 1, 2$*. We decompose the characteristic function of the cone* $\mathfrak{a}$ *as* $\chi(\mathfrak{a}) = \chi(\mathfrak{a}_2) + \chi(\mathfrak{a}_1) - \chi(\mathbb{R}_+ u)$ *or* $\chi(\mathfrak{a}) = \chi(\mathfrak{a}_2) - \chi(\mathfrak{a}_1) + \chi(\mathbb{R}_+ v_1)$*, depending on whether* u *belongs to* $\mathfrak{a}$ *or not. Using the valuation property, case (a) and the relation*

$$\frac{1}{1 - e^x} + \frac{1}{1 - e^{-x}} = 1,$$

we obtain in both cases

$$S^L(\mathfrak{a})(\xi) = -\frac{1}{\langle \xi, u \rangle} \left(\frac{1}{1 - e^{\frac{\langle \xi, v_2 \rangle}{\det(u,v_2)}}} - \frac{1}{1 - e^{\frac{\langle \xi, v_1 \rangle}{\det(u,v_1)}}} \right). \tag{5}$$

In this example, we see that $\langle \xi, v_1 \rangle \langle \xi, v_2 \rangle S^L(\mathfrak{a})(\xi)$ *is indeed analytic near* $\xi = 0$*.*

In the following theorem and its applications, we will consider the functions $S^L(\mathfrak{p})$ when the space V is replaced with a quotient space W. We denote by $\mathcal{C}(W)$ the set of affine cones in W. Thus if $\mathfrak{a} \in \mathcal{C}(W)$, and L a rational subspace of W, the function $S^L(\mathfrak{a})$ is a meromorphic function on W^*. We are going to show that the function $S^L(\mathfrak{a})$ has a local Euler-Maclaurin expansion, which generalizes the case $L = \{0\}$ of [4].

Theorem 8 *Let V be a rational space and Q a rational scalar product on V^*. There exists a unique family of maps μ_W^L, indexed by pairs (W, L) where W is a rational quotient space of V and L is a rational vector subspace of W such that the family enjoys the following properties:*

(a) μ_W^L maps $\mathcal{C}(W)$ to $\mathcal{H}(W^*)$, the space of analytic functions on W^*.

(b) If $W = \{0\}$, then $\mu_{\{0\}}^{\{0\}}(\{0\}) = 1$.

(c) For $\dim W > 0$ and $L = W$, then $\mu_W^W(\mathfrak{a}) = 0$.

(d) If the affine cone $\mathfrak{a} \in \mathcal{C}(W)$ contains a straight line, then $\mu_W^L(\mathfrak{a}) = 0$.

(e) For any affine cone $\mathfrak{a}$ in W, the following formula holds

$$S^L(\mathfrak{a}) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{a})} \mu_{W/\operatorname{lin}(\mathfrak{f})}^{L+\operatorname{lin}(\mathfrak{f})/\operatorname{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f})) I(\mathfrak{f})$$

where the sum is over all faces of the cone $\mathfrak{a}$.

In this last formula, the function $\mu_{W/\operatorname{lin}(\mathfrak{f})}^{L+\operatorname{lin}(\mathfrak{f})/\operatorname{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}))$ is considered as a function on W^* itself by means of the orthogonal projection $W^* \to (W/\operatorname{lin}(\mathfrak{f}))^* = (\operatorname{lin}(\mathfrak{f}))^\perp$ with respect to the scalar product on $W^* \subset V^*$.

Proof. The proof is entirely similar to the case $L = \{0\}$ studied in [4]. Note that $\mu_W^{\{0\}}$ coincides with the map denoted by μ_W in [4]. The only new item is (c). It follows immediately from the relation $S^W(\mathfrak{a}) = I(\mathfrak{a})$.

Remark 9 *Let $\mathfrak{a}$ be a solid cone in W, and let $\mathfrak{f}$ be a face of $\mathfrak{a}$ such that $\dim \mathfrak{f} < \dim W$. If L is transverse to the face $\mathfrak{f}$, that is, if $L + \operatorname{lin}(\mathfrak{f}) = W$, then $\mu_{W/\operatorname{lin}(\mathfrak{f})}^{L+\operatorname{lin}(\mathfrak{f})/\operatorname{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f})) = 0$. This follows from (c).*

From now on we omit the subscript W, thus we write μ^L in place of μ_W^L. The next theorem and its proof are also entirely similar to the case $L = \{0\}$ in [4].

Theorem 10 *The analytic functions defined in Theorem 8 have the following properties:*

(a) For any $x \in \Lambda$*, one has* $\mu^L(x + \mathfrak{a}) = \mu^L(\mathfrak{a})$*.*

(b) The map $(\mathfrak{a}, L) \mapsto \mu^L(\mathfrak{a})$ *is equivariant with respect to lattice-preserving isometries. In other words, let* g *be an isometry of* W *which preserves the lattice* Λ_W*. Then* $\mu^{g(L)}(g(\mathfrak{a}))({}^t g^{-1}\xi) = \mu^L(\mathfrak{a})(\xi)$*.*

(c) If W *is an orthogonal sum* $W = W_1 \oplus W_2$ *of rational spaces,* $L_i \subsetneqq W_i$ *and* $\mathfrak{a}_i$ *is an affine cone in* W_i *for* $i = 1, 2$*, then*

$$\mu^{L_1 \oplus L_2}(\mathfrak{a}_1 + \mathfrak{a}_2) = \mu^{L_1}(\mathfrak{a}_1)\mu^{L_2}(\mathfrak{a}_2).$$

