# Example with 12-gon inside of 6-gon (aka “13 holes”)

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A *triangulation* of a point set $X \in \mathbb{R}^2$ is a maximal set $\mathcal{T}$ of non-intersecting straight line segments connecting pairs of points in $X$. By maximal, we mean that any other set $\mathcal{S}$ which also contains such segments will have the property that $\mathcal{S} \subseteq \mathcal{T}$. Such a set $\mathcal{T}$ is considered to be maximal *with respect to inclusion.* A triangulation may be specified by either a listing of the edges used or by a listing of the triangles. When we speak of the *combinatorial type* of a triangulation, we mean the listing of the triangles used in that triangulation, or, equivalently, a listing of the edges. We define the *length* or *weight* of a triangulation of $X$ to be the sum of the lengths of the edges used in the triangulation. A *minimum weight triangulation* of a point set $X$ is a triangulation which has length less than or equal to the length of every other triangulation of $X$. We note that such a triangulation is not necessarily unique.

Now we move into notation and terms which may not be as standard. Denote the weight of the minimum weight triangulation of $X$ by $mwt(X)$. We say that a point set $X$ is *reducible* if there exists a point $p = (x, y) \notin X$ such that $mwt(X \cup \{p\}) < mwt(X)$. Such a point $p$ is said to *reduce* the length of the triangulation, and we refer to $p$ as a *reducing point.* For a given point set, we are concerned with the region of the plane consisting of all reducing points, which we refer to as the *reducing region.* We seek to prove that this region may have arbitrarily complex topology. One step on the road to that result is to show that some sets of points admit reducing regions with holes. This paper will present a point set whose reducing region contains at least 13 holes.

We consider the point set: $\mathbb{P} = \mathbb{S} \cup \mathbb{T}$, where

$$\mathbb{S} = \left\{ \left( 83 \cos \frac{2\pi(2k-1)}{12}, 83 \sin \frac{2\pi(2k-1)}{12} \right) \middle| k = 1..6 \right\}, \text{ and}$$

$$\mathbb{T} = \left\{ \left( 20 \cos \frac{2\pi(2k-1)}{24}, 20 \sin \frac{2\pi(2k-1)}{24} \right) \middle| k = 1..12 \right\}.$$

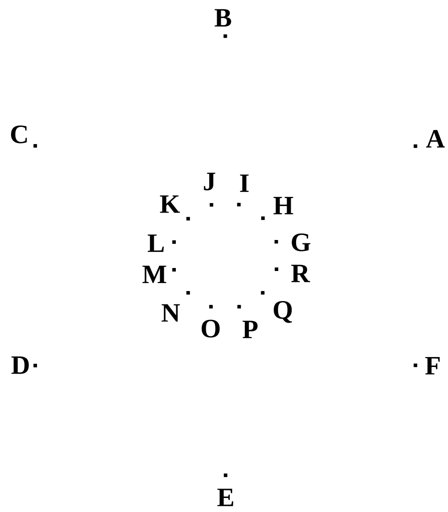


Figure 1: Our original point set, $\mathbb{P} = \mathbb{S} \cup \mathbb{T}$

We label the points of $\mathbb{S}$ by $A, \ldots, F$, for values of $k = 1..6$. We similarly label the points of $\mathbb{T}$ by $G, \ldots, R$, for values of $k = 1..12$.

Notice that our point set is preserved under the standard group action of $D_6$, the dihedral group of order 12. We will utilize this symmetry of our point set to reduce the number of cases we much consider. We may claim that certain cases are unique “up to symmetry” - by this we will mean that we are avoiding the consideration of duplicate cases that arise by the action of some element of $D_6$ which leaves the elements of our hypotheses fixed. We will say that edge $ST$ *is symmetric to* edge $UV$ if both segments are in the same orbit under the action of $D_6$.

When we refer to triangles in our triangulation, we mean triangles that contain no point from our original set. We will sometimes speak of "visibility constraints" or claim that certain results are forced "by visibility." This should be taken to mean that all other choices of triangles would either contain points from our set or would intersect some edge which must belong to the triangulation. As shown in Figure 2 below, if $AC$ belongs to our triangulation, the we say that visibility constraints imply that either $\triangle ACI$ or $\triangle ACJ$ must belong to our triangulation. Moreover, those two cases are the same, up to symmetry: reflecting along the line $\overline{BE}$ will fix $AC$ and map $\triangle ACI$ to $\triangle ACJ$.

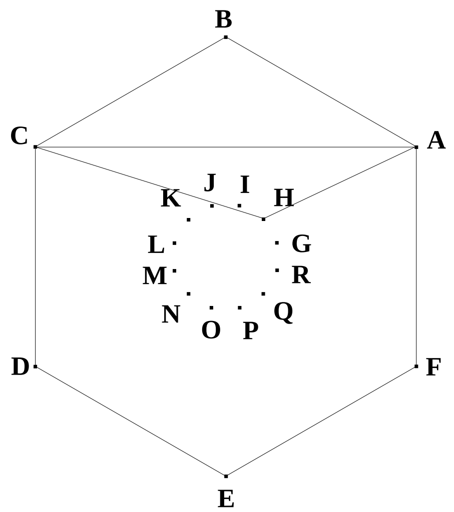


Figure 2: Triangle $\triangle ACH$ is disallowed by visibility, since point $I$ is in its interior.

We will rely heavily on proofs by contradiction when making claims about the structure of a given triangulation. Once we know (or if we assume) that a certain edge is included in the triangulation, then visibility constraints will give a set of possible triangulations that used the specified edge. We will seek to find local contradictions to minimality if possible: for example, pairs of triangles which share an edge that is the long diagonal of the 4-gon formed by their union. If, however, we assume that a certain edge is *not* present, then we know that some edge used in the triangulation must cross that segment. Let us state and prove that formally.

**Claim 1** *If edge $ST$ does not belong to a triangulation, then some edge in the triangulation intersects $ST$ in its interior.*

**Proof**: Assume this is not the case. Then edge $ST$ is not present, and segment $ST$ is not crossed by any other edge of the triangulation. This contradicts our definition of a triangulation as a maximal set of non-intersecting straight line segments connecting pairs of points in our point set. ■

We now establish, for our particular point set, a subset of the minimum weight triangulation that will simplify our task of finding the overall minimal triangulation of $\mathbb{P}$.

**Claim 2** *The minimum weight triangulation of $\mathbb{P}$ includes a minimum weight triangulation of the 12-gon formed by the points of $\mathbb{T}$.*

**Proof**: We note that if all edges of the convex hull of $\mathbb{T}$ are present in the minimum weight triangulation, then our claim must hold, for the interior of the 12-gon will be triangulated minimally. Assume that some edge of the 12-gon is not present. There are two types of edges in the convex hull of the 12-gon: those symmetric to $GH$ (edges $IJ$, $KL$, $MN$, $OP$, and $QR$) and those symmetric to $HI$ (edges $JK$, $LM$, $NO$, $PQ$, and $RG$).

