Volume and Ehrhart polynomials of polytopes

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Outline

- Preliminaries
- Ehrhart polynomials of cyclic polytopes and lattice-face polytopes
- Formula for the volume of the Birkhoff polytope

PART I:

Preliminaries

***Summary*: We will go over some basic definitions and theory we need for this talk.**

Basic definitions related to polytopes

Definition 1 ($\mathcal{V}$-representation)**.** A *convex polytope* P in the d-dimensional Euclidean space $\mathbb{R}^d$ is the convex hull of finitely many points $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^d$. In other words,

$$P = \text{conv}(V) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \ : \ \text{all } \lambda_i \geq 0, \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}.$$

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There is an alternative definition of polytopes in terms of halfspaces.

Definition 2 ($\mathcal{H}$-representation)**.** A *convex polytope* $P \subset \mathbb{R}^d$ is a bounded intersection of halfspaces:

$$P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{z}\},$$

for some $A \in \mathbb{R}^{m \times d}, \mathbf{z} \in \mathbb{R}^m$.

The set of all affine combinations of points in some set $S \subset \mathbb{R}^d$ is called the *affine hull* of S, and denoted as $\mathrm{aff}(S)$:

$$\mathrm{aff}(S) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \ : \ v_1, v_2, \dots, v_n \in S, \text{ all } \lambda_i \in \mathbb{R}, \text{ and } \sum_{i=1}^{n} \lambda_i = 1\}.$$

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Definition 3. Let $P \subset \mathbb{R}^d$ be a convex polytope. A linear inequality $\mathbf{cx} \leq c_0$ is *valid* for P if it is satisfied for all points $\mathbf{x} \in P$. A *face* of P is any set of the form

$$F = P \cap \{\mathbf{x} \in \mathbb{R}^d \ : \ \mathbf{cx} = c_0\},$$

where $\mathbf{cx} \leq c_0$ is a valid inequality for P. The *dimension* of a face is the dimension of its affine hull: $\dim(F) := \dim(\mathrm{aff}(F))$.

The faces of dimension $0, 1$, and $\dim(P) - 1$ are called *vertices, edges,* and *facets,* respectively.

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Definition 4. For any polytope $P \subset \mathbb{R}^d$ and some positive integer $m \in \mathbb{N}$, the *mth dilated polytope* of P is $mP = \{m\mathbf{x} : \mathbf{x} \in P\}$. We denote by

$$i(m, P) = |\mathcal{L}(mP)|$$

the number of lattice points in mP.

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(ii) When $d = 2$, P is an integral polygon, and so is mP. By Pick's theorem:

$$\begin{aligned} i(P, m) &= \operatorname{area}(mP) + \frac{1}{2}|\partial(mP) \cap \mathbb{Z}^d| + 1 \\ &= \operatorname{area}(P)m^2 + \frac{1}{2}|\partial(P) \cap \mathbb{Z}^d|m + 1 \end{aligned}$$

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(iii) For any d, let P be the convex hull of the set $\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i = 0 \text{ or } 1\}$, i.e. P is the *unit cube* in $\mathbb{R}^d$. Then it is obvious that

$$i(P, m) = (m + 1)^d.$$

Theorem of Ehrhart

Theorem 5. *(Ehrhart) Let P be a d-dimensional integral polytope, then $i(P, m)$ is a polynomial in m of degree d.*

Therefore, we call $i(P, m)$ the *Ehrhart polynomial* of P.

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- The second coefficient equals $1/2$ times the sum of volumes of each facet, each normalized with respect to the sublattice in the hyperplane spanned by the facet.
- The constant term of $i(P, m)$ is always 1.
- No results for other coefficients for general polytopes.

PART II:

Ehrhart polynomials of cyclic polytopes and lattice-face polytopes

***Summary*:** In this part, we introduce families of polytopes, the coefficients of whose Ehrhart polynomials can be described in terms of volumes.

Motivation

De Loera conjectured that the Ehrhart polynomial of an integral cyclic polytope has a simple formula.

Recall that given $T = \{t_1, \dots, t_n\}_<$ a linearly ordered set, a d-dimensional *cyclic polytope* $C_d(T) = C_d(t_1, \dots, t_n)$ is the convex hull $\mathrm{conv}\{v_d(t_1), v_d(t_2), \dots, v_d(t_n)\}$ of $n > d$ distinct points $\nu_d(t_i), 1 \leq i \leq n$, on the moment curve.

