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X-rays characterizing some classes of discrete sets

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Abstract

In this paper, we study the problem of determining discrete sets by means of their X-rays. An X-ray of a discrete set F in a direction u counts the number of points in F on each line parallel to u . A class \mathcal{F} of discrete sets is characterized by the set U of directions if each element in \mathcal{F} is determined by its X-rays in the directions of U . By using the concept of switching component introduced by Chang and Ryser [Comm. ACM 14 (1971) 21; Combinatorial Mathematics, The Carus Mathematical Monographs, No. 14, The Mathematical Association of America, Rahway, 1963] and extended in [Discrete Comput. Geom. 5 (1990) 223], we prove that there are some classes of discrete sets that satisfy some connectivity and convexity conditions and that cannot be characterized by any set of directions. Gardner and Gritzmann [Trans. Amer. Math. Soc. 349 (1997) 2271] show that any set U of four directions having cross ratio that does not belong to $\{4/3, 3/2, 2, 3, 4\}$, characterizes the class of convex sets. We prove the converse, that is, if U 's cross ratio is in $\{4/3, 3/2, 2, 3, 4\}$, then the hv-convex sets cannot be characterized by U . We show that if the horizontal and vertical directions do not belong to U , Gardner and Gritzmann's result cannot be extended to hv-convex polyominoes. If the horizontal and vertical directions belong to U and U 's cross ratio is not in $\{4/3, 3/2, 2, 3, 4\}$, we believe that U characterizes the class of hv-convex polyominoes. We give experimental evidence to support our conjecture. Moreover, we prove that there is no number δ such that, if $|U| \geq \delta$, then U characterizes the hv-convex polyominoes. This number

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exists for convex sets and is equal to 7 (see [Trans. Amer. Math. Soc. 349 (1997) 2271]).
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1. Introduction

A *discrete set* is a finite subset of the integer lattice \mathbb{Z}^2 . An *X-ray* of a discrete set in a direction u is a function giving the number of its points on each line parallel to u . The problem of reconstructing a discrete set from its X-rays along a given set of directions is motivated by some interesting applications in image processing [21], statistical data security [15], biplane angiography [19] and graph theory [1]. Moreover, the problem is of primary importance in reconstructing three-dimensional crystals from two-dimensional projections taken by an electron microscope [9,13,16,22]. Several authors [2,6,17,20] have been studying this theory and have proposed various algorithms for determining a discrete set starting out from its X-rays in horizontal and vertical directions. One of the main difficulties involved in this reconstruction is the “ambiguity” deriving from the fact that, in some cases, many different discrete sets have the same X-rays. In [10], Fishburn et al. gave several characterizations of the finite subsets of \mathbb{Z}^n that are uniquely determined by their X-ray in the coordinate directions. The characterization shows that this class of sets is small with respect to the class of all discrete sets, and so in an effort to reduce this ambiguity and facilitate the reconstruction, many authors suggest the following two methods:

1. more than two X-rays are assigned (algorithms using multiple projections are described in [9,21]);
2. some of the properties of the discrete set to be reconstructed are given “a priori” (for example: convexity, connectivity, symmetry) and the algorithms take advantage of this further information to reconstruct the set (see [2,5,7,17]).

In this paper, we study the ambiguity problem with respect to some classes of discrete sets on which some connectivity constraints are imposed. In particular, given a class \mathcal{F} of discrete sets, we want to know if a set U of directions exists such that among all the elements of \mathcal{F} , each element in \mathcal{F} is determined by its X-rays in the directions of U . If the set U exists, we say that the class \mathcal{F} is characterized by U . By extending the concept of switching component introduced by Chang and Ryser [6,11,20], we prove that there are some classes of discrete sets that cannot be characterized by any set of directions. One of these classes is the set of vertically convex polyominoes (i.e., discrete sets which are convex with respect to the vertical direction). Gardner and Gritzmann [12] studied the problem for the class of convex sets (i.e., sets which are convex with respect to all the directions). They show that if U is a set of four directions having cross ratio $\rho(U) \notin \{4/3, 3/2, 2, 3, 4\}$, then the class of convex sets is characterized by U . We prove that if the cross ratio

$\rho(U) \in \{4/3, 3/2, 2, 3, 4\}$, then the convex sets cannot be characterized by U . We then try to find out whether these results can be extended to the class of horizontally and vertically convex polyominoes (i.e., polyominoes which are only convex with respect to the horizontal and vertical directions). We call these discrete sets *hv-convex polyominoes* for brevity's sake. We prove that if the horizontal and vertical directions do not belong to U , Gardner and Gritzmann's result cannot be extended to hv-convex polyominoes. If $U = \{(1, 0), (0, 1), u_3, u_4\}$, where $(1, 0)$ and $(0, 1)$ are the horizontal and vertical directions and the cross ratio $\rho(U) \notin \{4/3, 3/2, 2, 3, 4\}$, we believe that U can characterize the class of hv-convex polyominoes. We wish to point out that, as shown in [8], there is an exponential number of hv-convex polyominoes having the same horizontal and vertical X-rays. In order to give experimental evidence for this conjecture, we use an algorithm that reconstructs hv-convex polyominoes from their discrete X-rays. The algorithm is illustrated in Section 5. Finally, we show that no number δ exists such that if $|U| \geq \delta$, then U characterizes the hv-convex polyominoes. This number exists for convex sets and is equal to 7 (see [12]).

2. Definitions and notations

A *discrete set* can be represented by a binary matrix or a set of *cells* (unit squares whose centers are the points in the set itself). This correspondence is illustrated in Fig. 1. Let \mathcal{F} be a class of discrete sets. As usual, \mathbf{E}^2 denotes the Euclidean plane. A *direction* is a vector of \mathbf{E}^2 . If u is a direction, we denote the line through the origin parallel to u by l_u . A *discrete direction* is a direction $u = (u_x, u_y)$ such that u_y/u_x is a rational number. If F is an element of \mathcal{F} , and u a discrete direction, then the *discrete X-ray of F in the direction u* is the function $X_u F$ defined as

$$X_u F(x) = |F \cap (x + l_u)|$$

for $x = (n, 0)$ or $x = (0, n)$ with $n \in \mathbb{Z}$ (see Fig. 2). We will also refer to $X_u F$ as to the *projection of F along the direction u* .