(d) For a fixed $s \in W_{\mathbb{Q}}$*, the map* $\mathfrak{c} \to \mu^L(s + \mathfrak{c})$ *is a valuation on the set of cones in* W*.*

(e) Let $\mathfrak{p} \subset W$ *be a rational polyhedron, then*

$$S^L(\mathfrak{p}) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p})} \mu^{L + \operatorname{lin}(\mathfrak{f})/\operatorname{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))I(\mathfrak{f}). \tag{6}$$

Example 11 *Let us compute the function* μ^L *for the various transverse cones of Example 7. We define a function* $B(u)$ *on* $\mathbb{C}$*, holomorphic near* 0*, by*

$$B(u) = \frac{1}{1 - e^u} + \frac{1}{u}.$$

We have

$$I(\mathfrak{a})(\xi) = \frac{|\det(v_1, v_2)|}{\langle \xi, v_1 \rangle \langle \xi, v_2 \rangle}.$$

Consider case (b) where L *is transverse to both edges* $\mathfrak{f}_i = \mathbb{R}_+ v_i$ *of* $\mathfrak{a}$*.*

Using the equation $\det(v_1, v_2)u = \det(u, v_2)v_1 - \det(u, v_1)v_2$*, we have*

$$I(\mathfrak{a})(\xi) = \frac{1}{\langle \xi, u \rangle}\left(\frac{\det(u, v_2)}{\langle \xi, v_2 \rangle} - \frac{\det(u, v_1)}{\langle \xi, v_1 \rangle}\right).$$

Thus we can rewrite (5) as

$$S^L(\mathfrak{a}) = \mu^L(\mathfrak{a}) + \sum_{i=1,2} \mu^{L + \operatorname{lin}(\mathfrak{f}_i)/\operatorname{lin}(\mathfrak{f}_i)}((\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_i))I(\mathfrak{f}_i) + I(\mathfrak{a}),$$

with

$$\mu^L(\mathfrak{a})(\xi) = \frac{1}{\langle \xi, u \rangle}\left[B\left(\frac{\langle \xi, v_1 \rangle}{\det(u, v_1)}\right) - B\left(\frac{\langle \xi, v_2 \rangle}{\det(u, v_2)}\right)\right], \tag{7}$$
$$\mu^{L + \operatorname{lin}(\mathfrak{f}_i)/\operatorname{lin}(\mathfrak{f}_i)}((\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_i)) = 0 \text{ for } i = 1, 2,$$

Observe that (7) is indeed regular at $\xi = 0$.

In case (a) where $L = \mathbb{R}v_1$, *we have* $L + \operatorname{lin}(\mathfrak{f}_1)/\operatorname{lin}(\mathfrak{f}_1) = \{0\}$. *Let us assume that* $\det(v_1, v_2) > 0$. *Then we have, by [4],*

$$\mu^{\{0\}}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_1))(\xi) = B\left(\frac{-C_1\langle \xi, v_1\rangle + \langle \xi, v_2\rangle}{\det(v_1, v_2)}\right)$$

with $C_1 = \frac{Q(v_1,v_2)}{Q(v_1,v_1)}$.

As $I(\mathfrak{f}_1)(\xi) = -\frac{1}{\langle \xi, v_1\rangle}$, *we can rewrite (4) as*

$$S^L(\mathfrak{a}) = \mu^L(\mathfrak{a}) + \mu^{\{0\}}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_1))I(\mathfrak{f}_1) + \ \mu^{L+\operatorname{lin}(\mathfrak{f}_2)/\operatorname{lin}(\mathfrak{f}_2)}((\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_2))I(\mathfrak{f}_2) + I(\mathfrak{a}),$$

with

$$\mu^L(\mathfrak{a})(\xi) = \frac{1}{\langle \xi, v_1\rangle}\left[B\left(\frac{-C_1\langle \xi, v_1\rangle + \langle \xi, v_2\rangle}{\det(v_1, v_2)}\right) - B\left(\frac{\langle \xi, v_2\rangle}{\det(v_1, v_2)}\right)\right],$$

which is indeed regular at $\xi = 0$, *and, again,*

$$\mu^{L+\operatorname{lin}(\mathfrak{f}_2)/\operatorname{lin}(\mathfrak{f}_2)}((\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_2)) = 0.$$

3 Barvinok valuations

Let $\mathcal{L}$ be a finite family of rational vector subspaces of V, and let $\rho(L)$, $L \in \mathcal{L}$, be a set of rational coefficients. The linear combination $\sum_{L\in\mathcal{L}} \rho(L)S^L(\mathfrak{p})$ is again a valuation on the set of polyhedra $\mathfrak{p} \subset V$, with values in $\mathcal{M}_\ell(V^*)$. By taking linear combinations, we obtain a local Euler-Maclaurin expansion for the function $\sum_{L\in\mathcal{L}} \rho(L)S^L(\mathfrak{p})(\xi)$.