The *moment curve* (also known as *rational normal curve*) in $\mathbb{R}^d$ is defined by

$$\nu_d : \mathbb{R} \to \mathbb{R}^d, t \mapsto \nu_d(t) = \begin{pmatrix} t \\ t^2 \\ \vdots \\ t^d \end{pmatrix}.$$

Example: $T = \{1, 2, 3, 4\}, d = 3$:

$C_d(T)$ is the convex polytope whose vertices are $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 27 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \\ 64 \end{pmatrix}.$

Theorem 6. *For any d-dimensional integral cyclic polytope $C_d(T)$,*

$$i(C_d(T), m) = \mathrm{Vol}(mC_d(T)) + i(C_{d-1}(T), m).$$

Hence,

$$\begin{aligned} i(C_d(T), m) &= \sum_{k=0}^{d} \mathrm{Vol}_k(mC_k(T)) \\ &= \sum_{k=0}^{d} \mathrm{Vol}_k(C_k(T)) m^k, \end{aligned}$$

where $\mathrm{Vol}_k(mC_k(T))$ is the volume of $mC_k(T)$ in k-dimensional space, and by convention we let $\mathrm{Vol}_0(mC_0(T)) = 1$.

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- $C_{d-3}(T) = \mathbb{R}^0$: $i(C_{d-3}(T), m) = 1.$
- $2, 4, 3$ and 1 are the volumes of $C_3(T)$, $C_2(T)$, $C_1(T)$ and $C_0(T)$, respectively.

Note that if we define $\pi^k : \mathbb{R}^d \longrightarrow \mathbb{R}^{d-k}$ to be the map which ignores the last k coordinates of a point, then $\pi^k(C_d(T)) = C_{d-k}(T)$. So when $P = C_d(T)$ is an integral cyclic polytope, we have that

$$i(P,m) = \mathrm{Vol}(mP) + i(\pi(P),m) = \sum_{k=0}^{d} \mathrm{Vol}_k(\pi^{d-k}(P))m^k, \tag{7}$$

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Question: Are there other integral polytopes which have the same form of Ehrhart polynomials as cyclic polytopes? In other words, what kind of integral d-polytopes P are there whose Ehrhart polynomials will be in the form of (7)?

Properties of integral cyclic polytopes

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For $d \geq 2$, for any d-subset $T' \subset T$, let $U = \nu_d(T')$ be the corresponding d-subset of the vertex set $V = \nu_d(T)$ of $C_d(T)$. Then:

a) $\pi(\mathrm{conv}(U)) = \pi(C_d(T')) = C_{d-1}(T')$ is an integral cyclic polytope, and

b) $\pi(\mathcal{L}(\mathrm{aff}(U))) = \mathbb{Z}^{d-1}$. In other words, after dropping the last coordinate of the lattice of $\mathrm{aff}(U)$, we get the $(d-1)$-dimensional lattice.

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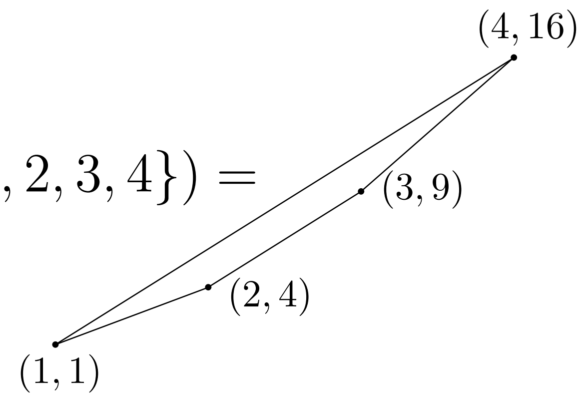

