DIMACS Series in Discrete Mathematics
and Theoretical Computer Science
Volume 6, 1991

Ehrhart Polynomials of Convex Polytopes, h-Vectors of Simplicial Complexes, and Nonsingular Projective Toric Varieties

TAKAYUKI HIBI

ABSTRACT. We develop the theory of combinatorics on Ehrhart polynomials of convex polytopes by means of some fundamental results on Cohen-Macaulay rings and nonsingular projective toric varieties.

Introduction

Let $\mathscr{P} \subset \mathbb{R}^N$ be a *rational* convex polytope, i.e., a convex polytope, any of whose vertices has rational coordinates. Given a positive integer n we write $i(\mathscr{P}, n)$ for the number of those rational points $(\alpha_1, \alpha_2, \ldots, \alpha_N)$ in $\mathscr{P}$ such that each $n\alpha_i$ is an integer. In other words,

$$i(\mathscr{P}, n) := \#(n\mathscr{P} \cap \mathbb{Z}^N).$$

Here $n\mathscr{P} := \{n\alpha; \alpha \in \mathscr{P}\}$ and $\#(X)$ is the cardinality of a finite set X.

Even though the history of the research on enumeration of certain rational points in convex polytopes goes back to the nineteenth century, the systematic study of $i(\mathscr{P}, n)$ originated in the work of Ehrhart (who was a teacher in a *lycée*) beginning around 1955. The monograph [**Ehr1**] is an exposition of Ehrhart's research over a period of many years. Since Ehrhart built up the foundation on $i(\mathscr{P}, n)$, this interesting topic has been studied by, e.g., Macdonald [**Mac1**, **Mac2**], McMullen [**Mc1**, **Mc2**], and Stanley [**Sta4**].

Nowadays, the technique of commutative algebra and algebraic geometry is recognized as one of the basic and powerful tools for the study of combinatorics. Consult, e.g., [**Hoc2, Rei, Sta2, Sta5, Sta9, Sta11, Sta13, Bil, B-R, H1**]. Such algebraic technique can be also applied to the investigation of $i(\mathscr{P}, n)$. In particular, the theory of canonical modules [**Sta3**] of Cohen-Macaulay rings generated by monomials [**Hoc1**] plays an important role in our study of $i(\mathscr{P}, n)$.

1991 *Mathematics Subject Classification*. Primary 05E25; Secondary 13H10.

The purpose of this paper is to invite the reader to a short tour for a survey of recent development on $i(\mathscr{P}, n)$ with some concrete problems that might stimulate further study of the topic. Our treatment will be rather sketchy; in a subsequent paper, a more comprehensive account will be given.

Our main research object is the functions $i(\mathscr{P}, n)$ of *integral* convex polytopes $\mathscr{P}$ (i.e., convex polytopes $\mathscr{P}$ such that each vertex of $\mathscr{P}$ has integer coordinates). Ehrhart established that, when $\mathscr{P}$ is integral, the function $i(\mathscr{P}, n)$ possesses the following fundamental properties:

(0.1) $i(\mathscr{P}, n)$ is a polynomial in n of degree $d(=\dim\mathscr{P})$. (Thus $i(\mathscr{P}, n)$ can be defined for *every* integer n.)

(0.2) $i(\mathscr{P}, 0) = 1$.

(0.3) ("loi de réciprocité" **[Ehr2]**) $(-1)^d i(\mathscr{P}, -n) = \#(n(\mathscr{P} - \partial\mathscr{P}) \cap \mathbb{Z}^N)$ for every integer $n > 0$.

We say that $i(\mathscr{P}, n)$ is the *Ehrhart polynomial* of $\mathscr{P}$. Consult, e.g., **[Sta8, pp. 235–241]** for an introduction to Ehrhart polynomials.