Let U be a finite set of discrete directions. We say that $F \in \mathcal{F}$ is *determined* by the discrete X-rays in the directions of U if whenever $F' \in \mathcal{F}$ and $X_u F = X_u F'$ for all $u \in U$, we have $F = F'$. Otherwise, F is *ambiguous* with respect to U in \mathcal{F} . We say that the class \mathcal{F} is *characterized* by the set U of directions if each set in \mathcal{F} is determined by the discrete X-rays in the directions of U .

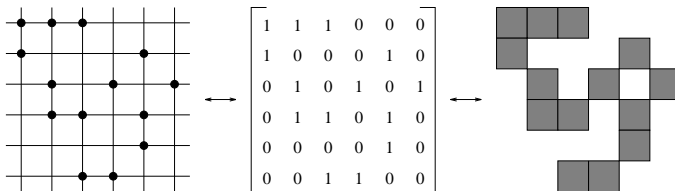


Fig. 1. A discrete set with the corresponding binary matrix and set of cells.

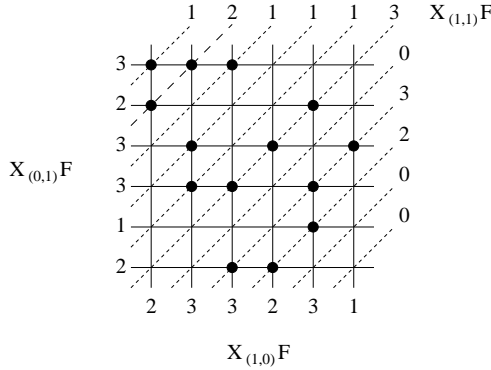


Fig. 2. The discrete X-rays of a discrete set in the directions parallel to $(1, 1)$, $(0, 1)$ and $(1, 0)$.

We can now define the following problem:

Determination of discrete sets by X-rays

Instance: A class \mathcal{F} of discrete sets.

Question: Is there a finite set U of discrete directions characterizing \mathcal{F} ?

We approach this problem by defining some classes by means of the correspondence between the discrete sets and sets of cells, both of unrestricted cardinality. A *polyomino* F is a discrete set in which for any pair of cells of F there exists a lattice path in F connecting them (see Fig. 3). A lattice path is a path made up of horizontal and vertical unit steps. Let F be a discrete set. A *column* (*row*) of F is the set of cells whose center (x, y) satisfies $x = n$ ($y = n$), $n \in \mathbb{Z}$.

Notations. We say that a set F satisfies the properties **p**, **v** and **h** if:

- p:** F is a polyomino;
- h:** every row of F is a connected set;
- v:** every column of F is a connected set.

Moreover, a polyomino satisfies property **d** if it is obtained by starting out from a cell called a *source* and by adding the other cells in two predetermined directions, such as East and North, i.e., to the right of, or over, the existing cells.

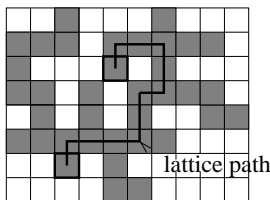


Fig. 3. A polyomino.

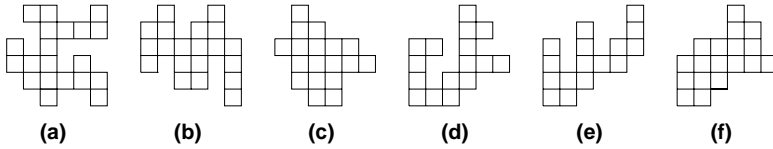


Fig. 4. A polyomino (a) and vertically convex (b), hv-convex (c), directed (d), directed vertically convex (e) and directed hv-convex polyomino (f).

A set F belongs to the class (\mathbf{x}) ($F \in (\mathbf{x})$) if and only if it satisfies the property \mathbf{x} . We denote the whole set of discrete sets by (\emptyset) .

We now introduce the following definitions:

- F is a vertically convex (respectively, horizontally convex) polyomino if $F \in (\mathbf{p}, \mathbf{v})$ (respectively, $F \in (\mathbf{p}, \mathbf{h})$);
- F is a directed polyomino if $F \in (\mathbf{p}, \mathbf{d})$;
- F is a hv-convex polyomino if $F \in (\mathbf{p}, \mathbf{h}, \mathbf{v})$;
- F is a directed vertically convex (respectively, horizontally convex) polyomino if $F \in (\mathbf{p}, \mathbf{d}, \mathbf{v})$ (respectively, $F \in (\mathbf{p}, \mathbf{d}, \mathbf{h})$);
- F is a directed hv-convex polyomino if $F \in (\mathbf{p}, \mathbf{d}, \mathbf{h}, \mathbf{v})$.

Such polyominoes are illustrated in Fig. 4.

3. State of the art

Some classes of discrete sets can be determined by their X-rays. In [7], it is shown that:

Theorem 3.1. *The pair of horizontal and vertical directions (i.e., $U = \{(1, 0), (0, 1)\}$) characterizes the following classes: $(\mathbf{p}, \mathbf{d}, \mathbf{h})$, $(\mathbf{p}, \mathbf{d}, \mathbf{v})$, $(\mathbf{p}, \mathbf{d}, \mathbf{h}, \mathbf{v})$.*

Proof. In these cases the horizontal (vertical) X-rays determine the size of the rows (columns) and the vertical (horizontal) ones indicate how they must be connected. □

Many authors have studied the problem of determining discrete sets by their X-rays using the horizontal and vertical directions only. Chang [6] and Ryser [20] introduced the idea of a *switching component* (Ryser calls it an *interchange*). A discrete set F has a switching component if, in the corresponding binary matrix, there is a four-element configuration arranged as shown in Fig. 5(a), where $a = \bar{b}$.

For example, in Fig. 5(b) a binary matrix containing the switching component of Fig. 5(a) is shown. By complementing the elements of a switching component (Fig. 5(c)), we obtain a different matrix that has the same X-rays in the horizontal and vertical directions. Therefore, if a discrete set F has a switching component, then F is ambiguous with respect to $\{(1, 0), (0, 1)\}$.

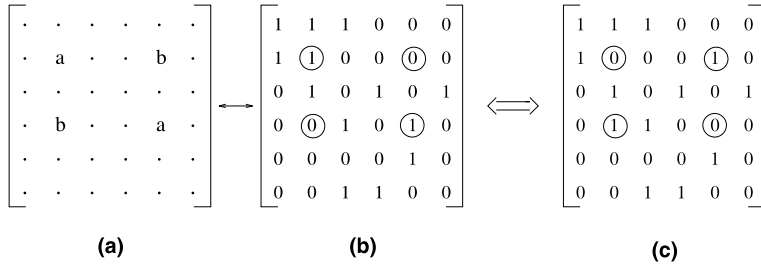


Fig. 5. A switching component along $\{(1, 0), (0, 1)\}$.