Definition 12 *Let* $\mathfrak{p} \subset V$ *be a polyhedron.*

(a) We denote

$$S^{(\mathcal{L},\rho)}(\mathfrak{p}) = \sum_{L\in\mathcal{L}} \rho(L)S^L(\mathfrak{p}).$$

(b) We define the $\mathfrak{f}$*-term in the local Euler-Maclaurin expansion of* $S^{(\mathcal{L},\rho)}(\mathfrak{p})$ *to be*

$$\mu^{(\mathcal{L},\rho)}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f})) = \sum_{L\in\mathcal{L}} \rho_{\mathcal{L}}(L)\mu^{L+\operatorname{lin}(\mathfrak{f})/\operatorname{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f})).$$

Thus we have

$$S^{(\mathcal{L},\rho)}(\mathfrak{p}) = \sum_{\mathfrak{f}\in\mathcal{F}(\mathfrak{p})} \mu^{(\mathcal{L},\rho)}(\mathfrak{t}(\mathfrak{p},\mathfrak{f}))I(\mathfrak{f}).$$

We are going to compute the $\mathfrak{f}$-term in the case of the following particular linear combinations introduced by Barvinok [2].

Definition 13 *The valuation $S^{(\mathcal{L},\rho)}$ is called a Barvinok valuation if*

(a) the family of subspaces $\mathcal{L}$ is stable under sum,

(b) ρ is an integer valued function on the set $\mathcal{L}$ such that the characteristic function of the union of the subspaces $L^\perp \subseteqq V^*$ can be written as the linear combination

$$\chi(\cup_{L\in\mathcal{L}} L^\perp) = \sum_{L\in\mathcal{L}} \rho(L)\chi(L^\perp). \tag{8}$$

Definition 14 *We call a function $\mathcal{L} \to \mathbb{Z}$ which satisfies (8) a patchwork function on $\mathcal{L}$.*

As the set of orthogonal subspaces $L^\perp \subseteqq V^*$ is stable under intersection, a particular function $\rho_{\mathcal{L}}$ with this property can be computed in terms of the Moebius function of the partially ordered set $\mathcal{L}$ ([7], vol I, section 3.7), as explained in [2].

Let us compute a patchwork function ρ in the following case. $V = \mathbb{R}^d$ with standard basis $e_i, i = 1, \ldots, d$, and $\mathcal{L}_{d,q}$ is the set of subspaces $L_I = \oplus_{i\in I}\mathbb{R}e_i$ with cardinal $|I| \geq q$. The function $\rho_{d,q}$ defined below is actually the one associated to the Moebius function, but we will not need this fact.

We denote the binomial coefficient $\frac{m!}{k!(m-k)!}$ by C_m^k.

Lemma 15 *The function $\rho_{d,q}$ on $\mathcal{L}_{d,q}$ defined by*

$$\rho_{d,q}(L_I) = (-1)^{n-q}C_{n-1}^{q-1} \quad \textit{if } |I| = n,$$

satisfies Equation (8).

Proof. If e^i is the dual basis, the orthogonal space $L_I^\perp$ is equal to $\sum_{i\notin I}\mathbb{R}e^i$. Let $\xi = \sum_{i=1}^d \xi_i e^i \in \cup_{L\in\mathcal{L}_q} L^\perp$. Let I_0 be the set of indices $i \in [1, \ldots, d]$ such that $\xi_i = 0$. Then $|I_0| \geq q$, and $\xi \in L_I^\perp$ if and only $I \subseteq I_0$. Let $|I_0| = N$. The value at ξ of the right-hand side of (8) is equal to

$$E(N,q) = \sum_{I\subseteq I_0} \rho_{d,q}(L_I) = \sum_{n=q}^{N} (-1)^{n-q} C_N^n C_{n-1}^{q-1}.$$

We want to prove that $E(N,q)=1$. Writing $n=q+i$, we have

$$E(N,q)=\sum_{i=0}^{N-q}(-1)^i\frac{N!}{(q+i)(N-q-i)!i!(q-1)!}.$$

Let us compute $(q-1)!\Big(E(N+1,q)-E(N,q)\Big)$. This is equal to

$$(-1)^{N+1-q}\frac{(N+1)!}{(N+1)(N+1-q)!}+\sum_{i=0}^{N-q}(-1)^i\frac{1}{(q+i)i!}\left(\frac{(N+1)!}{(N+1-q-i)!}-\frac{N!}{(N-q-i)!}\right)$$

$$=N!\sum_{i=0}^{N+1-q}(-1)^i\frac{N!}{i!(N+1-q-i)!}=0.$$

We obtain $E(N,q)=E(q,q)=1$. □

In the case of a Barvinok valuation, it turns out that the $\mathfrak{f}$-term in the Euler-Maclaurin expansion of $S^{(\mathcal{L},\rho)}(\mathfrak{p})$ coincides with that of $S(\mathfrak{p})$, if the vector subspace $\mathrm{lin}(\mathfrak{f})$ belongs to the set $\mathcal{L}$. This is the crucial result of the present article. It is an easy consequence of the following combinatorial lemma.

Lemma 16 *Let $\mathcal{L}$ be a finite family of vector subspaces of V, stable under sum and let ρ be a patchwork function on $\mathcal{L}$.*

(a) Let $L_0\in\mathcal{L}$. Then

$$\sum_{\{L\in\mathcal{L},\ L\subseteq L_0\}}\rho(L)=1.$$

(b) Let $L_0\subsetneqq L_1$ be two subspaces in the family $\mathcal{L}$. Then

$$\sum_{\{L\in\mathcal{L},\ L+L_0=L_1\}}\rho(L)=0.$$

Proof. There exists a $\xi_0\in L_0^\perp$ such that, for $L\in\mathcal{L}$, $\xi_0\in L^\perp$ if and only if $L\subseteq L_0$. We obtain *(a)* by evaluating both sides of (8) at this particular element ξ_0. Next, we deduce *(b)* from *(a)*, by induction on $\dim L_1-\dim L_0$.