We organize this paper as follows. First, in §1, we define a certain combinatorial sequence $\delta(\mathscr{P}) = (\delta_0, \delta_1, \ldots, \delta_d) \in \mathbb{Z}^{d+1}$, called the *δ-vector* of $\mathscr{P}$, arising from the generating function for $i(\mathscr{P}, n)$ of an integral convex polytope $\mathscr{P}$ of dimension d (see equation (1)). We consider what can be said about the δ-vector of an arbitrary integral convex polytope (cf. Theorem (1.3)) and then turn to the problem of finding integral convex polytopes that possess symmetric δ-vectors (cf. Theorem (1.4)). On the other hand, the purpose of §2 is to discuss a relation between $\delta(\mathscr{P})$ and the h-vector $h(\Delta)$ of some triangulation Δ of the boundary $\partial\mathscr{P}$ of $\mathscr{P}$. Finally, in §3, we study a class of integral convex polytopes that are related with finite partially ordered sets (cf. equation (7)). Via the theory of nonsingular projective toric varieties (e.g., **[Sta5,]**), we prove that certain combinatorial sequences arising from enumeration on linear extensions of finite partially ordered sets are unimodal (Corollary (3.4)).

The author would like to thank Professor Richard P. Stanley for exciting discussions on $i(\mathscr{P}, n)$ and some related topics while the author was staying at Massachusetts Institute of Technology during the 1988–89 academic year.

1. Ehrhart polynomial

Let $\mathscr{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d and $\partial\mathscr{P}$ the boundary of $\mathscr{P}$. We define the sequence $\delta_0, \delta_1, \delta_2, \ldots$ of integers by the formula

$$(1) \qquad (1-\lambda)^{d+1}\left[1 + \sum_{n=1}^{\infty} i(\mathscr{P}, n)\lambda^n\right] = \sum_{i=0}^{\infty} \delta_i \lambda^i .$$

Then, thanks to the basic facts (0.1) and (0.2) on $i(\mathscr{P}, n)$, a fundamental result on generating functions, e.g., **[Sta8, Corollary 4.3.1]** guarantees that $\delta_i = 0$ for every $i > d$. When $\mathscr{P} \subset \mathbb{R}^N$ is an integral convex polytope of dimension d, we say that the sequence $\delta(\mathscr{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$, which appears in equation (1), is the *δ-vector* of $\mathscr{P}$. Thus, in particular,

$$\delta_0 = 1 \quad \text{and} \quad \delta_1 = \#(\mathscr{P} \cap \mathbb{Z}^N) - (d+1). \tag{2}$$

One of the fundamental results on δ-vectors of integral convex polytopes obtained earlier is the following

(1.1) PROPOSITION ([**Sta4, Theorem 2.1**]). *The δ-vector* $\delta(\mathscr{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ *of an integral convex polytope* $\mathscr{P} \subset \mathbb{R}^N$ *of dimension* d *is nonnegative, i.e.,* $\delta_i \geq 0$ *for every* $0 \leq i \leq d$.

Are these True for rational polytopes?

We refer the reader to [**Sta4, p. 337**] for a historical comment on the above Proposition (1.1). Also, see [**B-M, p. 254**].

On the other hand, it follows easily from (0.3) that

$$\delta_d = \#((\mathscr{P} - \partial\mathscr{P}) \cap \mathbb{Z}^N). \tag{3}$$

Moreover, when $N = d$, the leading coefficient of $i(\mathscr{P}, n)$ coincides with the volume (= Lebesgue measure) $\mathrm{vol}(\mathscr{P})$ of $\mathscr{P}$ (cf. [**Sta8, Proposition 4.6.30**]), i.e.,

$$(\delta_0 + \delta_1 + \cdots + \delta_d)/d! = \mathrm{vol}(\mathscr{P}). \tag{4}$$

Our final goal for the study of Ehrhart polynomials is to find a complete (combinatorial) characterization of the δ-vectors of integral convex polytopes.