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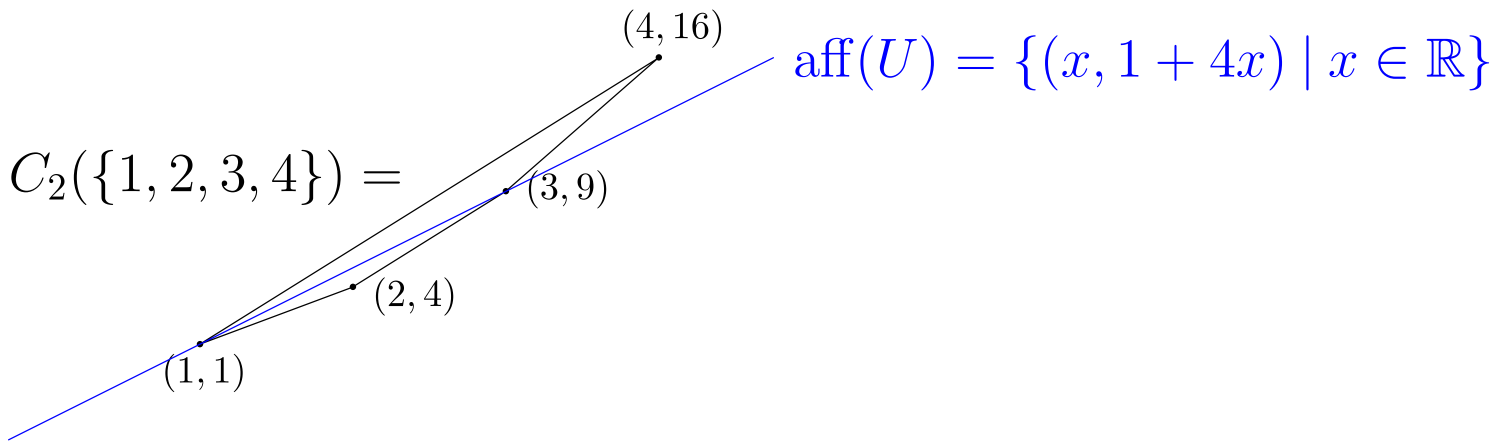

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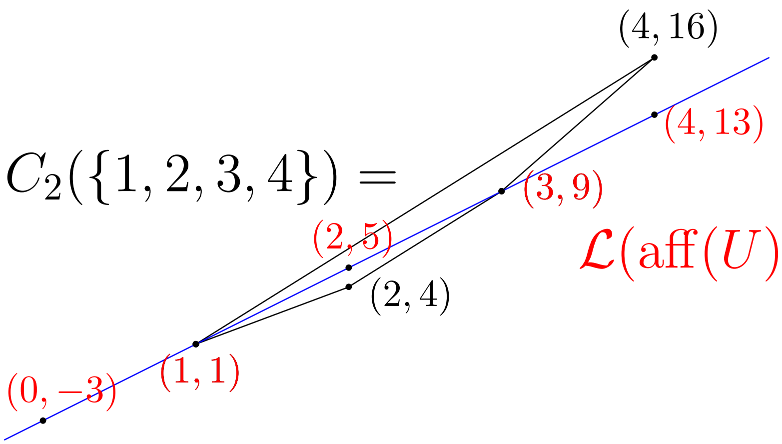

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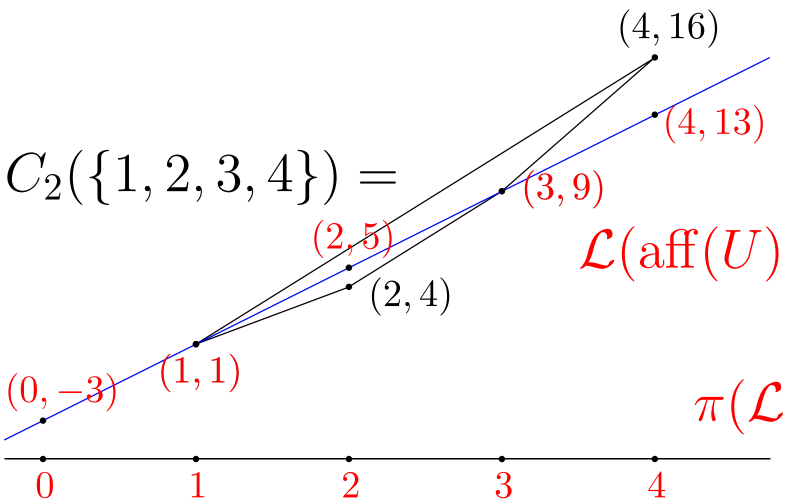

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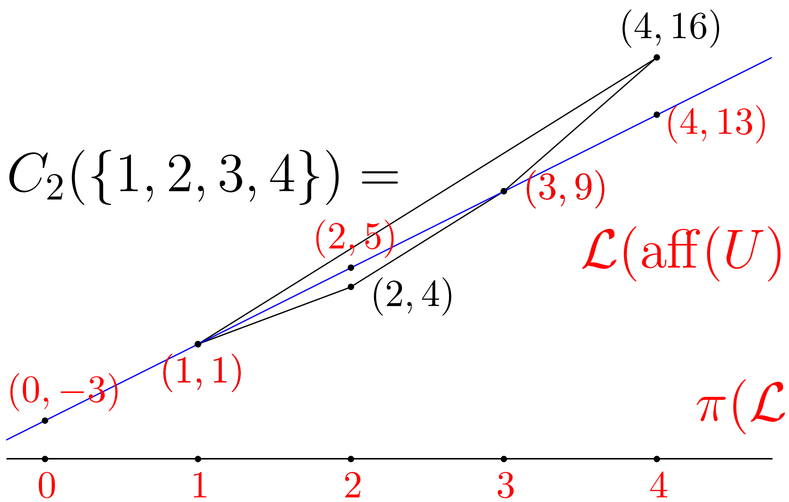

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Remark: Condition b) is equivalent to saying that for any lattice point $y \in \mathbb{Z}^{d-1}$, we have that $\pi^{-1}(y) \cap \mathrm{aff}(U)$, the intersection of $\mathrm{aff}(U)$ with the inverse image of y under π, is a lattice point.