Assume towards a contradiction that edge $GH$ is not in the minimum weight triangulation. Then by Claim 1 there must be some edge that passes between $G$ and $H$. There are three such possible edges, up to symmetry: $AM, BR$ and $AD$. Assume $AM$ is in the minimum weight triangulation. Then it must belong to two triangles. Visibility constraints then require that $\triangle AHM$ will then be in the minimum weight triangulation, and also one of $\triangle AGM$, $\triangle AMN$. Now, if $\triangle AGM$ is in the triangulation as shown in the left side of Figure 3, then $AGHM$ will use diagonal $AM$ instead of the shorter $GH$, a contradiction. Likewise, the use of $\triangle AMN$ forces $AHMN$ to use diagonal $AM$ instead of the shorter $HN$. Thus $AM$ does not belong to the minimum weight triangulation. Now assume that $BR$ is in the minimum weight triangulation. Then $\triangle BGR$ is forced to belong to the triangulation, as is $\triangle BHR$. This means that $BGRH$ uses $BR$ instead of the shorter $GH$, a contradiction. (See the right side of Figure 3.)

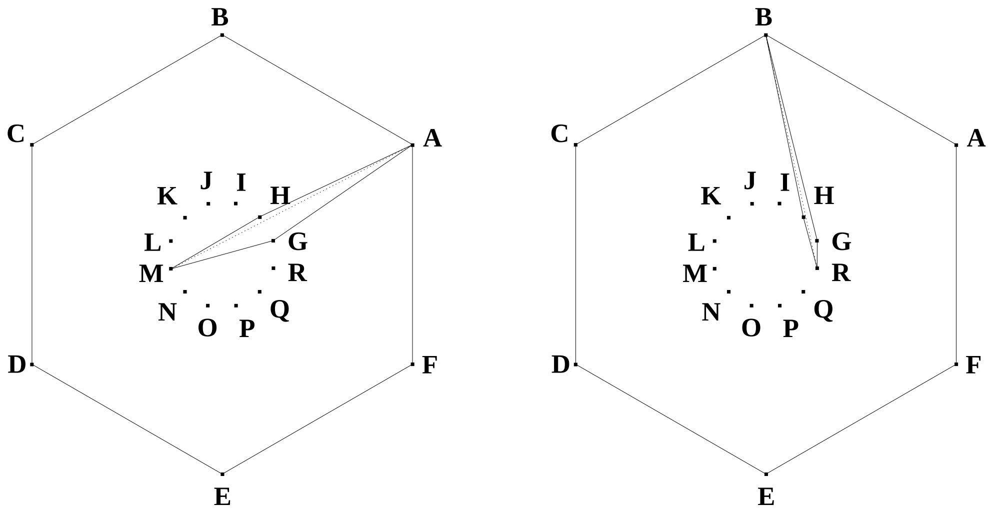


Figure 3: Edges $AM$ and $BR$ do not belong to the minimum weight triangulation of $\mathbb{P}$.

Lastly, assume that $AD$ is in the minimum weight triangulation. This forces triangles which in turn give two possible quadrilaterals (up to symmetry) which would be triangulated by $AD$ in the minimum weight triangulation: $AHDG$ and $AHDN$. See Figure 4 below.

Note that $AD$ is longer than $GH$ and $HN$, the other diagonals of those 4-gons. This implies that $AD$ does not belong to any minimal triangulation.

It follows that edge $GH$ must belong to the minimum weight triangulation of $\mathbb{P}$, and by symmetry, so must edges $IJ$, $KL$, $MN$, $OP$, and $QR$.

Now we assume, also towards a contradiction, that edge $HI$ is not in the minimum weight triangulation. Then there must be a segment that passes between $H$ and $I$. The only two possible such edges are $AL$ and $BQ$, which are symmetric to one another. Assume then, that $AL$ is in the minimum weight triangulation. This forces the inclusion of $\triangle AIL$ in the triangulation, as well as forcing $\triangle AHL$. Then $AILH$ uses $AL$ and not the shorter $HI$. It follows that edges $HI$, $JK$, $LM$, $NO$, $PQ$, and $RG$ are in the minimum weight triangulation of $\mathbb{P}$. We have established that the edges in the convex hull of $\mathbb{T}$ are also edges of the minimum weight triangulation of $\mathbb{P}$. ■

We now note that the following sets of segments are orbits under the action of $D_6$, and therefore define equivalence classes based on length.

$$\Gamma := \{AG, AH, BI, BJ, CK, CL, DM, DN, EO, EP, FQ, FR\}$$

$$\Phi := \{AR, AI, BH, BK, CJ, CM, DL, DO, EN, EQ, FP, FG\}$$

$$\Psi := \{AQ, AJ, BG, BL, CI, CN, DK, DP, EM, ER, FO, FH\}$$

All segments in $\Gamma$ have length $\sqrt{20^2 + 83^2 - 2 \cdot 20 \cdot 83 \cos\left(\frac{2\pi}{12} - \frac{2\pi}{24}\right)} = \sqrt{7289 - 3320 \cos\left(\frac{\pi}{12}\right)} \approx 63.8915$ units, segments in $\Phi$ have length $\sqrt{20^2 + 83^2 - 2 \cdot 20 \cdot 83 \cos\left(\frac{2\pi \cdot 5}{24} - \frac{2\pi}{12}\right)} = \sqrt{7289 - 3320 \cos\left(\frac{\pi}{4}\right)} \approx 70.2951$,

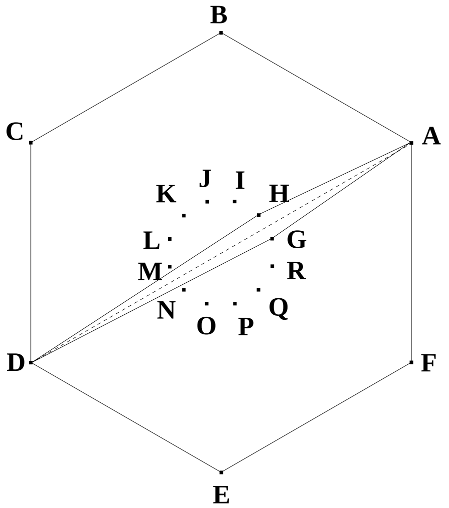


Figure 4: Edge $AD$ does not belong to the minimum weight triangulation of $\mathbb{P}$.

and segments in $\Psi$ have length $\sqrt{20^2 + 83^2 - 2 \cdot 20 \cdot 83 \cos\left(\frac{2\pi \cdot 7}{24} - \frac{2\pi}{12}\right)} = \sqrt{7289 - 3320 \cos\left(\frac{5\pi}{12}\right)} \approx 80.1855.$

**Claim 3** *A minimal triangulation of* $\mathbb{P}$ *includes all edges in the set* $\Gamma$ *and one edge each from the following six pairs of edges:* $(AI, BH)$, $(BK, CJ)$, $(CM, DL)$, $(DO, EN)$, $(EQ, FP)$, $(FG, AR)$.