Let us now generalize the concept of switching component to a set of discrete directions as in [4].

Definition 3.2. A discrete set F has a *switching component* along a set U of discrete directions if there is a set of elements S in the corresponding binary matrix such that:

- $S = S_a \cup S_b, S_a \cap S_b = \emptyset$ and $|S_a| = |S_b|$;
- if $a \in S_a$ and $b \in S_b$, then $a = \bar{b}$ (i.e., if $b = 0$, then $a = 1$ and vice versa);
- for each $u \in U, X_u S_a(x) = X_u S_b(x)$ (i.e., each line parallel to one direction of U contains the same number of elements in S_a and S_b).

We call S a switching component along U .

In Fig. 6, for instance, there is a discrete set having a switching component along a set of four directions parallel to the vectors $(1, 0), (0, 1), (1, 1), (-1, 2)$.

From the definition of a switching component, we can deduce that, if F has a switching component S along U , then by complementing the elements of S , we obtain another matrix such that the corresponding discrete set has the same X-rays as F in the directions of U (see Fig. 6). We denote this discrete set by $F_{\bar{S}}$. Furthermore, we call the switching component \bar{S} obtained by complementing S 's elements the *dual* of S .

Property 3.3. Let (\mathbf{x}) be a class of discrete sets. A discrete set $F \in (\mathbf{x})$ is ambiguous with respect to a set of discrete directions U in (\mathbf{x}) if and only if F has a switching component S along U such that $F_{\bar{S}} \in (\mathbf{x})$.

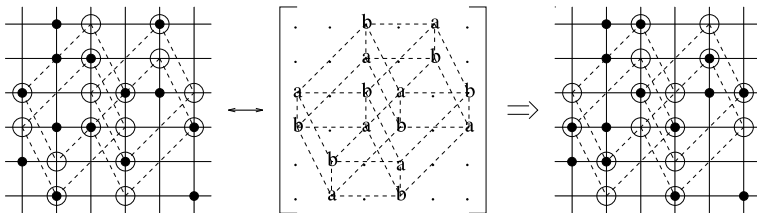


Fig. 6. A switching component along a set of four directions parallel to the vectors $(1, 0), (0, 1), (1, 1), (-1, 2)$.

Proof. If F has a switching component S along U such that $F_{\bar{S}} \in (\mathbf{x})$, then F and $F_{\bar{S}}$ have the same X-rays in the directions of U and F is ambiguous with respect to U in (\mathbf{x}) . Vice versa, if F is ambiguous with respect to U in (\mathbf{x}) , then there is a discrete set $F' \in (\mathbf{x})$ having the same X-rays of F in U 's directions. Let $F_M = (f_{i,j})$ and $F'_M = (f'_{i,j})$ be the corresponding binary matrices of F and F' . It is easy to prove that the set of F_M 's elements such that $f_{i,j} \neq f'_{i,j}$ is a switching component along U . Moreover, $F' = F_{\bar{S}} \in (\mathbf{x})$. \square

Therefore, if we can define a switching component S along a set of discrete directions U such that S is a switching component of a discrete set $F \in (\mathbf{x})$ and $F_{\bar{S}} \in (\mathbf{x})$, then the class (\mathbf{x}) cannot be characterized by U .

Let us now introduce a simple method for constructing some switching components along an assigned set of directions. It is easy to prove that:

Property 3.4. *Let S' be a switching component along a set of discrete directions $\{u_1, u_2, \dots, u_{k-1}\}$. If S'' is obtained by means of a translation of S' along a direction u_k and $S' \cap S'' = \emptyset$, then $S = S' \cup S''$ is a switching component along $\{u_1, u_2, \dots, u_{k-1}, u_k\}$.*

Consequently, we can construct a switching component along $U = \{u_1, u_2, \dots, u_k\}$ in the recursive way shown in Fig. 7.

Since we can construct a switching component along every U , the class (\emptyset) cannot be characterized by any set of discrete directions (this result was firstly proved by Lorentz [18]).

Remark 3.5. There are some switching components along a set of directions U that cannot be obtained by means of this recursive construction. For example, the switching component along $\{(1, 0), (0, 1)\}$ in Fig. 8 cannot be constructed by this method.

The elements of a switching component S are isolated if for each $x \in S$ the eight elements around x do not belong to S . We can easily define a switching component S along a set of directions U having isolated elements by means of the previous

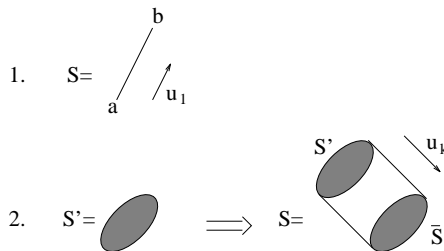


Fig. 7. A recursive construction of a switching component along $U = \{u_1, u_2, \dots, u_k\}$.

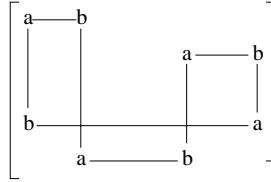


Fig. 8. A switching component along $U = \{(1, 0), (0, 1)\}$.

recursive method: we only have to translate \bar{S}' far enough at any step. However, if S 's elements are isolated, then there is a discrete set $F \in (\mathbf{p}, \mathbf{d})$ such that S is a switching component of F and $F_{\bar{S}} \in (\mathbf{p}, \mathbf{d})$ (see Fig. 9). F can be obtained by adding some elements to S in such a way that any element in S is connected to the source by a directed path. Therefore, since $(\mathbf{p}, \mathbf{d}) \subset (\mathbf{p})$ the classes (\mathbf{p}) and (\mathbf{p}, \mathbf{d}) cannot be characterized by any such set U .