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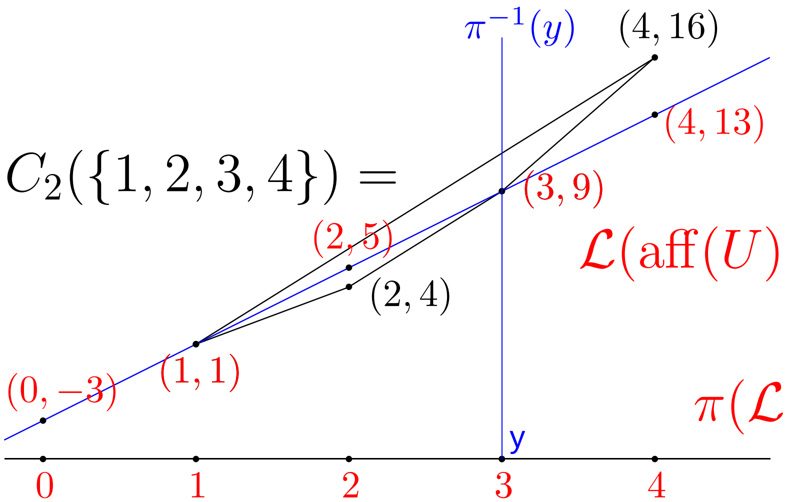

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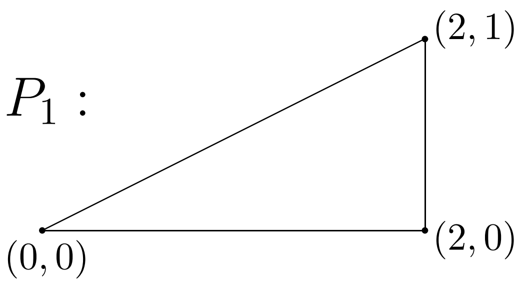

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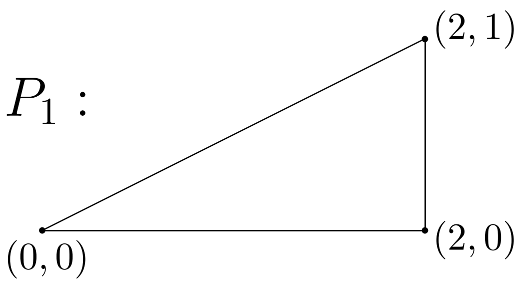

$U_1 = \{v_1, v_2\}, \text{aff}(U_1)$ is $\{(x,0) \mid x \in \mathbb{R}\}$. So $\pi(\mathcal{L}(\text{aff}(U_1))) = \mathbb{Z}$.

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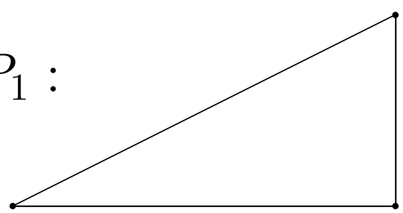

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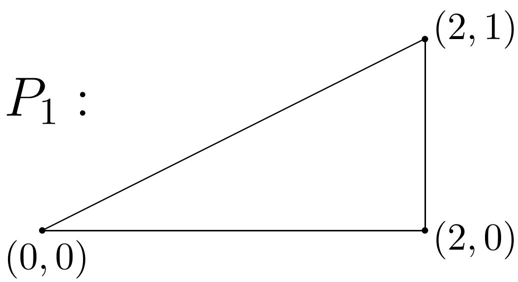

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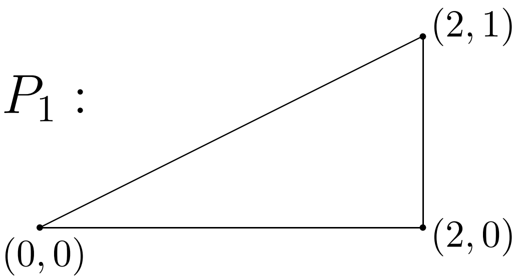