**Proof**: Other potential edges in a triangulation of $\mathbb{P}$ are: $AC$ (or one of the symmetric edges $BD$, $CE$, $DF$, $AE$, $BF$) and $AQ$ (or one of the symmetric edges from set $\Psi$). If we can show that none of these two equivalence classes of edges are used, then our above claim about the structure of the minimal triangulation will be true. Our proofs will continue to be structured to look for contradictions of the form of a quadrilateral which uses the long diagonal instead of the short diagonal.

Assume that edge $AC$ is in a minimum weight triangulation of point set $\mathbb{P}$. Then $AC$ forms a triangle also with one of $I, J$. WLOG, assume $\triangle ACI$ is in this triangulation of $\mathbb{P}$. (Note that $\triangle ACI$ is symmetric to $\triangle ACJ$.) Then $ABCI$ is triangulated with $AC$ instead of the shorter diagonal $BI$, a contradiction. It follows that neither $AC$ nor any edges symmetric to $AC$ belong to the minimum weight triangulation of $\mathbb{P}$.

Similarly, assume $AQ$ is in a minimum weight triangulation of $\mathbb{P}$. This edge must belong to two triangles. The only two possible such triangles are $\triangle AQR$ and $\triangle AFQ$. (Note the use of $\triangle AEQ$ would imply the use of edge $AE$, which is symmetric to $AC$ and therefore not in any minimum weight triangulation by the above argument.) This means $AFQR$ uses diagonal $AQ$ and not the shorter $FR$. It follows that neither $AQ$ nor any edges symmetric to $AQ$ belong to the minimum weight triangulation of $\mathbb{P}$.

We have therefore established that one minimum weight triangulation of $\mathbb{P}$ uses the following edge set between the convex hulls of $\mathbb{S}$ and $\mathbb{T}$:

$$\Omega := \{AG, AH, AI, BI, BJ, BK, CK, CL, CM, DM, DN, DO, EO, EP, EQ, FQ, FR, FG\}. \quad \blacksquare$$

We now seek to establish several convex regions, the union of which will be a planar region with at least 13 holes. There are five regions, up to symmetry, which we must consider. These regions are bounded by lines extended from the edges of the interior 12-gon. This line arrangement defines zones of visibility for our new point $Z$ that is to be added. We have established that the 12-gon which is the convex hull of $\mathbb{T}$ is included in the minimum weight triangulation of $\mathbb{P}$. A new point $Z$ can only be connected to points that do not require segments to intersect conv($\mathbb{T}$). Recall that a point of $\mathbb{T}$ that does not require the ray to $Z$ to intersect the 12-gon is said to be *visible* to $Z$.

The convexity of these regions will be established by the following lemma.

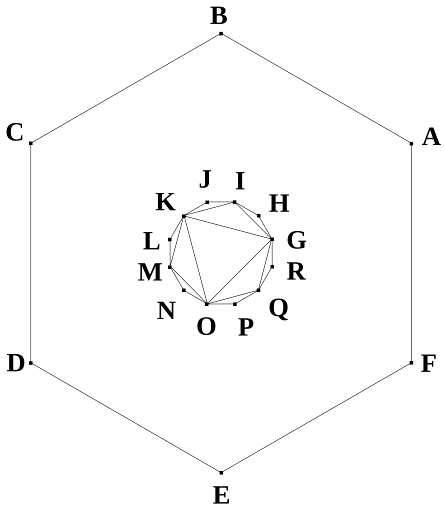


Figure 5: $\mathbb{P}$ with minimally triangulated 12-gon.

**Lemma 4** *Let $z = (x, y)$ be a point in $\mathbb{R}^2$ and $P = \{(x_i, y_i)\}_{i=1}^n$ a set of $n$ distinct points in $\mathbb{R}^2 - \{z\}$. Then the function $\sum_{p\in P} dist(z, p)$ is convex.*

**Proof**: We pick two distinct points $q_\alpha = (x_\alpha, y_\alpha)$, $q_\beta = (x_\beta, y_\beta)$ in $\mathbb{R}^2 - P$ and show that a point $z = (x, y)$ on the segment $q_\alpha q_\beta$ will have

$$\sum_{p\in P} \text{dist}(z, p) \le \max\left(\sum_{p\in P} \text{dist}(q_\alpha, p), \sum_{p\in P} \text{dist}(q_\beta, p)\right).$$

Let the points on the segment be denoted by $q_t = (x_t, y_t) := \Big(tx_\alpha + (1-t)x_\beta, ty_\alpha + (1-t)y_\beta\Big)$ for $0 \le t \le 1$. Note that $q_0 = q_\beta$ and $q_1 = q_\alpha$.

Now a comment about some unfortunate notational duplication. Please forgive my insistence that $(x_1, y_1)$ be the name for two distinct points - the point in $P$ with index $i = 1$, and when $t = 1$, the parameterized point $q_1$, which we recall is actually point $q_\alpha$. In the following proof I do not refer to $(x_1, y_1)$ explicitly, so the point to which I am referring should be clear by context. I do mean for these points to be distinct from one another.

Now, onto the proof!

We define

$$\begin{aligned} g(t) &= \sum_{p\in P} \text{dist}(q_t, p) \\ &= \sum_{i=1}^n \text{dist}\big(q_t, (x_i, y_i)\big) \\ &= \sum_{i=1}^n \sqrt{(tx_\alpha + (1-t)x_\beta - x_i)^2 + (ty_\alpha + (1-t)y_\beta - y_i)^2} \end{aligned}$$

If we can show that $g(t)$ is concave up for $0 \le t \le 1$, then the lemma will hold. Without further

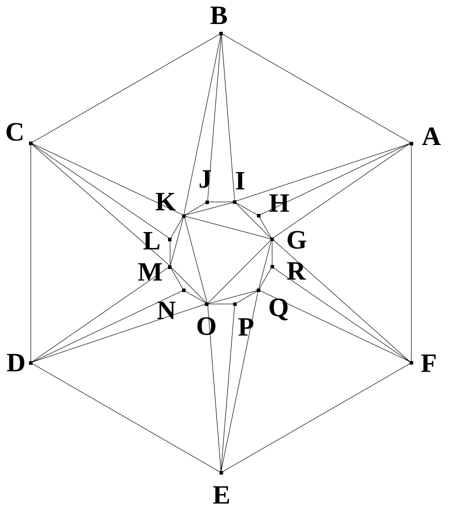


Figure 6: $\mathbb{P}$ triangulated minimally with the edges in $\Omega$.

ado, we take some derivatives.