For two subspaces $M\subseteq M'$ in the family $\mathcal{L}$, let us denote

$$f(M,M')=\sum_{\{L\in\mathcal{L},\ L+M=M'\}}\rho(L).$$

If $M = M'$, we have $f(M,M) = \sum_{\{L\in\mathcal{L}, L\subseteq M\}} \rho(L) = 1$ by *(a)*.

We apply this with $M = L_1$. Thus

$$\sum_{\{L\in\mathcal{L}, L\subseteq L_1\}} \rho(L) = 1.$$

In this sum, we group the $L \in \mathcal{L}$ such that $L + L_0$ is equal to a given $M_1 \in \mathcal{L}$ together.

First we consider the case when $\dim L_1 - \dim L_0 = 1$. Then M_1 is either L_0 or L_1, hence we obtain

$$f(L_0, L_0) + f(L_0, L_1) = 1.$$

Since $f(L_0, L_0) = 1$ by *(a)*, we obtain $f(L_0, L_1) = 0$ as required.

Next we consider the case when $\dim L_1 - \dim L_0 > 1$. We obtain

$$\sum_{\{M_1\in\mathcal{L},\ L_0\subseteq M_1\subseteq L_1\}} f(L_0, M_1) = 1.$$

For $M_1 = L_0$, we have $f(L_0, L_0) = 1$ by *(a)*. For $M_1 \subsetneq L_1$, we have $f(L_0, M_1) = 0$ by induction hypothesis. Hence there remains only the term $f(L_0, L_1)$ which must be equal to 0. □

We study now the Euler-Maclaurin expansion of a Barvinok valuation.

Theorem 17 *Let $\mathfrak{p} \subset V$ be a rational polyhedron and let $\mathfrak{f}$ be a face of $\mathfrak{p}$. Let $\mathcal{L}$ be a finite family of rational vector subspaces of V, stable under sum. Let ρ be a patchwork function on $\mathcal{L}$, and let $S^{(\mathcal{L},\rho)} = \sum_{L\in\mathcal{L}} \rho(L) S^L$.*

Assume that $\mathrm{lin}(\mathfrak{f})$ *belongs to $\mathcal{L}$. Then*

$$\mu^{(\mathcal{L},\rho)}(\mathfrak{t}(\mathfrak{p},\mathfrak{f})) = \mu^{\{0\}}(\mathfrak{t}(\mathfrak{p},\mathfrak{f})).$$

In other words, the $\mathfrak{f}$-term in the local Euler-Maclaurin expansion of $S^{(\mathcal{L},\rho)}(\mathfrak{p})$ coincides with that of $S(\mathfrak{p})$.

Proof. In the sum $\sum_{L\in\mathcal{L}} \rho(L) \mu^{L+\mathrm{lin}(\mathfrak{f})/\,\mathrm{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{p},\mathfrak{f}))$, we group the terms for which $L + \mathrm{lin}(\mathfrak{f})$ is equal to a given L_1 together. By Lemma 16 we obtain $\mu^{\{0\}}(\mathfrak{t}(\mathfrak{p},\mathfrak{f}))$ for $L_1 = \mathrm{lin}(\mathfrak{f})$ and 0 otherwise. □

Corollary 18 *Let $\mathfrak{p} \subset V$ be a rational polyhedron. Let $0 \leq k \leq d$. Let $\mathcal{L}$ be a finite family of rational vector subspaces of V, stable under sum, such that $\mathrm{lin}(\mathfrak{f}) \in \mathcal{L}$ for every k-dimensional face $\mathfrak{f}$ of $\mathfrak{p}$ and let ρ be a patchwork function on $\mathcal{L}$.*

- *Let $0 < k \leq d$. Then the meromorphic function*

$$S(\mathfrak{p})(\xi) - S^{(\mathcal{L},\rho)}(\mathfrak{p})(\xi)$$

has lowest degree $\geq -k+1$.

- *Let $k = 0$. Then*

$$S(\mathfrak{p})(\xi) = S^{(\mathcal{L},\rho)}(\mathfrak{p})(\xi).$$

Proof. By Theorem 17, the local Euler-Maclaurin expansion of the difference involves only faces of dimension $< k$.

$$S(\mathfrak{p})(\xi) - S^{(\mathcal{L},\rho)}(\mathfrak{p})(\xi) = \sum_{\{\mathfrak{f}\in\mathcal{F}(\mathfrak{p}),\ \dim \mathfrak{f}<k\}} \left(\mu^{\{0\}}(\mathfrak{t}(\mathfrak{p},\mathfrak{f}))(\xi) - \mu^{(\mathcal{L},\rho)}(\mathfrak{t}(\mathfrak{p},\mathfrak{f}))(\xi)\right) I(\mathfrak{f})(\xi).$$

For a face of dimension j, the function $I(\mathfrak{f})(\xi)$ is homogeneous of degree $-j$. Multiplied by the *holomorphic* function $\mu^{\{0\}}(\mathfrak{t}(\mathfrak{p},\mathfrak{f}))(\xi) - \mu^{\mathcal{L}}(\mathfrak{t}(\mathfrak{p},\mathfrak{f}))(\xi)$, its lowest degree can only increase. □

Remark that the statement of Corollary 18 above does not involve any scalar product. In the next section, we will show that this corollary implies our main Theorem (Theorem 20).