(1.2) EXAMPLE. Let $N = d = 3$ and $q > 0$ an integer. Also, let $\mathscr{P} \subset \mathbb{R}^3$ be the tetrahedron with the vertices $(0, 0, 0)$, $(q, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$. Then $\#(\mathscr{P} \cap \mathbb{Z}^3) = 4$ and $(\mathscr{P} - \partial\mathscr{P}) \cap \mathbb{Z}^3$ is empty. Also, the volume of $\mathscr{P}$ is $(q+1)/3!$. Hence, by (2), (3), and (4), we have $\delta(\mathscr{P}) = (1, 0, q, 0)$; thus, $i(\mathscr{P}, n) = ((q+1)n^3 + 6n^2 + (11-q)n + 6)/3!$.

When $d = 2$, thanks to [**Sco**], we can give a complete characterization of the δ-vectors of integral convex polytopes. In fact, the δ-vectors arising from integral convex polytopes of dimension 2 are the following: (i) $(1, n, 0)$, $n \geq 0$, (ii) $(1, n, 1)$, $1 \leq n \leq 7$, and (iii) $(1, n, m)$, $2 \leq m \leq n \leq 3m+3$.

Now, what can be said about the δ-vector of an arbitrary integral convex polytope?

(1.3) THEOREM. *Suppose that* $\mathscr{P} \subset \mathbb{R}^N$ *is an integral convex polytope of dimension* d *with the δ-vector* $\delta(\mathscr{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$.

(a) ([**H5, Theorem A**]) *We have the linear inequality*

$$\delta_d + \delta_{d-1} + \cdots + \delta_{d-i} \leq \delta_0 + \delta_1 + \cdots + \delta_i + \delta_{i+1} \tag{5}$$

for every $0 \leq i \leq [(d-1)/2]$.

(b) ([**Sta14, Proposition 4.1**]) *Assume that* $\delta_i \neq 0$ *and* $\delta_{i+1} = \delta_{i+2} = \cdots = \delta_d = 0$. *Then the inequality*

$$(6) \qquad \delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_j + \delta_{j-1} + \cdots + \delta_{j-i}$$

holds for every $0 \leq i \leq [j/2]$.

The above inequalities (5) and (6) in Theorem (1.3) are typical examples of the combinatorial consequences of some recent algebraic works toward the problem of finding a combinatorial characterization of the Hilbert functions of Cohen-Macaulay integral domains. See also **[H2]**.

We give here a sketch of proof of Theorem (1.3) for the reader who is familiar with the theory of canonical modules of Cohen-Macaulay rings. Consult **[Sta14]** and **[H9]** for the detailed information. First, let $X_1, X_2, \ldots, X_N$ and T be indeterminates over a field k. We write $A(\mathscr{P})_n$ for the vector space over k spanned by those monomials $X_1^{\alpha_1} \cdots X_N^{\alpha_N} T^n$ such that $(\alpha_1, \ldots, \alpha_N) \in n\mathscr{P} \cap \mathbb{Z}^N$. Also, set $A(\mathscr{P})_0 = k$. Then the direct sum $A(\mathscr{P}) := \bigoplus_{n \geq 0} A(\mathscr{P})_n$ of $\{A(\mathscr{P})_n\}_{n=0,1,2,\ldots}$ as a vector space over k turns out to be a noetherian graded ring of Krull dimension $d+1$ with the *Hilbert function* $H(A(\mathscr{P}), n) := \dim_k A(\mathscr{P})_n = i(\mathscr{P}, n)$. Now, Hochster **[Hoc1]** guarantees that $A(\mathscr{P})$ is a Cohen-Macaulay ring. Moreover, by virtue of **[Sta3, equation (21), p. 82]**, we can describe, explicitly, a graded ideal $I = \bigoplus_{n \geq 1}(I \cap A(\mathscr{P})_n) \subset A(\mathscr{P})_+ := \bigoplus_{n \geq 1} A(\mathscr{P})_n$ of $A(\mathscr{P})$ with $I \cong K_{A(\mathscr{P})}$. Here $K_{A(\mathscr{P})}$ is the canonical module of the Cohen-Macaulay ring $A(\mathscr{P})$. Let $\rho := \min\{n : I \cap A(\mathscr{P})_n \neq (0)\} (> 0)$ and $0 \neq a \in I \cap A(\mathscr{P})_\rho$. Then we have the exact sequence $(^{**})$ $0 \to A(\mathscr{P}) \to I \to I/aA(\mathscr{P}) \to 0$ since $A(\mathscr{P})$ is an integral domain. A fundamental fact **[H-K]** (see also **[H1, Lemma (1.7)]**) in the theory of canonical modules of Cohen-Macaulay rings implies that $A(\mathscr{P})/I$ is a Cohen-Macaulay ring of dimension d and that $I/aA(\mathscr{P})$ is a Cohen-Macaulay module of dimension d over $A(\mathscr{P})$. On the other hand, thanks to **[Sta3, equation (12), p. 71]**, we can compute the Hilbert functions of $A(\mathscr{P})/I$ and $I/aA(\mathscr{P})$ by means of the δ-vector of $\mathscr{P}$. Then the required inequalities (5) and (6) follow immediately from a standard (and well-known) fact, e.g., **[Sta3, Corollary 3.11]** on Hilbert functions of Cohen-Macaulay rings and modules. Q.E.D.