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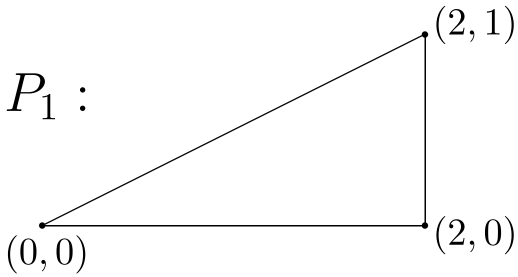

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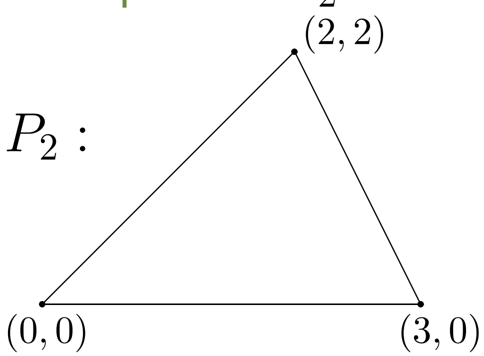

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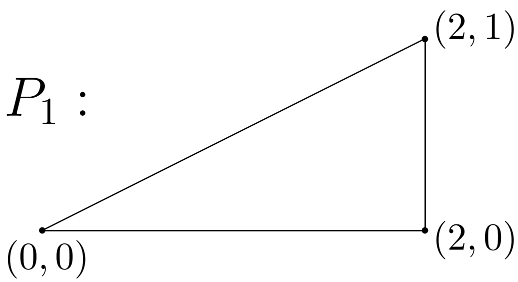

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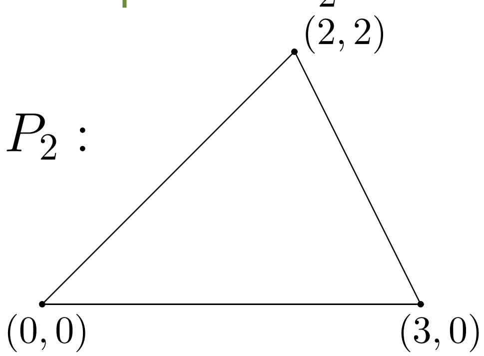

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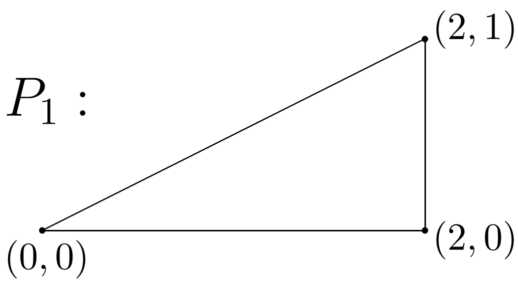

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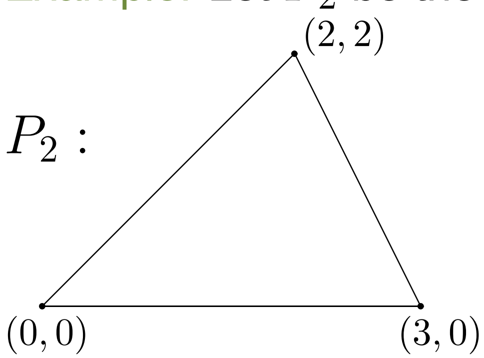

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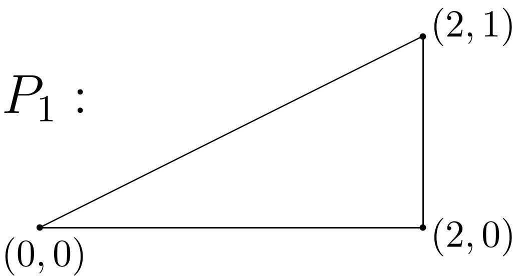

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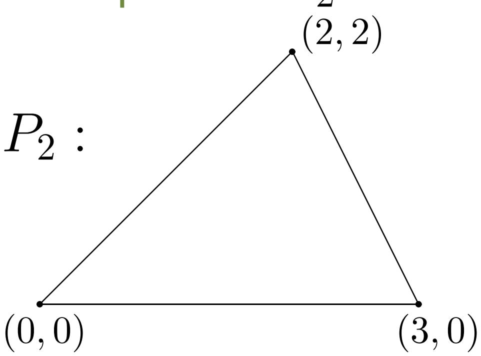