Proposition 3.6. *Let $U = \{u_1, u_2, \dots, u_k\}$ be a set of discrete directions. There is a switching component S along U such that every row and column of S contains at most two elements of S : one in S_a and one in S_b .*

Proof. If the horizontal and vertical directions belong to U , we arrange set U so that these directions are u_1 and u_2 . We construct the parallelogram switching component along $\{u_1, u_2\}$ (shown in Fig. 10(a)). This switching component satisfies the conditions of Proposition 3.6. We then use the previous recursive method to construct a switching component along U so that no row or column of S'' contains any of S 's elements (see Fig. 10(b)). If we perform this construction for each step, then we obtain a switching component S along U having two elements at most in every row and column: one in S_a and one in S_b . \square

If $U = \{u_1, u_2, \dots, u_k\}$, then S is made up of 2^{k-2} parallelogram switching components along $\{u_1, u_2\}$ (see Fig. 11(a)). We use S for the class (\mathbf{h}, \mathbf{v}) . Since all the rows and columns of the discrete sets in this class are connected, the connected components in $F \in (\mathbf{h}, \mathbf{v})$ are hv-convex polyominoes. Moreover, every row and

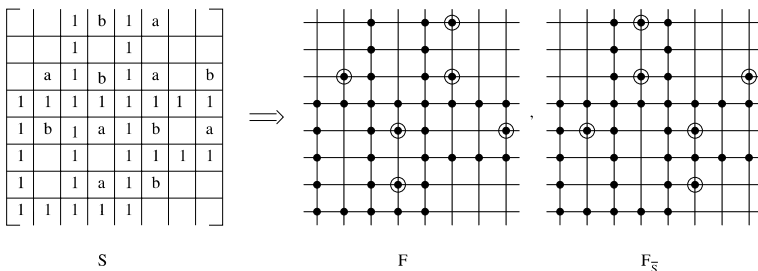


Fig. 9. A switching component having isolated elements and a directed polyomino containing this switching component.

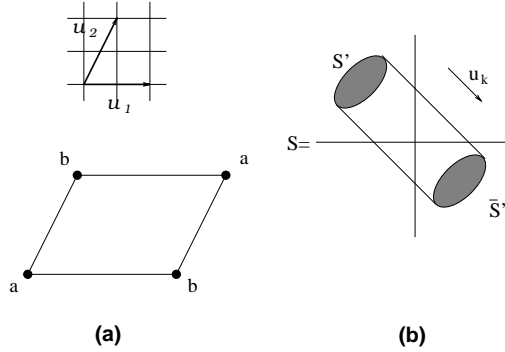


Fig. 10. A recursive construction of a switching component having at most two elements in every row and column.

column in F can belong to no more than one of these hv-convex polyominoes (see Fig. 11(b)). We can construct an hv-convex polyomino inside each of S 's parallelogram switching components. As a consequence, we have $F \in (\mathbf{h}, \mathbf{v})$ such that S is a switching component of F and $F_{\bar{S}} \in (\mathbf{h}, \mathbf{v})$. Since $(\mathbf{h}, \mathbf{v}) \subset (\mathbf{h})$ and $(\mathbf{h}, \mathbf{v}) \subset (\mathbf{v})$ it follows that the classes (\mathbf{h}) , (\mathbf{v}) and (\mathbf{h}, \mathbf{v}) cannot be characterized by any set U .

Finally, we consider the classes in (\mathbf{p}, \mathbf{h}) and (\mathbf{p}, \mathbf{v}) . By means of the recursive method, we can define a switching component S along U such that:

- every row and column of S contains at most two elements;
- there is at least one empty column between any pair of nonempty columns of S .

There is an $F \in (\mathbf{p}, \mathbf{v})$ such that S is a switching component of F and $F_{\bar{S}} \in (\mathbf{p}, \mathbf{v})$. For instance, we could proceed as we did for (\mathbf{h}, \mathbf{v}) and then add some columns in such a way that the union of these columns and the connected components is a polyomino. We proceed in an analogous way for (\mathbf{p}, \mathbf{h}) . Consequently, the classes (\mathbf{p}, \mathbf{h}) and (\mathbf{p}, \mathbf{v}) cannot be characterized by any set U .

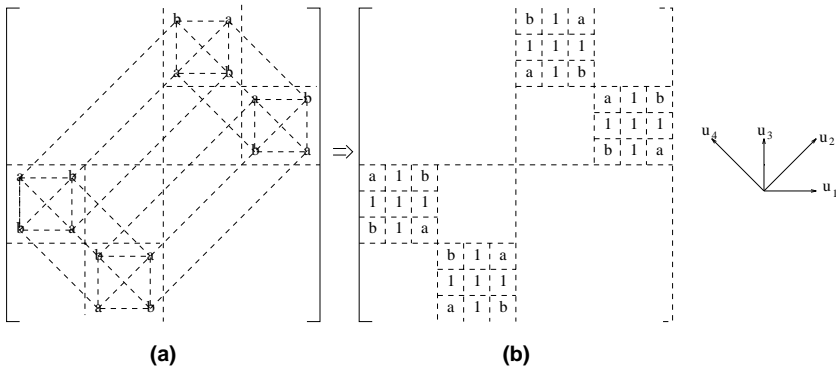


Fig. 11. A switching component along $U = \{u_1, u_2, u_3, u_4\}$ having at most two elements in every row and column and a discrete set in (\mathbf{h}, \mathbf{v}) containing this switching component.

As a results of the above discussion we obtain:

Theorem 3.7. *No finite set of directions characterizes the following classes: (\emptyset) , (\mathbf{p}) , (\mathbf{v}) , (\mathbf{h}) , (\mathbf{p}, \mathbf{v}) , (\mathbf{p}, \mathbf{h}) , (\mathbf{p}, \mathbf{d}) , (\mathbf{h}, \mathbf{v}) .*

4. The class of hv-convex polyominoes

Gardner and Gritzmann [12] studied the problem for the class of *convex sets*. A convex set can be considered to be the intersection between a planar convex body and the lattice \mathbb{Z}^2 . Let $U = \{u_1, u_2, u_3, u_4\}$ be made up of four discrete directions. The *cross ratio* of U is

$$\rho(U) = \frac{(h_3 - h_1)(h_4 - h_2)}{(h_3 - h_2)(h_4 - h_1)},$$

where $h_i = u_{iy}/u_{ix}$ for $i = 1, 2, 3, 4$ and $0 \leq h_1 < h_2 < h_3 < h_4$. If $u_i = (0, 1)$, then we only take the terms not containing h_i , that is, $\rho(U) = (h_j - h_k)/(h_j - h_l)$, with j, k, l not equal to i . The authors obtained the following results.