Note that, in the above proof, the noetherian graded ring $A(\mathscr{P})$ is not necessarily generated by $A(\mathscr{P})_1$. However, there exists a system of parameters for $A(\mathscr{P})$ consisting of elements of $A(\mathscr{P})_1$ (if k is infinite). It might be of interest to ask when $A(\mathscr{P})$ is generated by $A(\mathscr{P})_1$.

At present, there is little hope of giving a purely combinatorial proof for Theorem (1.3).

We now study an analogue of the Dehn-Sommerville equations of the h-vectors of simplicial convex polytopes (see, e.g., **[B-L, Sta5]**) for δ-vectors of integral convex polytopes.

In general, we say that a convex polytope $\mathscr{P}$ of dimension d is *of standard type* if $\mathscr{P} \subset \mathbb{R}^d$ and the origin of $\mathbb{R}^d$ is contained in the interior $\mathscr{P} - \partial\mathscr{P}$ of $\mathscr{P}$. When $\mathscr{P} \subset \mathbb{R}^d$ is of standard type, the *polar set* (or *dual polytope*)

$\mathscr{P}^*$ of $\mathscr{P}$ is defined to be

$$\mathscr{P}^* := \{(\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{R}^d ; \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_d\beta_d \le 1 \text{ for every } (\beta_1, \beta_2, \dots, \beta_d) \in \mathscr{P}\}.$$

Note that $\mathscr{P}^* \subset \mathbb{R}^d$ is also a convex polytope of standard type and $(\mathscr{P}^*)^* = \mathscr{P}$. Moreover, if $\mathscr{P}$ is rational, then $\mathscr{P}^*$ is also rational (cf. **[Grü, p. 47]**).

Fix an integer $d > 1$ and let $\mathscr{C}_0(d)$ be the set of integral convex polytopes $\mathscr{P} \subset \mathbb{R}^d$ of standard type. Also, we write $\mathscr{C}^*(d)$ for the set of those $\mathscr{P} \in \mathscr{C}_0(d)$ such that the polar set $\mathscr{P}^*$ of $\mathscr{P}$ is an integral convex polytope.

Even though the following Theorem (1.4) is easy to prove (based on the Ehrhart law of reciprocity (0.3)), this result plays an important role in our theory of combinatorics on δ-vectors.