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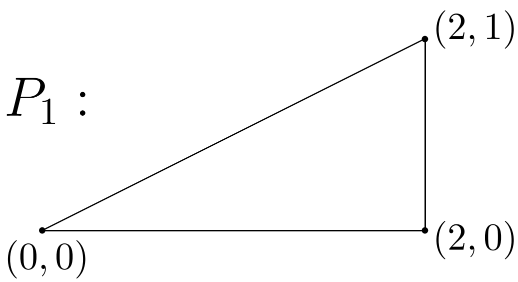

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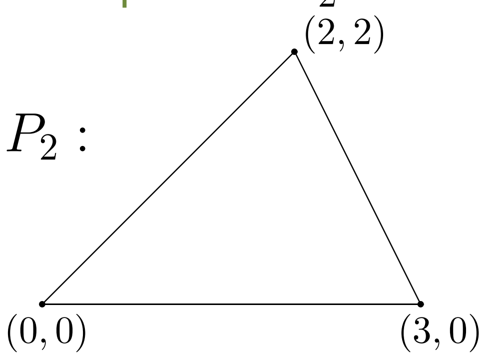

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For each U_i, condition $a)$ is always satisfied.

P_2 is a lattice-face polytope.

How big is the family of lattice-face polytopes?

The family of lattice-face polytopes is much bigger than that of cyclic polytopes. Cyclic polytopes are all simplicial polytopes, while lattice-face polytopes can be of any combinatorial type.

Theorem 10. *Let P be a lattice-face d-polytope, then*

$$i(P, m) = \mathrm{Vol}(mP) + i(\pi(P), m) = \sum_{k=0}^{d} \mathrm{Vol}_k(\pi^{d-k}(P))m^k.$$

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Example: Let $d = 3$, let P be the polytope with the vertex set $V = \{v_1 = (0,0,0), v_2 = (4,0,0), v_3 = (3,6,0), v_4 = (2,2,10)\}$. One can check that P is a lattice-face polytope.

$\mathrm{Vol}(P) = 40.$

$\pi(P) = \mathrm{conv}\{(0,0),(4,0),(3,6)\}$, and $\mathrm{Vol}(\pi(P)) = 12.$

$\pi^2(P) = [0,4]$, and $\mathrm{Vol}(\pi^2(P)) = 4.$

Thus, by the theorem, the Ehrhart polynomial of P is

$$i(P,m) = 40m^3 + 12m^2 + 4m + 1.$$

PART III:

Formula for the Volume of the Birkhoff polytope

***Summary*:** We give a formula for the volume of the Birkhoff polytope obtained by a calculation of its Ehrhart polynomial. This is joint work with Jesus De Loera and Ruriko Yoshida.

Birkhoff polytope

Definition 11. The *Birkhoff polytope*, denoted by B_n, is the convex polytope of $n \times n$ doubly-stochastic matrices; that is, the set of real nonnegative matrices with all row and column sums equal to one.

We consider B_n in the n^2-dimensional space $\mathbb{R}^{n^2} = \{n \times n$ real matrices $\}$. Below are some basic facts about B_n :

- The vertices of B_n are the $n \times n$ permutation matrices.
- B_n has n^2 facets: for each pair of (i, j) with $1 \leq i, j \leq n$, the doubly-stochastic matrices with (i, j) entry equal to 0 is a facet.

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It is a wide open problem to compute the volume of the Birkhoff polytopes. We only know the volume of B_n for $n \leq 10$. Our goal is to give a combinatorial formula of $\mathrm{Vol}(B_n)$.

Multivariate generating function

For any polyhedron $P \in \mathbb{R}^d$, we define the *multivariate generating function* (MGF) of P as

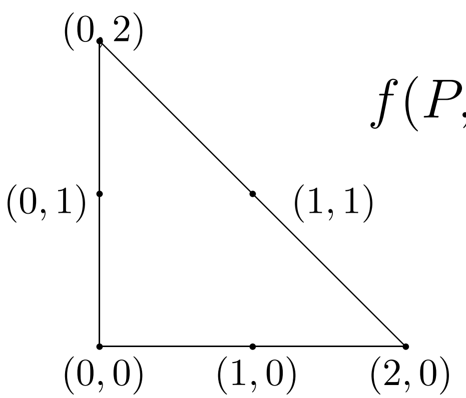
$$f(P, \mathbf{z}) = \sum_{\alpha \in P \cap \mathbb{Z}^d} \mathbf{z}^\alpha,$$

where $\mathbf{z}^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_d^{\alpha_d}$.

One sees that by setting $\mathbf{z} = (1, 1, \dots, 1)$, we get the number of lattice points in P if P is a polytope.