$$
\begin{aligned}
g'(t) &= \frac{d}{dt}\Big[\sum_{i=1}^{n} \text{dist}\big(q_t,(x_i,y_i)\big)\Big] \\
&= \sum_{i=1}^{n} \frac{d}{dt}\Big[\text{dist}\big(q_t,(x_i,y_i)\big)\Big] \\
&= \sum_{i=1}^{n} \frac{d}{dt}\Big[\sqrt{(x_t-x_i)^2+(y_t-y_i)^2}\Big] \\
&= \sum_{i=1}^{n} \frac{1}{2}\Big[(x_t-x_i)^2+(y_t-y_i)^2\Big]^{-\frac{1}{2}}\Big[\frac{d}{dt}\Big((x_t-x_i)^2+(y_t-y_i)^2\Big)\Big] \\
&= \sum_{i=1}^{n} \frac{1}{2}\Big[(x_t-x_i)^2+(y_t-y_i)^2\Big]^{-\frac{1}{2}}\Big[2(x_t-x_i)\Big(\frac{d}{dt}x_t\Big)+2(y_t-y_i)\Big(\frac{d}{dt}y_t\Big)\Big] \\
&= \sum_{i=1}^{n} \frac{(x_t-x_i)(x_\alpha-x_\beta)+(y_t-y_i)(y_\alpha-y_\beta)}{\Big[(x_t-x_i)^2+(y_t-y_i)^2\Big]^{\frac{1}{2}}}
\end{aligned}
$$

$$
\begin{aligned}
g''(t) &= \frac{d}{dt}\left[\sum_{i=1}^{n} \frac{(x_t-x_i)(x_\alpha-x_\beta)+(y_t-y_i)(y_\alpha-y_\beta)}{\big[(x_t-x_i)^2+(y_t-y_i)^2\big]^{\frac{1}{2}}}\right] \\
&= \sum_{i=1}^{n} \frac{\Big[(x_t-x_i)^2+(y_t-y_i)^2\Big]^{\frac{1}{2}}\Big((x_\alpha-x_\beta)^2+(y_\alpha-y_\beta)^2\Big) - \frac{\big[(x_t-x_i)(x_\alpha-x_\beta)+(y_t-y_i)(y_\alpha-y_\beta)\big]^2}{\big[(x_t-x_i)^2+(y_t-y_i)^2\big]^{\frac{1}{2}}}}{\Big[(x_t-x_i)^2+(y_t-y_i)^2\Big]} \\
&= \sum_{i=1}^{n} \frac{(x_\alpha-x_\beta)^2+(y_\alpha-y_\beta)^2}{\Big[(x_t-x_i)^2+(y_t-y_i)^2\Big]^{\frac{1}{2}}} - \frac{\Big[(x_t-x_i)(x_\alpha-x_\beta)+(y_t-y_i)(y_\alpha-y_\beta)\Big]^2}{\Big[(x_t-x_i)^2+(y_t-y_i)^2\Big]^{\frac{3}{2}}}
\end{aligned}
$$

We now ask if $g''(t) \geq 0$. This will certainly be the case if each term in the sum is greater than or equal to 0.
We note that the $i^{\text{th}}$ term

$$\frac{(x_\alpha - x_\beta)^2 + (y_\alpha - y_\beta)^2}{\Big[(x_t - x_i)^2 + (y_t - y_i)^2\Big]^{\frac{1}{2}}} - \frac{\Big[(x_t - x_i)(x_\alpha - x_\beta) + (y_t - y_i)(y_\alpha - y_\beta)\Big]^2}{\Big[(x_t - x_i)^2 + (y_t - y_i)^2\Big]^{\frac{3}{2}}} \geq 0$$

if and only if

$$\begin{aligned} \frac{(x_\alpha - x_\beta)^2 + (y_\alpha - y_\beta)^2}{\Big[(x_t - x_i)^2 + (y_t - y_i)^2\Big]^{\frac{1}{2}}} &\geq \frac{\Big[(x_t - x_i)(x_\alpha - x_\beta) + (y_t - y_i)(y_\alpha - y_\beta)\Big]^2}{\Big[(x_t - x_i)^2 + (y_t - y_i)^2\Big]^{\frac{3}{2}}} \\ \Big[(x_\alpha - x_\beta)^2 + (y_\alpha - y_\beta)^2\Big]\Big[(x_t - x_i)^2 + (y_t - y_i)^2\Big] &\geq \Big[(x_t - x_i)(x_\alpha - x_\beta) + (y_t - y_i)(y_\alpha - y_\beta)\Big]^2 \quad (1) \end{aligned}$$

Now for the sake of simplifying notation, we make the following substitutions:

$$\begin{aligned} A &= (x_\alpha - x_\beta)^2 \\ B &= (y_\alpha - y_\beta)^2 \\ C &= (x_t - x_i)^2 \\ D &= (y_t - y_i)^2 \end{aligned}$$

This turns inequality (1) into

$$\begin{aligned} (A^2 + B^2)(C^2 + D^2) &\geq [AC + BD]^2 \\ A^2C^2 + A^2D^2 + B^2C^2 + B^2D^2 &\geq A^2C^2 + 2ABCD + B^2D^2 \\ A^2D^2 + B^2C^2 &\geq 2ABCD \\ A^2D^2 - 2ABCD + B^2C^2 &\geq 0 \\ (AD - BC)^2 &\geq 0, \end{aligned}$$

which is always true.

So $g$ is concave up as desired. ∎

The strength of this lemma is that we may now find sets of points which reduce with a certain combinatorial type, and then know that the convex hull of those points will also reduce.

**Corollary 5** *Reducing regions for a fixed combinatorial type of triangulation are convex if no obstructions to visibility exist.*

**Proof**: The reducing point is connected to some set of

We define region 1 to be the bounded chamber formed by lines $HI, GH, KL$, and $JK$. (See left side of Figure 7.) Let

$$\begin{aligned} J_1 &= HI \cap JK \approx (0, 22.30710), \\ K_1 &= GH \cap JK \approx (7.07107, 26.38958), \\ A_1 &= GH \cap (y = 30.675) \approx (4.59688, 30.675), \\ T &= (y = 30.675) \cap JK \approx (-4.59688, 30.675), \text{ and} \\ O_1 &= HI \cap K \approx (-7.07107, 26.38958). \end{aligned}$$

We claim that the interior of the convex hull of $\bar{\mathcal{A}} = \{J_1, K_1, A_1, T, O_1\}$ is a reducing region when a new point $Z$ is connected to points in $\mathcal{A} := \{A, B, C, H, I, J, K\}$. The edges $ZA, ZB, ZC, ZH, ZI, ZJ, ZK$ will replace edges $AI, BI, BJ, BK$ from the original triangulation, which have a summed length of 268.374. Let $d_\mathcal{A}(Z)$ be the sum over points $P \in \mathcal{A}$ of the distance from $P$ to $Z$.

Then we have

$$\begin{aligned} d_{\mathcal{A}}(J_1) &= 254.103, \\ d_{\mathcal{A}}(K_1) &= 264.081, \\ d_{\mathcal{A}}(A_1) &= 268.349, \\ d_{\mathcal{A}}(T) &= 268.349, \text{ and} \\ d_{\mathcal{A}}(O_1) &= 264.081. \end{aligned}$$

Since all five of the above values are less than 268.374, any point added within the convex hull of $\bar{\mathcal{A}} = \{J_1, K_1, A_1, T, O_1\}$ will indeed reduce the length of the minimum weight triangulation.