(1.4) THEOREM (**[H6]**). *The δ-vector $\delta(\mathscr{P}) = (\delta_0, \delta_1, \dots, \delta_d)$ of $\mathscr{P} \in \mathscr{C}_0(d)$ is symmetric, i.e., $\delta_i = \delta_{d-i}$ for every $0 \le i \le d$, if and only if $\mathscr{P} \in \mathscr{C}^*(d)$.*

The above Theorem (1.4) can be generalized for the functions $i(\mathscr{P}, n)$ of rational convex polytopes $\mathscr{P}$ (see [H7]). Also, it would be of interest to compare our Theorem (1.4) (and **[H7]**) with **[Ish, Theorem 7.7]**.

If $\mathscr{P} \subset \mathbb{R}^N$ is an integral convex polytope of dimension d, then there exists an integral convex polytope $\mathscr{Q} \subset \mathbb{R}^d$ of dimension d with $\delta(\mathscr{P}) = \delta(\mathscr{Q})$. See, e.g., **[Sta8, pp. 238–239]**. Hence, for the combinatorial study of δ-vectors $\delta(\mathscr{P}) = (\delta_0, \delta_1, \dots, \delta_d)$ of integral convex polytopes $\mathscr{P}$ of dimension d with $\delta_d > 0$, thanks to equation (3), we have only to consider the integral convex polytopes $\mathscr{P} \subset \mathbb{R}^d$ of standard type.

Since $\mathscr{P} \in \mathscr{C}^*(d)$ implies $\mathscr{P}^* \in \mathscr{C}^*(d)$, it is quite reasonable to ask if we can compute $\delta(\mathscr{P}^*)$ in terms of $\delta(\mathscr{P})$. However, the following Example (1.5) falls short of our expectation.

(1.5) EXAMPLE. Let $d = 3$. First, we consider $\mathscr{P} \in \mathscr{C}^*(3)$ with the vertices $(1, 1, 1)$, $(-1, 0, 0)$, $(0, -1, 0)$, $(0, 0, -1)$, $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$. Then $\delta(\mathscr{P}) = (1, 4, 4, 1)$ and $\delta(\mathscr{P}^*) = (1, 19, 19, 1)$. On the other hand, let $\mathscr{Q} \in \mathscr{C}^*(3)$ be the bipyramid, which is the convex hull of $\{(1, 0, 0), (0, 1, 0), (1, 1, 0), (-1, 0, 0), (0, -1, 0), (0, 0, 1), (0, 0, -1)\}$ in $\mathbb{R}^3$, with $\delta(\mathscr{Q}) = (1, 4, 4, 1)$. Then $\mathscr{Q}^* \subset \mathbb{R}^3$ is the prism with $\delta(\mathscr{Q}^*) = (1, 20, 20, 1)$. Hence $\delta(\mathscr{P}^*) \ne \delta(\mathscr{Q}^*)$ even though $\delta(\mathscr{P}) = \delta(\mathscr{Q})$.

A somewhat interesting question related with Example (1.5) is the problem when convex polytopes $\mathscr{P} \in \mathscr{C}^*(d)$ satisfy $\delta(\mathscr{P}) = \delta(\mathscr{P}^*)$. We remark that the equality $\delta(\mathscr{P}) = \delta(\mathscr{P}^*)$ implies $\mathrm{vol}(\mathscr{P}) = \mathrm{vol}(\mathscr{P}^*)$ by equation (4).

On the other hand, during the DIMACS workshop on Polytopes and Convex Sets (Rutgers University, January 8—12, 1990), Stanley and the author discussed the following question: Let $\mathscr{P}$ and $\mathscr{Q}$ be integral convex polytopes in $\mathbb{R}^N$ of dimension d and suppose that each vertex of $\mathscr{P}$ is a vertex of $\mathscr{C}$ (thus, in particular, $\mathscr{P} \subset \mathscr{C}$). Then is $\delta(\mathscr{C}) \geq \delta(\mathscr{P})$? (Namley, is the ith component of $\delta(\mathscr{C})$ greater than or equal to the ith component of $\delta(\mathscr{P})$ for every $0 \leq i \leq d$?)[1]