Example: Let P be the polytope with vertices $v_1 = (0, 0)$, $v_2 = (2, 0)$ and $v_3 = (0, 2)$.

P :



$$\begin{aligned} f(P, \mathbf{z}) &= z_1^0 z_2^0 + z_1^1 z_2^0 + z_1^2 z_2^0 + z_1^0 z_2^1 + z_1^1 z_2^1 + z_1^0 z_2^2 \\ &= 1 + z_1 + z_1^2 + z_2 + z_1 z_2 + z_2^2. \end{aligned}$$

Why MGF?

Lemma 12 (Brion, 1988; Lawrence, 1991). *Let P be a rational polyhedron and let $V(P)$ be the vertex set of P. Then,*

$$f(P, \mathbf{z}) = \sum_{v \in V(P)} f(C(P, v), \mathbf{z}),$$

where $C(P, v)$ is the supporting cone of P at v, i.e., the smallest cone with vertex v containing P.

If K is a d-dimensional cone in $\mathbb{R}^e$, generated by vectors $\{r_i\}_{1 \le i \le d}$ such that the r_i's form a $\mathbb{Z}$-basis of the lattice $\mathrm{span}(\{r_i\}) \cap \mathbb{Z}^e$, then we say K is a *unimodular cone*.

Lemma 13. *If K is a d-dimensional unimodular cone at an integral vertex v generated by the vectors $\{r_i\}_{1 \le i \le d}$, then we have*

$$f(K, \mathbf{z}) = \mathbf{z}^v \prod_{i=1}^{d} \frac{1}{1 - \mathbf{z}^{r_i}}.$$

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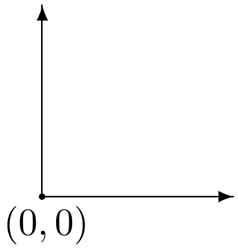

A unimodular cone generated by vectors $r_1 = (1,0)$ and $r_2 = (0,1)$.

$f(C(P, v_1), \mathbf{z}) = \mathbf{z}^{(0,0)} \prod_{i=1}^{2} \frac{1}{1-\mathbf{z}^{r_i}} = \frac{1}{(1-z_1)(1-z_2)}$.

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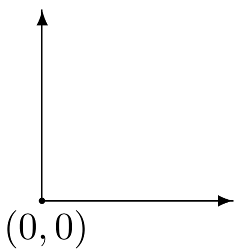

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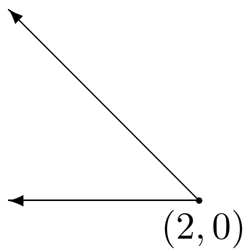

A unimodular cone generated by vectors $r_1 = (-1,0)$ and $r_2 = (-1,1)$.

$f(C(P,v_2),\mathbf{z}) = \mathbf{z}^{(2,0)} \prod_{i=1}^{2} \frac{1}{1-\mathbf{z}^{r_i}} = \frac{z_1^2}{(1-z_1^{-1})(1-z_1^{-1}z_2)} = \frac{z_1^4}{(z_1-1)(z_1-z_2)}$.

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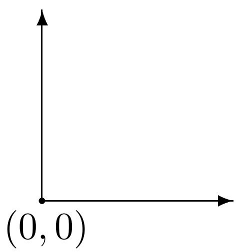

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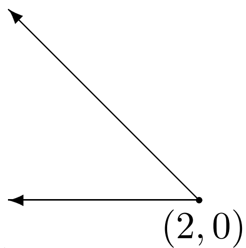

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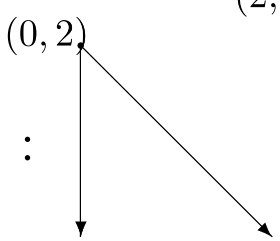

A unimodular cone generated by vectors $r_1 = (0,-1)$ and $r_2 = (1,-1)$.

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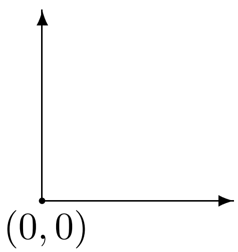

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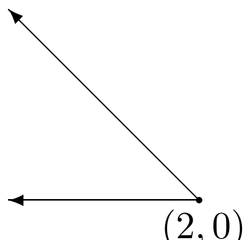

A unimodular cone generated by vectors $r_1 = (-1,0)$ and $r_2 = (-1,1)$.