4 Ehrhart quasi-polynomial

Let $\mathfrak{p}$ be a rational polytope and let $h(x)$ be a polynomial function on V. Let

$$S(\mathfrak{p}, h) = \sum_{x\in\mathfrak{p}\cap\mathbb{Z}^d} h(x).$$

When $\mathfrak{p}$ is dilated by a non negative integer n, we obtain the Ehrhart quasi-polynomial of the pair $(\mathfrak{p}, h)$

$$S(n\mathfrak{p}, h) = \sum_{m=0}^{d+N} E_m(\mathfrak{p}, h, n) n^m,$$

where $N = \deg h$. The coefficients $E_m(\mathfrak{p}, h, n)$ are periodic functions of $n \in \mathbb{N}$, with period the smallest integer q such that $q\mathfrak{p}$ is a lattice polytope.

If an integer $r \leq d$ is fixed, and $h = 1$, Barvinok [2] proved that the $r+1$ highest Ehrhart coefficients $E_d(\mathfrak{p}, 1, n), \ldots, E_{d-r}(\mathfrak{p}, 1, n)$ of $S(n\mathfrak{p}, 1)$ can be computed in polynomial time with respect to d, when $\mathfrak{p}$ is a rational simplex.

Let $L \subseteq V$ be a rational vector subspace. Denote the projected lattice on V/L by Λ_L. Consider the mixed valuation

$$S^L(\mathfrak{p}, h) = \sum_{y \in \Lambda_{V/L}} \int_{\mathfrak{p} \cap (y+L)} h(x) dx.$$

As shown by Barvinok, and as we will see here, we can use linear combination of these mixed valuations to approximate $S(n\mathfrak{p}, h)$ when n is big.

For any polyhedron $\mathfrak{a}$, we define the meromorphic function $S^L(\mathfrak{a}, h)(\xi) \in \mathcal{M}_\ell(V^*)$ similarly to $S^L(\mathfrak{a}, h)$. For $\xi \in V^*$ such that the sum converges, we have

$$S^L(\mathfrak{a}, h)(\xi) = \sum_{y \in \Lambda_{V/L}} \int_{\mathfrak{a} \cap (y+L)} h(x) e^{\langle \xi, x \rangle} dm_L(x).$$

Remark 19 *It is clear that* $S^L(\mathfrak{a}, h)(\xi) = h(\partial_\xi) \cdot S^L(\mathfrak{a})(\xi)$.

If $\mathfrak{p}$ *is a* polytope*, then* $S^L(\mathfrak{p}, h)(\xi)$ *is regular at* $\xi = 0$ *and* $S^L(\mathfrak{p}, h)(0) = S^L(\mathfrak{p}, h)$.

For a family $\mathcal{L}$ and a function $L \mapsto \rho(L), L \in \mathcal{L}$, we define

$$S^{(\mathcal{L},\rho)}(\mathfrak{a}, h)(\xi) = \sum_{L \in \mathcal{L}} \rho(L) S^L(\mathfrak{a}, h)(\xi).$$

If $\mathfrak{p}$ is a polytope, and we dilate by $n \in N$, we have again a quasi-polynomial

$$S^{(\mathcal{L},\rho)}(n\mathfrak{p}, h)(0) = \sum_{m=0}^{d+N} E_m(\mathcal{L}, \rho, \mathfrak{p}, h, n) n^m.$$

We can replace the quasi-polynomial $S^{(\mathcal{L},\rho)}(n\mathfrak{p}, h)(0)$ by q legal polynomials in the variable u, by splitting $\mathbb{N}$ into classes modulo q. Writing $n = qu+k$, for $k = 0, \ldots, q-1$, we obtain the polynomial function of u:

$$S^{(\mathcal{L},\rho)}((qu+k)\mathfrak{p}, h)(0) = \sum_{m=0}^{d+N} E_m^{(k)}(\mathcal{L}, \rho, \mathfrak{p}, h) u^m.$$

We briefly recall how the usual Ehrhart quasi-polynomial of a polytope can be computed using Brion's theorem. We will then use a similar method in order to compute efficiently the $r+1$ highest coefficients only, using Barvinok valuations.

Let $\mathcal{V}(\mathfrak{p})$ be the set of vertices of $\mathfrak{p}$. For each vertex s, let $\mathfrak{c}_s$ be the cone of feasible directions of $\mathfrak{p}$ at s, so that the supporting cone at s is $s+\mathfrak{c}_s$. By Brion's theorem [5], we have

$$S(\mathfrak{p},h)(\xi)=\sum_{s\in\mathcal{V}(\mathfrak{p})} S(s+\mathfrak{c}_s,h)(\xi).$$

Let $n\in\mathbb{N}$ and consider the dilated polytope $n\mathfrak{p}$. The supporting cone at the vertex ns is $ns+\mathfrak{c}_s$. Let $q\in\mathbb{N}$ such that $q\mathfrak{p}$ is a lattice polytope and fix $k\in\mathbb{N}$, $0\leq k\leq q-1$. Let $n=qu+k$. As qus is an integral point, we have

$$S((qu+k)s+\mathfrak{c}_s,h)(\xi)=e^{qu\langle\xi,s\rangle}S^{\mathcal{L}_s,\rho_s}(ks+\mathfrak{c}_s,h)(\xi).$$

Expanding in powers of u, we obtain

$$S((qu+k)\mathfrak{p},h)(\xi)=\sum_{m\geq 0} u^m\frac{q^m}{m!}\sum_{s\in\mathcal{V}(\mathfrak{p})}\langle\xi,s\rangle^m\ S(ks+\mathfrak{c}_s,h)(\xi).$$