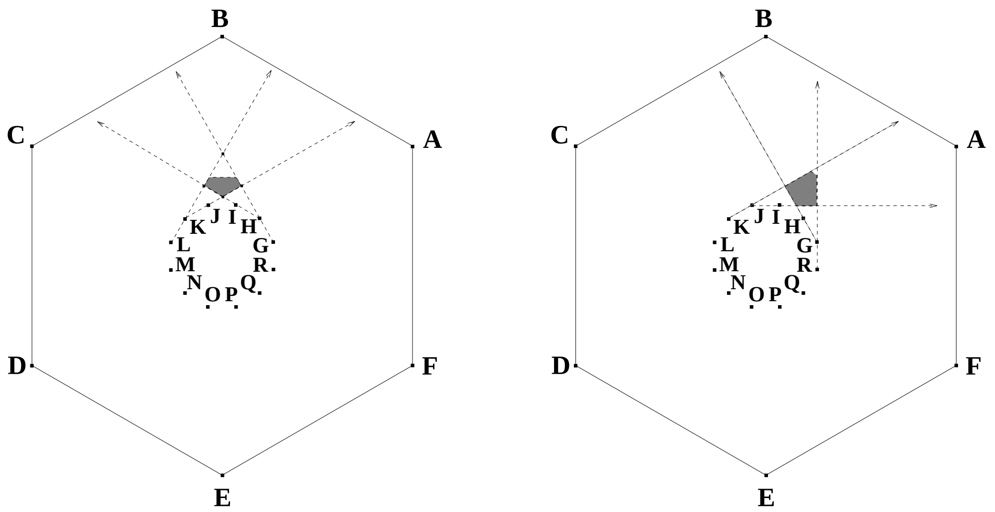


Figure 7: Regions 1 and 2 and their location within $\mathbb{P}$.

We define region 2 to be the bounded chamber formed by lines $GH, IJ, GR$, and $JK$. (See right side of Figure 7.) Let

$$\begin{aligned} B_1 &= JK \cap (y = -0.58307x + 41.77457) \approx (16.77621, 31.99285), \\ C_1 &= GR \cap (y = -0.58307x + 41.77457) \approx (19.31852, 30.51051), \\ K_1 &= GH \cap JK \approx (7.07107, 26.38958), \\ R_1 &= GH \cap IJ \approx (11.15355, 19.31852), \text{ and} \\ S_1 &= GR \cap IJ \approx (19.31852, 19.31852). \end{aligned}$$

We claim that the convex hull of $\bar{\mathcal{B}} = \{B_1, C_1, K_1, R_1, S_1\}$ is a reducing region when a new point $Z$ is connected to points in $\mathcal{B} := \{A, B, G, H, I, J\}$. The edges $ZA, ZB, ZG, ZH, ZI, ZJ$ will replace edges $AH, AI, BI$ from the original triangulation, which have a summed length of 198.079. Let $d_{\mathcal{B}}(Z)$ be the sum over points $P \in \mathcal{B}$ of the distance from $P$ to $Z$.

Then we have

$$\begin{aligned} d_{\mathcal{B}}(B_1) &= 197.124, \\ d_{\mathcal{B}}(C_1) &= 197.097, \\ d_{\mathcal{B}}(K_1) &= 183.697, \\ d_{\mathcal{B}}(R_1) &= 173.916, \text{ and} \\ d_{\mathcal{B}}(S_1) &= 183.697. \end{aligned}$$

Since all five of the above values are less than 198.079, any point added within the convex hull of $B_1, C_1, K_1, R_1, S_1$ will indeed reduce the length of the minimum weight triangulation.

We define region 3 to be the bounded chamber formed by lines $GH, KL, GR$, and $JK$. (See left side of Figure 8.) Let

$$\begin{aligned} D_1 &= KL \cap (y = -0.24958x + 41.81550) \approx (1.60397, 41.41519), \\ E_1 &= GR \cap (y = -0.24958x + 41.81550) \approx (19.31852, 36.99399), \\ K_1 &= GH \cap JK \approx (7.07107, 26.38958), \\ P_1 &= GH \cap KL \approx (0.00000, 38.63703), \text{ and} \\ Q_1 &= GR \cap JK \approx (19.31852, 33.46065). \end{aligned}$$

We claim that the convex hull of $D_1, E_1, K_1, P_1, Q_1$ is a reducing region when a new point $Z$ is connected to points in $\mathcal{C} := \{A, B, G, H, I, J, K\}$. That set of edges will replace the following edges from the original triangulation: $AH, AI, BI, BJ$. The summed length of those four edges is 261.971. Let $d_{\mathcal{C}}(Z)$ be the sum over points $P \in \mathcal{C}$ of the distance from $P$ to $Z$.

Then we have

$$\begin{aligned} d_{\mathcal{C}}(D_1) &= 259.236, \\ d_{\mathcal{C}}(E_1) &= 251.262, \\ d_{\mathcal{C}}(K_1) &= 208.192, \\ d_{\mathcal{C}}(P_1) &= 251.505, \text{ and} \\ d_{\mathcal{C}}(Q_1) &= 241.551. \end{aligned}$$

Since all five of the above values are less than 261.971, any point added within the convex hull of $D_1, E_1, K_1, P_1, Q_1$ will indeed reduce the length of the minimum weight triangulation.

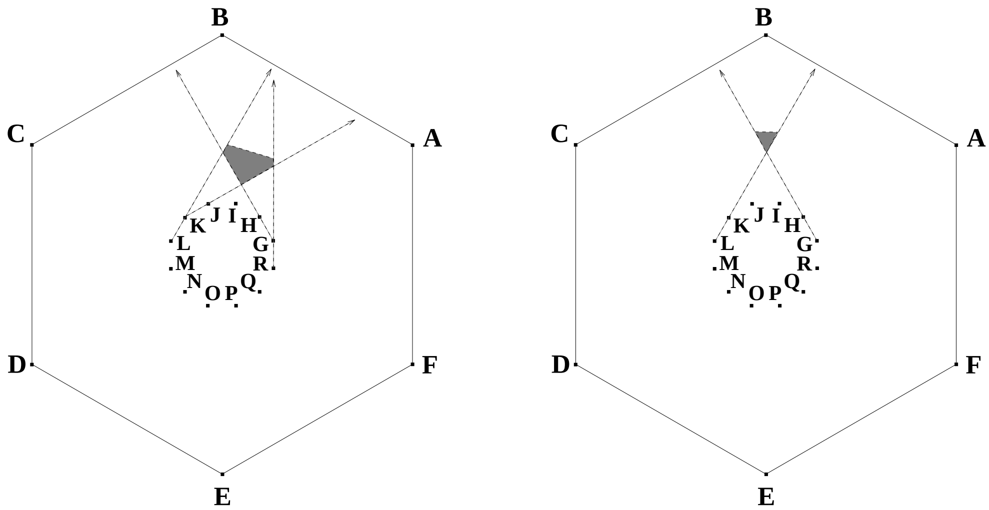


Figure 8: Regions 3 and 4 and their location within $\mathbb{P}$.