2. Simplicial complexes

Let us review the definition of f-vectors and h-vectors of simplicial complexes. Let Δ be a *simplicial complex* on the *vertex set* $V = \{x_0, x_1, \ldots, x_v\}$. Thus, Δ is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for every $0 \leq i \leq v$ and (ii) if $\tau \in \Delta$ and $\tau \subset \sigma$ then $\tau \in \Delta$. An element $\sigma \in \Delta$ is called an *i-face* of Δ if $\#(\sigma) = i + 1$. A *facet* of Δ is a maximal face (with respect to inclusion) of Δ. Let $d := \max\{\#(\sigma)\,;\, \sigma \in \Delta\}$. Then the *dimension* of Δ is $\dim(\Delta) := d - 1$. We write $f_i = f_i(\Delta)$, $0 \leq i < d$, for the number of i-faces of Δ. The vector $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ is called the *f-vector* of Δ. Define the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of Δ by the formula

$$\sum_{i=0}^{d} f_{i-1}(\lambda - 1)^{d-i} = \sum_{i=0}^{d} h_i \lambda^{d-i}$$

with $f_{-1} = 1$. In particular, $h_0 = 1$, $h_1 = \#(V) - d$, and

$$h_0 + h_1 + \cdots + h_d = f_{d-1}.$$

We now study a certain triangulation Δ of the boundary $\partial\mathscr{P}$ of a convex polytope $\mathscr{P} \in \mathscr{K}^*(d)$ and discuss a relation between the δ-vector of $\mathscr{P}$ and the h-vector of Δ.

Let $\mathscr{P} \in \mathscr{K}^*(d)$ and set $V := \partial\mathscr{P} \cap \mathbb{Z}^d$. We write $\mathscr{T}$ for the set of simplices $\sigma \subset \mathbb{R}^d$ such that each vertex of σ is contained in V. Thus in particular $\{x\} \in \mathscr{T}$ for each $x \in V$. We say that a subset Δ of $\mathscr{T}$ is a *triangulation of $\partial\mathscr{P}$ with the vertex set V* if the following conditions are satisfied:

(i) $\{x\} \in \Delta$ for each $x \in V$,
(ii) if $\sigma \in \Delta$ and τ is a face of σ, then $\tau \in \Delta$,
(iii) if $\sigma, \tau \in \Delta$, then $\sigma \cap \tau$ is a common face of both σ and τ,
(iv) $\bigcup_{\sigma \in \Delta} \sigma = \partial\mathscr{P}$.

(2.1) LEMMA. *The boundary $\partial\mathscr{P}$ of every $\mathscr{P} \in \mathscr{K}^*(d)$ possesses a triangulation with the vertex set $V = \partial\mathscr{P} \cap \mathbb{Z}^d$.*

A triangulation Δ of the boundary $\partial\mathscr{P}$ of $\mathscr{P} \in \mathscr{K}^*(d)$ with the vertex set $V = \partial\mathscr{P} \cap \mathbb{Z}^d$ might be regarded as a simplicial complex on V of dimension $d - 1$ whose geometric realization is $\partial\mathscr{P}$. Since $\partial\mathscr{P}$ is homeomorphic to the $(d - 1)$-sphere, the h-vector $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of the simplicial

[1] Stanley [Sta15] answered this question affirmatively by the use of a modification of the Cohen-Macaulay ring $k[\mathscr{P}]$, which appears in the sketch of the proof of Theorem (1.3).

complex Δ on V satisfies the *Dehn-Sommerville equation* $h_i = h_{d-i}$ for every $0 \le i \le d$. Consult, e.g., **[Hoc2]** and **[Sta6]**.