It follows that for each m, the sum of meromorphic functions

$$\frac{q^m}{m!}\sum_{s\in\mathcal{V}(\mathfrak{p})}\langle\xi,s\rangle^m\ S(ks+\mathfrak{c}_s,h)(\xi)$$

is actually analytic. Its value at $\xi=0$ is obtained by taking the zero degree term. We obtain

$$S((qu+k)\mathfrak{p},h)(0)=\sum_{m\geq 0} E_m^{(k)}(\mathfrak{p},h)\ u^m,$$

with

$$E_m^{(k)}(\mathfrak{p},h)=\frac{q^m}{m!}\sum_{s\in\mathcal{V}(\mathfrak{p})}\langle\xi,s\rangle^m\ S(ks+\mathfrak{c}_s,h)_{[-m]}(\xi).$$

The right-hand side of this relation, a priori a meromorphic function of ξ, is actually constant. Moreover, we have $S(ks+\mathfrak{c}_s,h)_{[-m]}(\xi)=0$ if $m>d+N$, hence the Ehrhart quasi-polynomial has degree $\leq d+N$.

We apply now Brion's theorem to $S^{\mathcal{L},\rho}(\mathfrak{p},h)(\xi)$. We obtain

$$S^{\mathcal{L},\rho}(\mathfrak{p},h)(\xi)=\sum_{s\in\mathcal{V}(\mathfrak{p})}S^{\mathcal{L},\rho}(s+\mathfrak{c}_s,h)(\xi).$$

For reasons to be explained later on, instead of one family $\mathcal{L}$, we take a family of subspaces $\mathcal{L}_s$ for each vertex s. Let $\rho_s:\mathcal{L}_s\to\mathbb{Z}$ be a function on $\mathcal{L}_s$. We denote now by $(\mathcal{L},\rho)$ the map $s\mapsto(\mathcal{L}_s,\rho_s)$.

We define:

$$\mathcal{B}^{\mathcal{L},\rho}(\mathfrak{p},h)(\xi)=\sum_{s\in\mathcal{V}(\mathfrak{p})}S^{\mathcal{L}_s,\rho_s}(s+\mathfrak{c}_s,h)(\xi). \tag{9}$$

If the family does not depend on s, $(\mathcal{L}_s,\rho_s)=(\mathcal{L}_0,\rho_0)$ for every vertex s, then, by Brion's theorem, we have

$$\mathcal{B}^{\mathcal{L},\rho}(\mathfrak{p},h)(\xi)=S^{\mathcal{L}_0,\rho_0}(\mathfrak{p},h)(\xi).$$

We dilate (9). Let $n=qu+k$. We obtain

$$\mathcal{B}^{\mathcal{L},\rho}((qu+k)\mathfrak{p},h)(\xi)=\sum_{s\in\mathcal{V}(\mathfrak{p})}e^{qu\langle\xi,s\rangle}S^{\mathcal{L}_s,\rho_s}(ks+\mathfrak{c}_s,h)(\xi).$$

Expanding in powers of u, we obtain

$$\mathcal{B}^{\mathcal{L},\rho}((qu+k)\mathfrak{p},h)(\xi)=\sum_{m\geq 0}u^m E_m^{(k)}(\mathcal{L},\rho,\mathfrak{p},h)(\xi) \tag{10}$$

with

$$E_m^{(k)}(\mathcal{L},\rho,\mathfrak{p},h)(\xi)=\frac{q^m}{m!}\sum_{s\in\mathcal{V}(\mathfrak{p})}\langle\xi,s\rangle^m\,S^{\mathcal{L}_s,\rho_s}(ks+\mathfrak{c}_s,h)(\xi). \tag{11}$$

If the family does not depend on s, $(\mathcal{L}_s,\rho_s)=(\mathcal{L}_0,\rho_0)$ for all vertices, then $\mathcal{B}^{\mathcal{L},\rho}(\mathfrak{p},h)(\xi)=S^{\mathcal{L}_0,\rho_0}(\mathfrak{p},h)(\xi)$ is analytic near $\xi=0$, and so are the coefficients (11).

On the contrary, if we take a different family $\mathcal{L}_s$ for each vertex s, the coefficient $E_m^{(k)}(\mathcal{L},\rho,\mathfrak{p},h)(\xi)$ of u^m in (10) is no longer analytic near $\xi=0$, in general. However, the meromorphic function $\xi\mapsto E_m^{(k)}(\mathcal{L},\rho,\mathfrak{p},h)(\xi)$ belongs to $\mathcal{M}_\ell(V^*)$, thus it has a term of degree 0 with respect to ξ, given by

$$E_m^{(k)}(\mathcal{L},\rho,\mathfrak{p},h)_{[0]}(\xi)=\frac{q^m}{m!}\sum_{s\in\mathcal{V}(\mathfrak{p})}\langle\xi,s\rangle^m\,S^{\mathcal{L}_s,\rho_s}(ks+\mathfrak{c}_s,h)_{[-m]}(\xi). \tag{12}$$

For a family $(\mathcal{L}, \rho)$ as described in the next theorem, it turns out that, for large m, this zero-degree part $E_m^{(k)}(\mathcal{L}, \rho, \mathfrak{p}, h)_{[0]}(\xi)$ is actually analytic, hence constant, and its value is equal to the m-th Ehrhart coefficient $E_m^{(k)}(\mathfrak{p}, h)$ of $S((k+qu)\mathfrak{p}, h)(0)$.