$f(C(P,v_2), \mathbf{z}) = \mathbf{z}^{(2,0)} \prod_{i=1}^{2} \frac{1}{1-\mathbf{z}^{r_i}} = \frac{z_1^2}{(1-z_1^{-1})(1-z_1^{-1}z_2)} = \frac{z_1^4}{(z_1-1)(z_1-z_2)}$.

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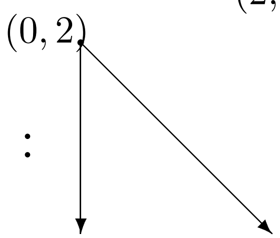

A unimodular cone generated by vectors $r_1 = (0,-1)$ and $r_2 = (1,-1)$.

$f(C(P,v_3), \mathbf{z}) = \mathbf{z}^{(0,2)} \prod_{i=1}^{2} \frac{1}{1-\mathbf{z}^{r_i}} = \frac{z_2^2}{(1-z_2^{-1})(1-z_1 z_2^{-1})} = \frac{z_2^4}{(z_2-1)(z_2-z_1)}$.

$$\sum_{i=1}^{3} f(C(P,v_i), \mathbf{z}) = \frac{(z_1-z_2) - z_1^4(1-z_2) + z_2^4(1-z_1)}{(1-z_1)(1-z_2)(z_1-z_2)} = 1 + z_1 + z_1^2 + z_2 + z_1 z_2 + z_2^2 = f(P, \mathbf{z}).$$

Barvinok's algorithm

- Barvinok gave an algorithm to decompose a cone C as a signed sum of simple unimodular cones.
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i. Find a dual cone K to C.

ii. Apply the Barvinok decomposition to K and get a set of unimodular cones K_i.

iii. Find dual cone C_i of each K_i. (Note C_i is unimodular as well.)

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We will use this idea to find the MGF of the Birkhoff polytopes.

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iii. We show that any triangulation of K_n gives a set of unimodular cones. Instead of using Barvinok's method, we use the idea of Gröbner bases of toric ideals to produce triangulations. For any $\ell \in [n] = \{1, 2, \ldots, n\}$, we can give a triangulation Tri_ℓ of C_n into n^{n-2} cones. In fact, the set of cones in Tri_ℓ is in bijection with $\mathbf{Arb}(\ell, n)$, the set of all ℓ-arborescences on the nodes $[n]$.

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iv. By finding the dual cones to all of the cones in the Tri_ℓ, we give the MGF of C_n.

The MGF of the dilation mB_n

The multivariate generating function of C_n is given by

$$f(C_n, \mathbf{z}) = \sum_{T \in \mathbf{Arb}(\ell, n)} \mathbf{z}^I \prod_{e \notin E(T)} \frac{1}{(1 - \prod \mathbf{z}^{W^{T,e}})},$$

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Theorem 14. *The multivariate generating function of mB_n is given by*

$$f(mB_n, \mathbf{z}) = \sum_{\sigma \in S_n} \sum_{T \in \mathbf{Arb}(\ell,n)} \mathbf{z}^{m\sigma} \prod_{e \notin E(T)} \frac{1}{(1 - \prod \mathbf{z}^{W^{T,e}\sigma})},$$

From MGF to Ehrhart polynomial and volume

Corollary 15. *The Ehrhart polynomial* $i(B_n, m)$ *of* B_n *is given by the formula*

$$i(B_n, m) = \sum_{k=0}^{(n-1)^2} m^k \frac{1}{k!} \sum_{\sigma \in S_n} \sum_{T \in \mathbf{Arb}(\ell, n)} \frac{(\langle c, \sigma \rangle)^k \operatorname{td}_{(n-1)^2-k}(\{\langle c, W^{T,e}\sigma \rangle,\ e \notin E(T)\})}{\prod_{e \notin E(T)} \langle c, W^{T,e}\sigma \rangle}.$$

The symbol $\operatorname{td}_j(S)$ *is the* j*-th Todd polynomial evaluated at the numbers in the set* S*. The vector* $c \in \mathbb{R}^{n^2}$ *is any vector such that* $\langle c, W^{T,e}\sigma \rangle$ *is non-zero for all pairs* (T, e) *of an* ℓ*- arborescence* T *and an arc* $e \notin E(T)$ *and all* $\sigma \in S_n$*.*

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As a special case, the normalized volume of B_n *is given by*

$$\operatorname{Vol}(B_n) = \frac{1}{((n-1)^2)!} \sum_{\sigma \in S_n} \sum_{T \in \mathbf{Arb}(\ell, n)} \frac{\langle c, \sigma \rangle^{(n-1)^2}}{\prod_{e \notin E(T)} \langle c, W^{T,e}\sigma \rangle}.$$