We define region 4 to be the bounded chamber formed by lines $KL, GH$ and convex hull edges $AB, BC$. (See right side of Figure 8.) Let

$$\begin{aligned} F_1 &= GH \cap (y = 44.6) \approx (-3.12136, 44.6), \\ G_1 &= KL \cap (y = 44.6) \approx (3.12136, 44.6), \text{ and} \\ P_1 &= GH \cap KL \approx (0.00000, 38.63703). \end{aligned}$$

We claim that the convex hull of $F_1, G_1, P_1$ is a reducing region when a new point $Z$ is connected to points in $\mathcal{D} := \{A, B, C, G, H, I, J, K, L\}$. That set of edges will replace the following edges from the original triangulation: $AH, AI, BI, BJ, BK, CK$. The summed length of those six edges is 396.158. Let $d_{\mathcal{D}}(Z)$ be the sum over points $P \in \mathcal{D}$ of the distance from $P$ to $Z$.

Then we have

$$\begin{aligned} d_{\mathcal{D}}(F_1) &= 389.779, \\ d_{\mathcal{D}}(G_1) &= 389.779, \text{ and} \\ d_{\mathcal{D}}(P_1) &= 362.079. \end{aligned}$$

Since all three of the above values are less than 396.158, any point added within the convex hull of $F_1, G_1, P_1$ will indeed reduce the length of the minimum weight triangulation.

We define region 5 to be the bounded chamber formed by lines $JK, GR$ and convex hull edge $AB$. Let

$$\begin{aligned} H_1 &= GR \cap (y = -0.56463x + 50.38075) \approx (19.31852, 39.47297), \\ I_1 &= JK \cap (y = -0.56463x + 50.38075) \approx (24.58335, 36.50030), \text{ and} \\ Q_1 &= JK \cap GR \approx (19.31852, 33.46065). \end{aligned}$$

We claim that the convex hull of $H_1, I_1, Q_1$ is a reducing region when a new point $Z$ is connected to points in $\mathcal{E} := \{A, B, G, H, I, J, K, R\}$. That set of edges will replace the following edges from the original triangulation: $AG, AH, AI, BI, BJ$. The summed length of those five edges is 325.863. Let $d_{\mathcal{E}}(Z)$ be the sum over points $P \in \mathcal{E}$ of the distance from $P$ to $Z$.

Then we have

$$\begin{aligned} d_{\mathcal{E}}(H_1) &= 303.332, \\ d_{\mathcal{E}}(I_1) &= 303.605, \text{ and} \\ d_{\mathcal{E}}(Q_1) &= 280.188. \end{aligned}$$

Since all three of the above values are less than 325.863, any point added within the convex hull of $H_1, I_1, Q_1$ will indeed reduce the length of the minimum weight triangulation.

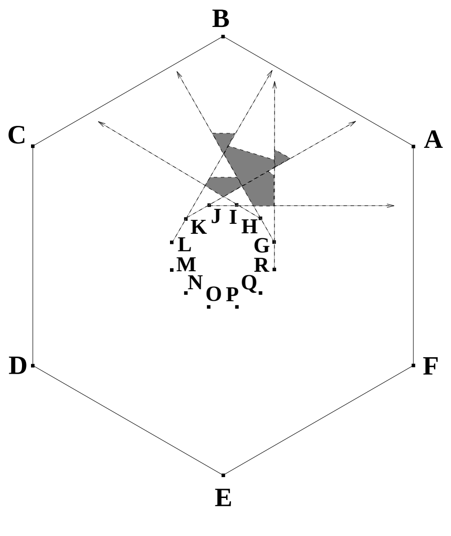


Figure 9: Regions 1 through 5 and their locations within $\mathbb{P}$.

We have now established a reducing region that appears to be connected but not simply connected. We now proceed to prove the existence of 13 holes within this reducing region. We will do so by finding points in the interior of the holes that do not reduce, combined with polygonal reducing paths around the holes.

**Claim 6** *The point* $G_2 = (0.00000, 35.08709)$ *will not reduce.*

**Proof**: We first must establish the minimum weight triangulation of $\mathbb{P} \cup \{G_2\}$, and then we will calculate the length of that triangulation. We claim that the minimum weight triangulation connects $G_2$ to points $A, B, C, H, I, J, K$.

Note that the use of edge $AC$ would imply that $\triangle ABC$ and $\triangle ACG_2$ are both in the minimum weight triangulation, with the latter triangle forced by visibility. Edge $BG_2$ is shorter than edge $AC$, a contradiction to minimality. Thus edge $AC$ will not be used in this minimum weight triangulation.

We claim that edge $IJ$ must be in the minimum weight triangulation. Otherwise, an edge from $G_2$ must cross it, and there is one type of such edge up to symmetry, edge $PG_2$. The inclusion of this edge forces triangle $\triangle PIG_2$ to be in the triangulation, as well as one of $\triangle PJG_2, \triangle POG_2$. In the case where $\triangle PJG_2$ is used, we have $PJG_2I$ using $PG_2$ instead of the shorter $IJ$. In the case where $\triangle POG_2$ is used, we have $POG_2I$ using $PG_2$ instead of the shorter $IO$. Thus it follows that edge $IJ$ must be included in the new minimum weight triangulation.

The edge $IJ$ can connect to two possible points, up to symmetry: $G_2$ and $A$. If $IJ$ connects to $A$, then $AJ$ is forced by visibility to connect to $G_2$. This implies that the shorter edge $G_2I$ should have been used instead of $AJ$. Thus the triangle $\triangle IJG_2$ is in the minimum weight triangulation.

Edge $G_2I$ can connect to $A, B$, or $H$. If we connect it to $A$, then we have $G_2AI$ in the minimum weight triangulation, and edge $G_2A$ must connect to $B$. (It cannot connect to $C$ by an earlier comment above.) If $G_2A$ connects to $B$, then the shorter edge $BI$ should have been used instead of $G_2A$. Thus $G_2AI$ is not in the minimum weight triangulation. If we connect $B$ to $G_2I$, then $BI$ must connect to $A$ or $H$. Connecting $BI$ to $A$ implies the use of $\triangle AHI$, which puts us in an interesting position. Now, trapezoid $ABIH$ can be triangulated with either $BH$ or the equal-length $AI$. If we flip edge $AI$ to $BH$, then we are back in the above situation of using $\triangle BHI$, which gave us a contradiction. Thus we cannot connect $G_2I$ to $B$, so we must attach it to $H$ and include $\triangle HIG_2$ in the minimum weight triangulation.

Edge $HG_2$ can connect to $B$ or to $A$. If it connects to $A$, then edge $G_2A$ must connect to $B$, but we note that $AG_2 > BH$, so we should have used $BH$ instead of $AG_2$. Thus triangle $\triangle AHG_2$ does not belong to the minimum weight triangulation, but $\triangle BHG_2$ will be in the minimum weight triangulation. Moreover, edge $BH$ belonged to an original minimum weight triangulation.