Theorem 4.1. *The class (c) of convex sets is such that:*

1. *If U is a set of four discrete directions having cross ratio $\rho(U) \notin \{4/3, 3/2, 2, 3, 4\}$, then (c) is characterized by U .*
2. *(c) is characterized by any set of seven mutually nonparallel directions.*
3. *There is a set of six discrete directions not characterizing (c).*
4. *(c) cannot be characterized by any set of three discrete directions.*

We note that a convex set may not be a polyomino. The class of convex sets which are also polyominoes is a subclass of hv-convex polyominoes (i.e., $(\mathbf{p}, \mathbf{c}) \subset (\mathbf{p}, \mathbf{h}, \mathbf{v})$).

Let us now take the class $(\mathbf{p}, \mathbf{h}, \mathbf{v})$ into consideration.

The directions which determine the convex sets cannot characterize the class of hv-convex polyominoes and we prove that result (2) of Theorem 4.1 cannot be extended to hv-convex polyominoes.

We report a lemma proved in [12] since its proof constitutes the first step towards the proof of Proposition 4.5. A switching component along three discrete directions whose elements are made up of a convex hexagon is called a *hexagonal switching component*.

Lemma 4.2. *Let u_1, u_2, u_3 be three discrete directions. There is a hexagonal switching component along $U = \{u_1, u_2, u_3\}$.*

Proof. We build a parallelogram switching component having the sides parallel to u_1 and u_2 and a diagonal parallel to u_3 (see Fig. 12(a)). Such a parallelogram

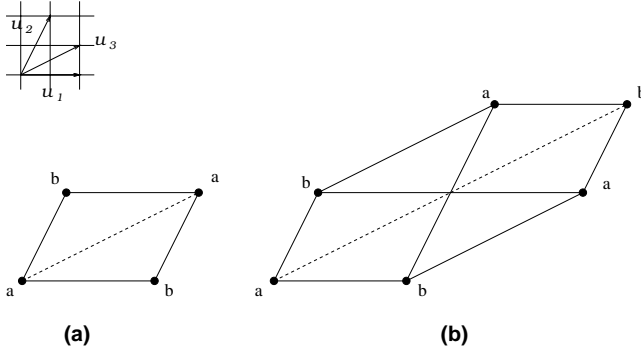


Fig. 12. A hexagonal switching component switching component along $\{u_1, u_2, u_3\}$.

switching component whose vertices are lattice points always exists if u_1, u_2 and u_3 are discrete directions (see also [12]). Then we connect two copies of this parallelogram in such a way that the diagonals parallel to u_3 lie on the same line and have a common vertex (see Fig. 12(b)). As a result we obtain a hexagonal switching component along $U = \{u_1, u_2, u_3\}$. \square

If we can define a switching component S along U such that S is a convex polygon, we can construct a hv-convex polyomino inside S so that the corresponding discrete set $F \in (\mathbf{p}, \mathbf{h}, \mathbf{v})$, S is a switching component of F and $F_{\bar{S}} \in (\mathbf{p}, \mathbf{h}, \mathbf{v})$. As a consequence, class $(\mathbf{p}, \mathbf{h}, \mathbf{v})$ cannot be characterized by any set $\{u_1, u_2, u_3\}$, and the result (4) of Theorem 4.1 holds for $(\mathbf{p}, \mathbf{h}, \mathbf{v})$.

Lemma 4.3. *The class $(\mathbf{p}, \mathbf{h}, \mathbf{v})$ cannot be characterized by a finite set of discrete directions U such that:*

- (i) $\{(1, 0), (0, 1)\} \not\subset U$;
- (ii) *all the directions of U belong to the same quadrant.*

Proof. We obtain the proof of Lemma 4.3 as in the previous cases. Let us now assume that U 's directions belong to the first quadrant. If the horizontal or vertical direction belongs to U , we denote this direction by u_1 . We define a switching component along U in the recursive way shown in Fig. 10. Since U 's directions belong to the first quadrant, we obtain a switching component S made up of 2^{k-2} parallelogram switching components along $\{u_1, u_2\}$ and these parallelograms can be connected by some paths only having north and east steps (see Fig. 13). We construct a hv-convex polyomino inside each of S 's parallelogram switching components. If $(0, 1) \notin U$, then we can connect these polyominoes with the paths using only the north and east directions so that the union of all the polyominoes, paths and S corresponds to a discrete set $F \in (\mathbf{p}, \mathbf{h}, \mathbf{v})$ and $F_{\bar{S}} \in (\mathbf{p}, \mathbf{h}, \mathbf{v})$. This construction is illustrated in Fig. 13. We proceed in the same way if $(0, 1) \in U$. Therefore, the class $(\mathbf{p}, \mathbf{h}, \mathbf{v})$ cannot be characterized by U . \square

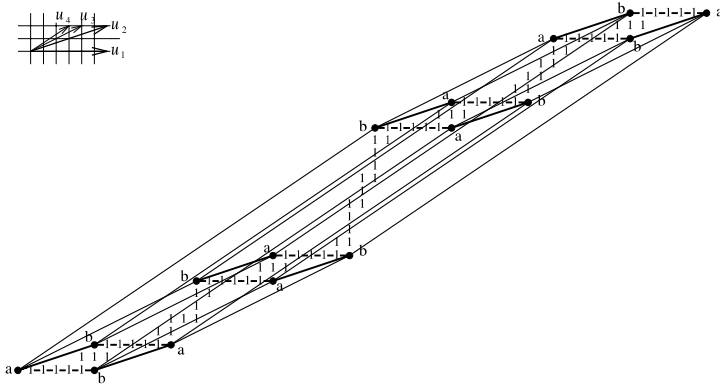


Fig. 13. A switching component along a set of discrete directions in the first quadrant.

Lemma 4.3 gives us:

Theorem 4.4. *No number δ exists such that if $|U| \geq \delta$, then U characterizes the class $(\mathbf{p}, \mathbf{h}, \mathbf{v})$.*

Moreover, the result (1) of Theorem 4.1 cannot be extended to hv-convex polyominoes. The set $U = \{(1, 0), (1, 1), (1, 3), (1, 5)\}$ characterizes the convex sets ($\rho(U) = 6/5$), but, from Lemma 4.3, we deduce that U cannot characterize hv-convex polyominoes (U 's directions belong to the first quadrant).