Theorem 20 *Let $\mathfrak{p}$ be a rational polytope in a rational vector space of dimension d. For each vertex s of the polytope $\mathfrak{p}$, let $\mathfrak{c}_s$ be the cone of feasible directions of $\mathfrak{p}$ at s, so that the supporting cone at s is $s + \mathfrak{c}_s$. For each vertex s, let $\mathcal{L}_s$ be a finite family of rational vector subspaces of V, stable under sum, such that $\operatorname{lin}(\mathfrak{f})$ belongs to $\mathcal{L}_s$ for every face $\mathfrak{f}$ of codimension r of the cone $\mathfrak{c}_s$, and let ρ_s be a patchwork function on $\mathcal{L}_s$. Let $q \in \mathbb{N}$ such that $q\mathfrak{p}$ is a lattice polytope and fix $k \in \mathbb{N}$, $0 \leq k \leq q-1$. Let $h(x)$ be a homogeneous polynomial of total degree N.*

Then, for $m \geq d+N-r$, the zero-degree term $E_m^{(k)}(\mathcal{L}, \rho, \mathfrak{p}, h)_{[0]}(\xi)$ defined by (12) is regular near $\xi = 0$, hence constant. Its value is the coefficient $E_m^{(k)}(\mathfrak{p}, h)$ of u^m in the Ehrhart quasi-polynomial

$$S((k+qu)\mathfrak{p}, h)(0) = \sum_{x \in ((k+qu)\mathfrak{p}) \cap \Lambda} h(x) = \sum_{m=0}^{d+N} u^m E_m^{(k)}(\mathfrak{p}, h).$$

Proof. We first consider the case $h(x) = 1$. We have, for every $m \geq 0$,

$$E_m^{(k)}(\mathfrak{p}, 1) = \frac{q^m}{m!} \sum_{s \in \mathcal{V}(\mathfrak{p})} \langle \xi, s \rangle^m \, S(ks + \mathfrak{c}_s)_{[-m]}(\xi)$$

where the right-hand side is actually a constant function of ξ. For $m > d - r - 1$, we have, by Corollary 18,

$$S(ks + \mathfrak{c}_s)(\xi)_{[-m]} = S^{\mathcal{L}_s, \rho_s}(ks + \mathfrak{c}_s)_{[-m]}(\xi).$$

This proves the theorem when $h(x) = 1$. The case of a non constant polynomial $h(x)$ is quite similar. If $h(x) = x_1^{N_1} \dots x_d^{N_d}$, we just have to replace the meromorphic functions $S(ks + \mathfrak{c}_s)(\xi)$ and $S^{\mathcal{L}_s, \rho_s}(ks + \mathfrak{c}_s)(\xi)$ by their derivatives under $\partial_{\xi_1}^{N_1} \dots \partial_{\xi_d}^{N_d}$. $\square$

If for each vertex s, we take $\mathcal{L}_s = \mathcal{L}$, the full collection generated by all r codimensional faces of $\mathfrak{p}$, we obtain Corollary 21 below, that is Barvinok's theorem [2], with an extension to the sum of values of any polynomial $h(x)$ over the set of integral points of a rational polytope (Barvinok considers only the case $h(x) = 1$).

Corollary 21 *Let* $\mathfrak{p} \subset V$ *be a rational polytope and let* $h(x)$ *be a polynomial function on* V*. Let* $\mathcal{L}$ *be a finite family of rational vector subspaces of* V*, stable under sum. Assume that* $\mathrm{lin}(\mathfrak{f})$ *belongs to* $\mathcal{L}$ *for every face* $\mathfrak{f}$ *of codimension* r *of* $\mathfrak{p}$*. Let* ρ *be a patchwork function on* $\mathcal{L}$ *and let* $S^{\mathcal{L},\rho} = \sum_{L\in\mathcal{L}} \rho(L) S^L$*. Then the* $r+1$ *highest Ehrhart coefficients of* $S(t\mathfrak{p}, h)(0)$ *and* $S^{\mathcal{L},\rho}(t\mathfrak{p}, h)(0)$ *are equal.*

The point in taking a family $\mathcal{L}_s$ which depends on the vertex s lies in the case where $\mathfrak{p}$ is *simplicial*. In this case, we can take $\mathcal{L}_s$ to be just the set of subspaces $\mathrm{lin}(\mathfrak{f})$, for all faces $\mathfrak{f}$ of codimension $\leq r$ of the supporting cone $\mathfrak{c}_s$ at vertex s. This family is stable under sum. Moreover the patchwork function on $\mathcal{L}_s$ is simple, (Lemma 15) and the computation of the function $S^L(ks + \mathfrak{c}_s)(\xi)$, when $L \in \mathcal{L}_s$, is immediately reduced (Example 6) to the computation of a function $S(\mathfrak{a})(\xi)$ for a simplicial cone $\mathfrak{a}$ in a rational vector space of dimension **smaller or equal** than r. When $\mathfrak{p}$ is a simplex, we obtain in this way a method for computing the $r+1$ highest Ehrhart coefficients for the pair $(\mathfrak{p}, h)$.

5 Local Euler-Maclaurin formula for mixed sums

Finally in this last section, we discuss an application of the existence of the coefficients μ^L (Theorem 8) in the line of [4].