Edge $BG_2$ can connect to $C, J$, or $K$. If we connect to $C$ and form triangle $\triangle BG_2C$, then edge $G_2C$ can connect to $J, K$, or $D$. If $G_2C$ connects to $J$, then we should have used the shorter $BJ$ instead of $G_2C$. If $G_2C$ connects to $K$, then we should have used the shorter $BK$ instead of $G_2C$. If we connect $G_2C$ to $D$, this forces $\triangle G_2DK$, which implies we should have used the shorter $CK$ as opposed to $G_2D$. So we should not use triangle $\triangle BCG_2$. If we connect $BG_2$ to $J$, then we find ourselves considering connecting edge $BJ$ to one of points $C$ or $K$, which is a case symmetric to our consideration of connecting edge $BI$ to $A$ or $H$. Recall from arguments above that both of those choices led to contradictions. Thus we are forced to include triangle $\triangle BG_2K$ in our minimum weight triangulation. Note this also implies that triangle $\triangle JKG_2$ is in our triangulation.

Now we notice that edges $BK$ and $BH$ are both included in a minimum weight triangulation of our original point set. Therefore our previous work tells us how to triangulate the rest of the point set. We may now consider the length of this new triangulation. We compare the length of the new edges within the non-convex pentagon $BHIJK$ to the length of the edges that originally triangulated $BHIJK$. The new edges are $G_2B, G_2H, G_2I, G_2J$, and $G_2K$, and these have a summed length of 154.2164. They replace edges $BI$ and $BJ$, which have a summed length of 127.78. Therefore the addition of point $G_2$ does *not* reduce the length of the minimum weight triangulation, as desired. ■

We now note that there will actually be a small neighborhood around point $G_2$ in which no point will reduce.

**Lemma 7** *If a point $p = (x, y)$ in the interior of a visibility region does not reduce the length of the minimum weight triangulation, then there will be a small open neighborhood around that point in which no point will reduce the length of the minimum weight triangulation.*

**Proof**: Since $p$ is in the interior of the visibility region, there must be a ball $B(p, \delta)$ of radius $\delta$ around $p$, such that all points inside of $B(p, \delta)$ can be connected to the same set of points to which $p$ may be legally connected. An arbitrary point $q$ within $B(p, \delta)$ may or may not give rise to the same combinatorial type of minimum weight triangulation as the addition of $p$ would imply. We know that distance is a continuous function, as is the sum of multiple distance functions. It follows that the length of the minimum weight triangulation cannot change too drastically within $B(p, \delta)$. Specifically, there must exist an $\epsilon \leq \delta$ such that no point within $B(p, \epsilon)$ will reduce the length of the triangulation. ■

The following corollary follows directly from the above lemma, and the fact that $G_2$ is contained entirely inside a visibility region.

**Corollary 8** *There is a non-reducing neighborhood around point $G_2$.*

We now work to establish a reducing polygonal path around this hole. We rely on lemma (CONVEXITY LEMMA) to build this path. If we can find two points which reduce, then the segment between them will also reduce.

**Claim 9** *The boundary of the triangle formed by points $M_2 = (-6.1021, 28.79429), O_2 = (0, 40.61712)$, and $L_2 = (6.1021, 28.79429)$ will reduce.*

**Proof**: We must show that the points on segments $M_2L_2, M_2O_2$, and $L_2O_2$ all reduce. We note that segment $M_2O_2$ is symmetric to segment $L_2O_2$, so we only have to work to show that two segments reduce.

For a point on the segment $M_2O_2$, we claim that connecting that point to the points of $\mathcal{F} = \{B, C, H, I, J, K, L\}$ will give a reduction in the length of the triangulation. Let $d_{\mathcal{F}}(Z)$ be the sum over points $P \in \mathcal{F}$ of the distance from $P$ to $Z$. We have $d_{\mathcal{F}}(M_2) = 214.5609$ and $d_{\mathcal{F}}(O_2) = 258.501$. We note that connecting our new point ($M_2$ or $O_2$) to the points of $\mathcal{F}$ replaces the edges $CK, BK, BJ, BI$ and forces edge $AI$ to flip to $BH$,an edge of equal length. We are replacing edges from our original triangulation that have summed length $(3 \cdot 63.8915) + 70.2951 = 261.9696$. Therefore both $M_2$ and $O_2$ reduce with this combinatorial type of triangulation, and so must all points on the edge $M_2O_2$ between them. By symmetry, all points on the edge $L_2O_2$ will also reduce.

For a point on the segment $M_2L_2$, we claim that connecting to the points of $\mathcal{A} = \{A, B, C, H, I, J, K\}$ will give a reduction in the length of the triangulation. This will replace edges $AI, BI, BJ, BK$ from the original triangulation, which together have summed length $(2 \cdot 63.8915) + (2 \cdot 70.2951) = 268.3732$. Once again, we let $d_{\mathcal{A}}(Z)$ be the sum over points $P \in \mathcal{A}$ of the distance from $P$ to $Z$. We see that $d_{\mathcal{A}}(M_2) = 266.5075$, and by symmetry, $d_{\mathcal{A}}(L_2) = 266.5075$. Note that this is because

$$\begin{aligned} \text{dist}(A, M_2) &= \text{dist}(C, L_2), \\ \text{dist}(B, M_2) &= \text{dist}(B, L_2), \\ \text{dist}(C, M_2) &= \text{dist}(A, L_2), \\ \text{dist}(H, M_2) &= \text{dist}(K, L_2), \\ \text{dist}(I, M_2) &= \text{dist}(J, L_2), \\ \text{dist}(J, M_2) &= \text{dist}(I, L_2), \text{ and} \\ \text{dist}(K, M_2) &= \text{dist}(H, L_2). \end{aligned}$$

It follows that both $M_2$ and $L_2$ reduce with this combinatorial type of triangulation, and so must all points on the edge $M_2L_2$ between them.

Thus the boundary of triangle $\triangle M_2L_2O_2$ will reduce as desired. ■

**Claim 10** *The point $H_2 = (18.47521, 32.00000)$ will not reduce.*

**Proof**: We first must establish the minimum weight triangulation of $\mathbb{P} \cup \{H_2\}$, and then we will calculate the length of that triangulation. (FINISH TYPING THIS PART.)

**Claim 11** *The boundary of the triangle formed by points $P_2 = (20.40390, 26.69670)$, $Q_2 = (13.15766, 31.08258)$, and $R_2 = (20.40390, 35.27778)$ will reduce.*

**Proof**: As above, we must show that the points on segments $P_2Q_2, P_2R_2$, and $Q_2R_2$ all reduce. It will suffice to show that, pairwise, the endpoints of those segments will reduce with the same combinatorial type.

For a point on the segment $P_2Q_2$, we claim that connecting that point to the points of $\mathcal{B} = \{A, B, G, H, I, J\}$ will give a reduction in the length of the triangulation. As before, let $d_{\mathcal{B}}(Z)$ be the sum over points $P \in \mathcal{B}$ of the distance from $P$ to $Z$. Then we have $d_{\mathcal{B}}(P_2) = 192.795$ and $d_{\mathcal{B}}(Q_2) = 192.570$. We note that connecting our new point ($P_2$ or $Q_2$) to the points of $\mathcal{B}$ replaces the edges $AH, AI, BI$. We are thus replacing edges from our original triangulation that have summed length $(2 \cdot 63.8915) + 70.2951 = 198.079$. Therefore both $P_2$ and $Q_2$ reduce with this combinatorial type of triangulation, and so must all points on the edge $P_2Q_2$ between them.