From the previous lemma, we deduce that the directions $\{(1, 0), (0, 1)\}$ are crucial for the characterization of hv-convex polyominoes. It is worth noting that hv-convex polyominoes are only convex along the horizontal and vertical directions. Therefore, let us take the case $\{(1, 0), (0, 1)\} \subset U$ into consideration. The question is now:

Is there a set of four discrete directions $U = \{(1, 0), (0, 1), u_3, u_4\}$ characterizing the class $(\mathbf{p}, \mathbf{h}, \mathbf{v})$?

Let $s = (s_x, s_y)$ be a vector parallel to u_3 and $s_x \in \mathbb{Z}$ and $s_y \in \mathbb{N}_0$. We denote $\bar{h} = (s_x, 0)$ and $\bar{v} = (0, s_y)$ (i.e., $u_3 = \bar{v} + \bar{h}$).

Proposition 4.5. *If the direction u_4 is parallel to one of the following linear combinations of \bar{v} and \bar{h} :*

$$\mathcal{A} = \{ \bar{v} - \bar{h}, 2\bar{v} + \bar{h}, \bar{v} + 2\bar{h}, 2\bar{v} - \bar{h}, \bar{v} - 2\bar{h}, 3\bar{v} + \bar{h}, \bar{v} + 3\bar{h}, 3\bar{v} + 2\bar{h}, \\ 2\bar{v} + 3\bar{h}, 4\bar{v} + 3\bar{h}, 3\bar{v} + 4\bar{h}, 4\bar{v} + \bar{h}, \bar{v} + 4\bar{h}, 3\bar{v} - \bar{h}, \bar{v} - 3\bar{h} \},$$

then the class $(\mathbf{p}, \mathbf{h}, \mathbf{v})$ cannot be characterized by $U = \{(1, 0), (0, 1), u_3, u_4\}$.

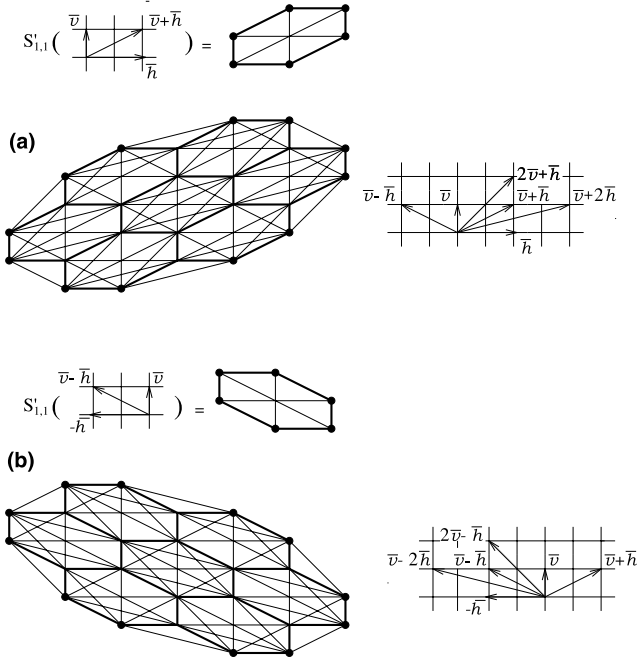


Fig. 14. Two dodecagonal switching components along $\{(1, 0), (0, 1), \bar{v} + \bar{h}, \bar{v} - \bar{h}, 2\bar{v} + \bar{h}, \bar{v} + 2\bar{h}\}$ and $\{(1, 0), (0, 1), \bar{v} + \bar{h}, \bar{v} - \bar{h}, 2\bar{v} - \bar{h}, \bar{v} - 2\bar{h}\}$.

The cross ratio of $U = \{(1, 0), (0, 1), u_3, u_4\}$, where u_4 is one of \mathcal{A} 's directions, is $\rho(U) \in \{4/3, 3/2, 2, 3, 4\}$. If we fix u_3 , there are 15 discrete directions u_4 such that $\rho(U) \in \{4/3, 3/2, 2, 3, 4\}$ and their union is exactly \mathcal{A} .

Let S be the set obtained by joining seven switching components $S'_{1,1}((1, 0), (0, 1), \bar{v} + \bar{h})$ as shown in Fig. 14(a). The set of points not belonging to any common edges of S (i.e., the boundary of S) is a convex dodecagon and a switching component along $\{(1, 0), (0, 1), \bar{v} + \bar{h}, \bar{v} - \bar{h}, 2\bar{v} + \bar{h}, \bar{v} + 2\bar{h}\}$. Therefore, the class $(\mathbf{p}, \mathbf{h}, \mathbf{v})$ cannot be characterized by $\{(1, 0), (0, 1), u_3, u_4\}$, with u_4 parallel to $\bar{v} - \bar{h}$ or $2\bar{v} + \bar{h}$ or $\bar{v} + 2\bar{h}$.

Let us now take $2\bar{v} - \bar{h}$ and $\bar{v} - 2\bar{h}$ into consideration. Let S be the set obtained by joining seven switching components $S'_{1,1}((1, 0), (0, 1), \bar{v} - \bar{h})$ as shown in Fig. 14(b). The set of points not belonging to any common edges of S is a convex dodecagon and a switching component along $\{(1, 0), (0, 1), \bar{v} + \bar{h}, \bar{v} - \bar{h}, 2\bar{v} - \bar{h}, \bar{v} - 2\bar{h}\}$. Therefore, the class $(\mathbf{p}, \mathbf{h}, \mathbf{v})$ cannot be characterized by $\{(1, 0), (0, 1), u_3, u_4\}$, with u_4 parallel to $2\bar{v} - \bar{h}$ or $\bar{v} - 2\bar{h}$.