Let $\mathfrak{p}$ be a rational polytope in a rational vector space V of dimension d and let $h(x)$ be a polynomial function on V. Let L be a rational subspace of V. Consider the mixed sum

$$S^L(\mathfrak{p}, h) = \sum_{y\in\Lambda_{V/L}} \int_{\mathfrak{p}\cap(y+L)} h(x)\,dm_L(x).$$

As in [4], we associate to the analytic function $\mu^L(\mathfrak{t}(\mathfrak{p},\mathfrak{f}))$ a constant coefficients differential operator (of infinite order) on V.

Definition 22 *Let* $\mathfrak{f}$ *be a face of* $\mathfrak{p}$*. We denote by* $D^L(\mathfrak{p},\mathfrak{f})$ *the differential operator on* V *associated to analytic function* $\mu^L(\mathfrak{t}(\mathfrak{p},\mathfrak{f}))$*:*

$$D^L(\mathfrak{p},\mathfrak{f})(\partial_\xi)\cdot e^{\langle \xi, x\rangle} = \mu^L(\mathfrak{t}(\mathfrak{p},\mathfrak{f})))(\xi) e^{\langle \xi, x\rangle}.$$

The operators $D^L(\mathfrak{p},\mathfrak{f})$ are **local**, that is they depend only of the transverse cone $\mathfrak{t}(\mathfrak{p},\mathfrak{f})$ of $\mathfrak{p}$ along $\mathfrak{f}$, and they involve only derivatives in directions orthogonal to the face $\mathfrak{f}$. We can state the following theorem with the same proof as in [4].

Theorem 23 *(Local Euler-Maclaurin formula) Let $\mathfrak{p}$ be a polytope in V. For any polynomial function $h(x)$ on V, we have*

$$S^L(\mathfrak{p},h) = \sum_{\mathfrak{f}\in\mathcal{F}(\mathfrak{p})} \int_{\mathfrak{f}} D^L(\mathfrak{p},\mathfrak{f})\cdot h \tag{13}$$

where the integral on the face $\mathfrak{f}$ is taken with respect to the Lebesgue measure on $<\mathfrak{f}>$ defined by the lattice $\Lambda\cap \mathrm{lin}(\mathfrak{f})$.

In particular, for $h=1$, we obtain

$$S^L(\mathfrak{p},1) = \sum_{\mathfrak{f}\in\mathcal{F}(\mathfrak{p})} \mu^L(\mathfrak{t}(\mathfrak{p},\mathfrak{f}))(0)\,\mathrm{vol}(\mathfrak{f}). \tag{14}$$

Let us dilate the polytope $\mathfrak{p}$ by a non negative integer n. If $\mathfrak{f}$ is a face of $\mathfrak{p}$, let $q_{\mathfrak{f}}$ be the smallest positive integer such that $q_{\mathfrak{f}}<\mathfrak{f}>$ contains integral points. Define $D(\mathfrak{p},\mathfrak{f},n)=D(n\mathfrak{p},n\mathfrak{f})$, if $n>0$, and $D(\mathfrak{p},\mathfrak{f},0)=D(q_{\mathfrak{f}}\mathfrak{p},q_{\mathfrak{f}}\mathfrak{f})$. The function $n\mapsto D(\mathfrak{p},\mathfrak{f},n)$ is periodic of period $q_{\mathfrak{f}}$.

Proposition 24 *Let $\mathfrak{p}$ be a rational polytope and h a polynomial function of degree N on V. Then, for any integer $n\geq 0$, we have*

$$S^L(n\mathfrak{p},h) = \sum_{\mathfrak{f}\in\mathcal{F}(\mathfrak{p})} \int_{n\mathfrak{f}} D^L(\mathfrak{p},\mathfrak{f},n)\cdot h. \tag{15}$$

Furthermore, if $\mathfrak{f}\in\mathcal{F}(\mathfrak{p})$, we have

$$\int_{n\mathfrak{f}} D^L(\mathfrak{p},\mathfrak{f},n)\cdot h = \sum_{i=\dim\mathfrak{f}}^{\dim\mathfrak{f}+N} E_i(\mathfrak{p},h,\mathfrak{f},n)\,n^i$$

where the coefficients $E_i(\mathfrak{p},h,\mathfrak{f},n)$ are periodic with period $q_{\mathfrak{f}}$.

Hence the Ehrhart coefficients are given by

$$E^L_m(\mathfrak{p},h,n) = \sum_{\mathfrak{f},\dim\mathfrak{f}\leq m} E_m(\mathfrak{p},h,\mathfrak{f},n).$$

When we apply the last proposition to the function $h(x) = 1$, we obtain

$$S^L(n\mathfrak{p}, 1) = \sum_{\mathfrak{f}\in\mathcal{F}(\mathfrak{p})} \mu^L(n\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(0)\,\mathrm{vol}(\mathfrak{f})n^{\dim \mathfrak{f}}. \tag{16}$$

As $\mu^L(n\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))$ is invariant by integral translations, the function $\mu^L(n\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(0)$ is of period $q_{\mathfrak{f}}$.

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Velleda Baldoni, Universita di Roma Tor Vergata, Dipartimento di Matematica, via della Ricerca Scientifica, 00133 Roma, Italy.

email: baldoni@mat.uniroma2.it

Nicole Berline, Ecole Polytechnique, Centre de mathématiques Laurent Schwartz, 91128, Palaiseau, France.

email: berline@math.polytechnique.fr

Michèle Vergne, Institut de Mathématiques de Jussieu, Théorie des Groupes, Case 7012, 2 Place Jussieu, 75251 Paris Cedex 05, France;

Ecole Polytechnique, Centre de mathématiques Laurent Schwartz, 91128, Palaiseau, France.

email: vergne@math.polytechnique.fr