For a point on the segment $P_2R_2$, we claim that connecting that point to the points of $\mathcal{G} = \{A, B, G, H, I, J, R\}$ will give a reduction in the length of the triangulation. We let $d_{\mathcal{G}}(Z)$ be the sum over points $P \in \mathcal{G}$ of the distance from $P$ to $Z$. Then we have $d_{\mathcal{G}}(P_2) = 224.4615$ and $d_{\mathcal{G}}(R_2) = 248.5943$. We note that connecting our new point ($P_2$ or $R_2$) to the points of $\mathcal{G}$ replaces the edges $AG, AH, AI, BI$. We are thus replacing edges from our original triangulation that have summed length $(3 \cdot 63.8915) + \cdot 70.2951 = 261.971$. Therefore both $P_2$ and $R_2$ reduce with this combinatorial type of triangulation, and so must all points on the edge $P_2R_2$ between them.

For a point on the segment $Q_2R_2$, we claim that a reduction in the length of the triangulation can be obtained by connecting to the points of $\mathcal{H} = \{A, B, G, H, I, J, K\}$. We let $d_{\mathcal{H}}(Z)$ be the sum over points $P \in \mathcal{H}$ of the distance from $P$ to $Z$. Then we have $d_{\mathcal{H}}(Q_2) = 224.924$ and $d_{\mathcal{H}}(R_2) = 248.6243$. We note that connecting our new point ($Q_2$ or $R_2$) to the points of $\mathcal{H}$ replaces the edges $AH, AI, BI, BJ$. We are thus replacing edges from our original triangulation that have summed length $(3 \cdot 63.8915) + 70.2951 = 261.971$. Therefore both $Q_2$ and $R_2$ reduce with this combinatorial type of triangulation, and so must all points on the edge $Q_2R_2$ between them.

Thus the boundary of triangle $\triangle P_2Q_2R_2$ will reduce as desired. ∎

The symmetry of our point set now grants us 12 distinct holes. We now aim for the lucky 13$^{\text{th}}$ hole in the center of our configuration.

**Claim 12** *A point added in the center of the 12-gon will not reduce.*

**Proof**: Let $Z = (0, 0)$ be the point in the center of the 12-gon. We now make some claims about which edges will not be included in the triangulation.

First, we assume towards a contradiction that edge $GL$ is in the minimum weight triangulation. This implies that one of $\triangle GHL$ or $\triangle GIL$ is included in the minimum triangulation. If $\triangle GHL$ is included, then edge $HL$ belongs to another triangle - one of $\triangle HIL, \triangle HJL$, or $\triangle HKL$. The combination of $\triangle GHL$ and $\triangle HIL$ gives $GHIL$ triangulated by $HL$, which is longer than $GI$. The combination of $\triangle GHL$ and $\triangle HJL$ gives $GHJL$ triangulated by $HL$, which is longer than $GJ$. If $\triangle HKL$ is used, then one of $\triangle HIK$ or $\triangle HJK$ is used. These cases are equivalent up to symmetry, so assume $\triangle HIK$ is used. Notice that $HL + HK + IK < GI + IL + IK$ in the triangulation of hexagon $GHIJKL$. It follows that edge $GL$ must not belong to the minimum weight triangulation, nor may any edge symmetric to $GL$, such as $HM, IN, JO, KP$, etc.

Now we assume towards a second contradiction that edge $GJ$ is in the minimum weight triangulation. Then one of $GI, HJ$ is also in the minimum weight triangulation, but these cases are symmetric. Without loss of generality, we will say that $GI$ (and therefore $\triangle GIJ$) is in the minimum weight triangulation. Now we look at possible triangles that use edge $GJ$ : $\triangle GJK, \triangle GJL, \triangle GJZ, \triangle GJQ$, and $\triangle GJR$, and we detail the contradictions these triangles create. Quadrilateral $GIJK$ is triangulated by $GJ$ instead of the shorter $IK$. The combination of $\triangle GIJ$ and $\triangle GJZ$ uses $GJ$ instead of the shorter $IZ$. Next, we note that $\triangle GJL$ forces $\triangle GLZ$, and then quadrilateral $GJLZ$ uses $GL$ instead of the shorter $JZ$. Similarly, $\triangle GJQ$ forces $\triangle JQZ$, and then quadrilateral $GJQZ$ uses $JQ$ instead of the shorter $GZ$. Lastly, if we use $\triangle GJR$, then pentagon $GHIJR$ should be triangulated by $HJ$ and $HR$ instead of the longer pair $GI$ and $GJ$. It follows that edge $GJ$ is not used in the minimum weight triangulation, nor is any edge symmetric to $GJ$.

We note that edges $GZ, HZ, IZ$, etc. have the same length as edges $GI, HJ, IK$, etc. No shorter edges exist in the interior of the 12-gon. Thus if a triangulation exists which uses only edges of that length, its triangulation length must be minimal. For this example, many such triangulations exist. Two such triangulations are:

$$\Lambda : \; = \; \{GI, IK, KM, MO, OQ, GQ, GZ, IZ, KZ, MZ, OZ, QZ\} \text{ and}$$
$$\Upsilon : \; = \; \{GZ, HZ, IZ, JZ, KZ, LZ, MZ, NZ, OZ, PZ, QZ, RZ\}. \quad \blacksquare$$

**Claim 13** *The boundary of the 12-gon formed by symmetric copies of $T_2U_2$, where $T_2 = (0, 30)$ and $U_2 = (14.56088, 25.22019)$ will reduce.*

**Proof**: It will suffice to show that edge $T_2U_2$ reduces, then the reduction of the 12-gon will follow by symmetry. It may be necessary to employ a third point $Y_2 = (7.28044, 27.61009)$, the midpoint of $T_2U_2$. The hope is that $T_2$ and $Y_2$ will both reduce using the connectivity of region 1, and that $U_2$ and $Y_2$ will both reduce using the connectivity of region 2.

Recall that points in region 1 reduced by connecting to $\mathcal{A} = \{A, B, C, H, I, J, K\}$. We have $d_{\mathcal{A}}(T_2) = 264.8235$ and $d_{\mathcal{A}}(Y_2) = 266.2503$. The length of edges we replace is 268.374, so the segment $T_2Y_2$ reduces.

Now recall that points in region 2 reduced by connecting to $\mathcal{B} := \{A, B, G, H, I, J\}$. We have $d_{\mathcal{B}}(Y_2) = 186.0355$. and $d_{\mathcal{B}}(U_2) = 182.5459$. The length of edges we replace when using this combinatorial type is 198.079, so segment $Y_2U_2$ reduces.

It follows that segment $T_2U_2$ reduces, and therefore there is a reducing 12-sided closed path around the interior 12 points of our original point set $\mathbb{P}$. $\blacksquare$