By proceeding in the same way, we can construct six other convex dodecagons and switching components along

$$\{(1, 0), (0, 1), \bar{v} + \bar{h}, \bar{v} - \bar{h}, 3\bar{v} + \bar{h}, 3\bar{v} - \bar{h}\} \quad (\text{see Fig. 15(a)},$$

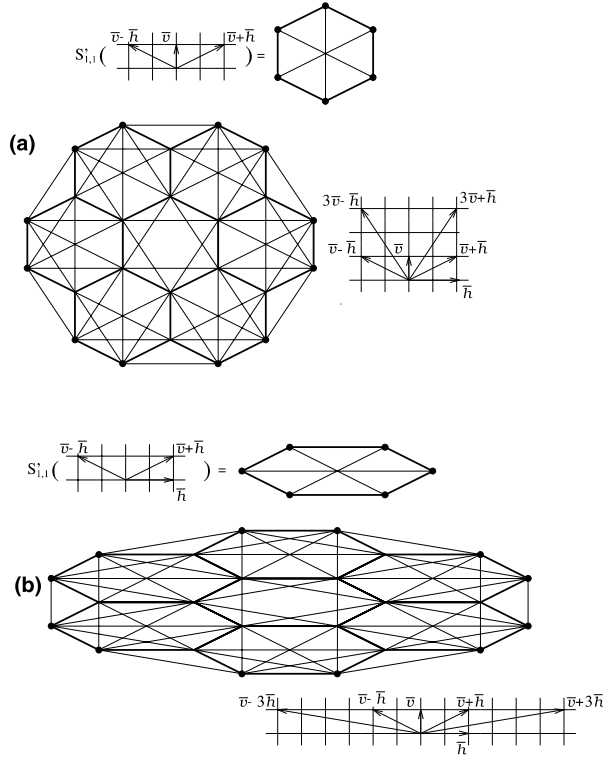


Fig. 15. Two dodecagonal switching components along $\{(1, 0), (0, 1), \bar{v} + \bar{h}, \bar{v} - \bar{h}, 3\bar{v} + \bar{h}, 3\bar{v} - \bar{h}\}$, and $\{(1, 0), (0, 1), \bar{v} + \bar{h}, \bar{v} - \bar{h}, \bar{v} + 3\bar{h}, \bar{v} - 3\bar{h}\}$.

- $\{(1, 0), (0, 1), \bar{v} + \bar{h}, \bar{v} - \bar{h}, \bar{v} + 3\bar{h}, \bar{v} - 3\bar{h}\}$ (see Fig. 15(b)),
- $\{(1, 0), (0, 1), \bar{v} + \bar{h}, \bar{v} + 2\bar{h}, \bar{v} + 4\bar{h}, \bar{v} - 2\bar{h}\}$ (see Fig. 16(a)),
- $\{(1, 0), (0, 1), \bar{v} + \bar{h}, 2\bar{v} + \bar{h}, 4\bar{v} + \bar{h}, 2\bar{v} - \bar{h}\}$ (see Fig. 16(b)),
- $\{(1, 0), (0, 1), \bar{v} + \bar{h}, \bar{v} + 2\bar{h}, 3\bar{v} + 2\bar{h}, 3\bar{v} + 4\bar{h}\}$ (see Fig. 17(a)),
- $\{(1, 0), (0, 1), \bar{v} + \bar{h}, 2\bar{v} + \bar{h}, 2\bar{v} + 3\bar{h}, 4\bar{v} + 3\bar{h}\}$ (see Fig. 17(b)).

We can now conclude that class $(\mathbf{p}, \mathbf{h}, \mathbf{v})$ cannot be characterized by $\{(1, 0), (0, 1), u_3, u_4\}$, with u_4 parallel to one of \mathcal{A} 's directions, and so Proposition 4.5 follows.

We now make the following conjecture:

Conjecture 4.6. The class of hv-convex polyominoes $(\mathbf{p}, \mathbf{h}, \mathbf{v})$ is characterized by a set of four discrete directions $U = \{(1, 0), (0, 1), u_3, u_4\}$ such that $\rho(U) \notin \{4/3, 3/2, 2, 3, 4\}$.

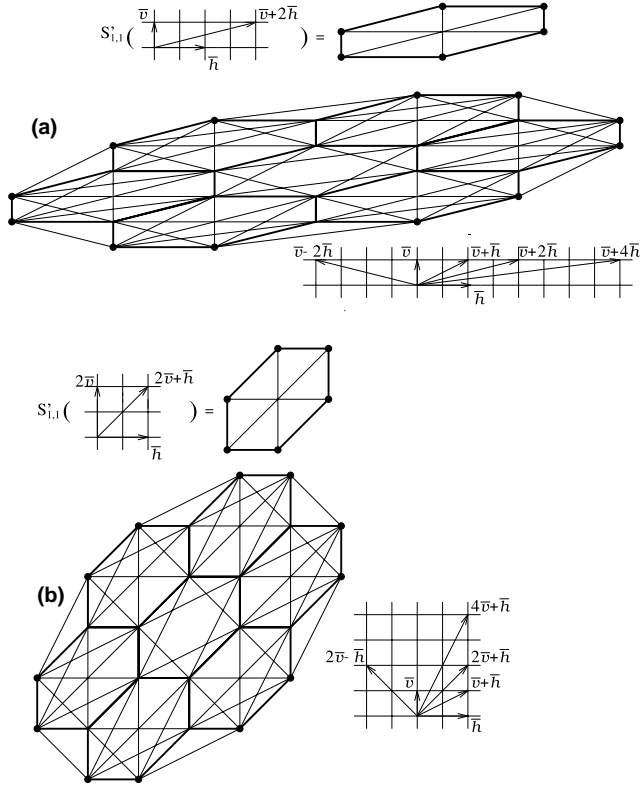


Fig. 16. Two dodecagonal switching components along $\{(1, 0), (0, 1), \bar{v} + \bar{h}, \bar{v} + 2\bar{h}, \bar{v} + 4\bar{h}, \bar{v} - 2\bar{h}\}$, and $\{(1, 0), (0, 1), \bar{v} + \bar{h}, 2\bar{v} + \bar{h}, 4\bar{v} + \bar{h}, 2\bar{v} - \bar{h}\}$.

We experimentally supported this conjecture by means of an algorithm that reconstructs hv-convex polyominoes from their discrete X-rays, as described in the previous section.

Gardner and Gritzmann [12] show that:

Theorem 4.7. *Let U be a finite set of two or more mutually nonparallel discrete directions. The following statements are equivalent:*

- *the class of convex sets is characterized by U ;*
- *there is no convex switching component along U .*

It is easy to extend the previous constructions to any set $\{u_1, u_2, u_3\}$. Therefore, if we fix u_1, u_2 and u_3 , there is a set \mathcal{A} of 15 discrete directions such that, if $u_4 \in \mathcal{A}$, then there is a dodecagonal switching component along $U = \{u_1, u_2, u_3, u_4\}$. Since these dodecagonal switching components are convex, from Theorem 4.7 we deduce that the class of convex sets cannot be characterized by $U = \{u_1, u_2, u_3, u_4\}$.

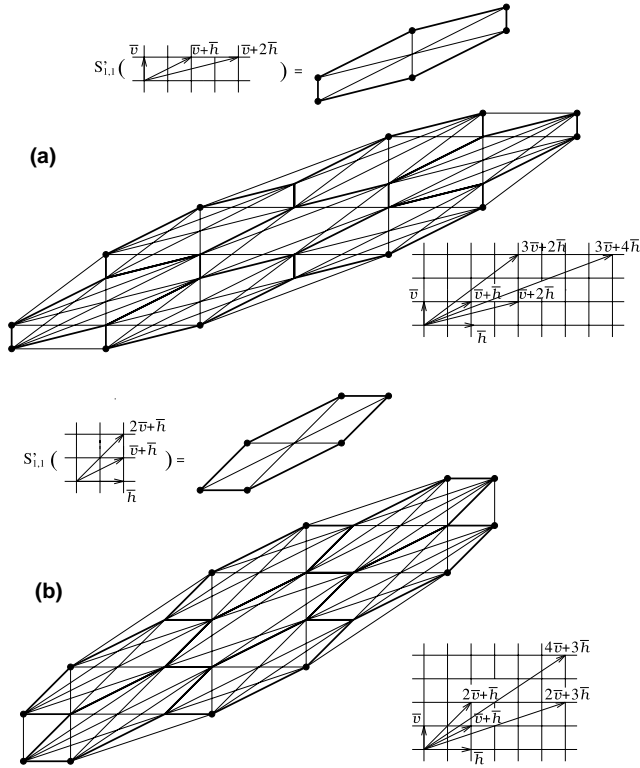


Fig. 17. Two dodecagonal switching components along $\{(1, 0), (0, 1), \bar{v} + \bar{h}, \bar{v} + 2\bar{h}, 3\bar{v} + 2\bar{h}, 3\bar{v} + 4\bar{h}\}$ and $\{(1, 0), (0, 1), \bar{v} + \bar{h}, 2\bar{v} + \bar{h}, 2\bar{v} + 3\bar{h}, 4\bar{v} + 3\bar{h}\}$.

Moreover, $\rho(U) \in \{4/3, 3/2, 2, 3, 4\}$. If we fix u_1, u_2 and u_3 , there are 15 discrete directions u_4 whose $\rho(\{u_1, u_2, u_3, u_4\}) \in \{4/3, 3/2, 2, 3, 4\}$. As a consequence, the set of these 15 directions are equal to \mathcal{A} , and so it follows that:

if U is a set of four discrete directions having cross ratio $\rho(U) \in \{4/3, 3/2, 2, 3, 4\}$, then the class of convex sets cannot be characterized by U .

By extending the proof of Proposition 4.5, we obtain:

Proposition 4.8. *If U is a set of four discrete directions having cross ratio $\rho(U) \in \{4/3, 3/2, 2, 3, 4\}$, then the class (c) of convex sets cannot be characterized by U .*

Consequently, from Theorem 4.1 it follows that:

Theorem 4.9. *Let U be a set of four discrete directions. The set U characterizes the class (c) of convex sets if and only if $\rho(U) \notin \{4/3, 3/2, 2, 3, 4\}$.*

5. Experimental evidence for the conjecture

We experimentally supported Conjecture 4.6 by means of an algorithm that reconstructs hv-convex polyominoes from their discrete X-rays. This checking algorithm consists of two steps:

1. the uniform random generation of a hv-convex polyomino F having a specified number of rows and columns and the determination of F 's X-rays in the four directions $U = \{(1, 0), (0, 1), u_3, u_4\}$;
2. the reconstruction of F by means of the four discrete X-rays determined in the previous step.

The first step is carried out by using a variant of Hochstätter–Loebl–Moll's algorithm [14], illustrated in [3]. The second step is performed by means of an algorithm that starts out from the four X-rays obtained in the second step and reconstructs the hv-convex polyominoes having these X-rays in U 's directions. The main steps of the algorithm are described below. If $U = \{(1, 0), (0, 1), u_3, u_4\}$ characterizes $(\mathbf{p}, \mathbf{h}, \mathbf{v})$, then for each randomly generated hv-convex polyomino F , the algorithm only reconstructs F from its X-rays in U 's directions. In order to give empirical evidence of our conjecture, we choose some quadruples $U = \{(1, 0), (0, 1), u_3, u_4\}$ such that $u_4 \notin \mathcal{A}$ (i.e., $\rho(U) \notin \{4/3, 3/2, 2, 3, 4\}$) and verify that the algorithm only reconstructs the randomly generated hv-convex polyomino F .

5.1. Main steps of the algorithm

We are going to illustrate the main steps of the algorithm. Let us consider the X-rays along the set of directions $U = \{(1, 0), (0, 1), u_3, u_4\}$. An hv-convex polyomino F that satisfies them is contained in a rectangle R of size $n \times m$, where m and n are the size of the X-ray of F along $(1, 0)$ and $(0, 1)$, respectively. We call any set α of cells such that $\alpha \subseteq F \subseteq R$ the *kernel*, and any set β of cells such that $F \subseteq \beta \subseteq R$ the *shell*. Consequently, the shell contains F , while the kernel is contained in F . Assuming that $\alpha = \emptyset$ and $\beta = R$, the basic idea of the algorithm that reconstructs F by starting out from the empty set is to reduce the shell and expand the kernel by means of the filling operations that take advantage of both the convexity constraint and the X-rays along $(1, 0)$, $(0, 1)$, u_3 , u_4 (for the definition of filling operations see [2]). The shell is reduced by eliminating the cells not belonging to F from β . Vice versa, the kernel is expanded by putting the cells belonging to F into α . If F does not exist, the reconstruction fails, that is, the filling operations produce a kernel α and a shell β such that $\alpha \not\subseteq \beta$. On the other hand, if only one polyomino F exists, that satisfies the given X-rays, the reconstruction algorithm stops when $\alpha = \beta = F$. However, if the quadruple $U = \{(1, 0), (0, 1), u_3, u_4\}$ does not characterize the class of hv-convex polyominoes, it is likely that the reconstruction algorithm stops when $\alpha \subset \beta$ and $\alpha \neq \beta$ for a certain set of X-rays along U . Consequently, F is ambiguous. In our tests, this never occurred.

We ran the algorithm thousands of times and it always reconstructed one hv-convex polyomino (the one randomly generated at step (1)) without any ambiguity. The results of these tests gave experimental evidence for both our conjecture and the good performance of the reconstruction algorithm.

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