On the other hand, a triangulation Δ of the boundary $\partial\mathscr{P}$ of $\mathscr{P} \in \mathscr{C}^*(d)$ with the vertex set $V = \partial\mathscr{P} \cap \mathbb{Z}^d$ is called *compressed* (cf. **[Sta4, p. 337]**) if, for each facet σ of Δ, the determinant of the matrix $(\mathbf{x}_{i1}, \mathbf{x}_{i2}, \dots, \mathbf{x}_{id})$ is equal to ± 1, where $\{\mathbf{x}_{ij}\}_{1 \le j \le d}$ is the set of vertices of σ. (Note that if $\mathscr{C} \subset \mathbb{R}^d$ is the simplex which is the convex hull of $\{(0, 0, \dots, 0)\} \cup \sigma$, then $d!\,\mathrm{vol}(\mathscr{C})$ coincides with the absolute value of the determinant of the matrix $(\mathbf{x}_{i1}, \mathbf{x}_{i2}, \dots, \mathbf{x}_{id})$.)

When $d \le 3$ and $\mathscr{P} \in \mathscr{C}^*(d)$, every triangulation of $\partial\mathscr{P}$ with the vertex set $V = \partial\mathscr{P} \cap \mathbb{Z}^d$ is compressed, because the volume of an integral convex polytope $\mathscr{C} \subset \mathbb{R}^2$ of dimension 2 with $\#(\mathscr{C} \cap \mathbb{Z}^2) = 3$ is equal to $1/2$.

(2.2) Proposition (cf. **[Sta4]** and **[B-M]**). *Suppose that Δ is a triangulation of the boundary $\partial\mathscr{P}$ of $\mathscr{P} \in \mathscr{C}^*(d)$ with the vertex set $V = \partial\mathscr{P} \cap \mathbb{Z}^d$. Let $h(\Delta) = (h_0, h_1, \dots, h_d)$ be the h-vector of Δ and $\delta(\mathscr{P}) = (\delta_0, \delta_1, \dots, \delta_d)$ the δ-vector of $\mathscr{P}$. Then $\delta(\mathscr{P}) \ge h(\Delta)$, i.e., $\delta_i \ge h_i$ for every $0 \le i \le d$. Moreover, $h(\Delta) = \delta(\mathscr{P})$ if and only if Δ is compressed.*

Rational?

(2.3) Example. Let $d = 4$ and $\mathscr{P} \in \mathscr{C}^*(4)$ the convex polytope with the vertex set $\{(0, 0, 0, 1), (1, 1, 0, 1), (1, 0, 1, 1), (0, 1, 1, 1), -(0, 0, 0, 1), -(1, 1, 0, 1), -(1, 0, 1, 1), -(0, 1, 1, 1)\}$. Then $\delta(\mathscr{P}) = (1, 4, 22, 4, 1)$, which is not an "0-sequence" (see, e.g., **[Sta3, H2]**), thus there exists no compressed triangulation of $\partial\mathscr{P}$ with the vertex set $V = \partial\mathscr{P} \cap \mathbb{Z}^4$. In fact, $\mathscr{P}$ is a simplicial convex polytope with the h-vector $h(\mathscr{P}) = (1, 4, 6, 4, 1)$ and the set of faces of $\mathscr{P}$ is the unique triangulation of $\partial\mathscr{P}$ with the vertex set V.

Now, let $\delta(\mathscr{P}) = (\delta_0, \delta_1, \dots, \delta_d)$ be the δ-vector of $\mathscr{P} \in \mathscr{C}^*(d)$ and suppose that $h(\Delta) = (h_0, h_1, \dots, h_d)$ is the h-vector of a triangulation Δ of $\partial\mathscr{P}$ with the vertex set $V = \partial\mathscr{P} \cap \mathbb{Z}^d$, whose existence is guaranteed by Lemma (2.1). Then the Lower Bound Theorem by Barnette **[Bar1, Bar2]** implies the inequality $h_i \ge h_1$ for every $1 \le i < d$. On the other hand, since we have $\delta_1 (= \#(\mathscr{P} \cap \mathbb{Z}^d) - (d+1)) = h_1 (= \#(V) - d)$, we obtain from Proposition (2.2) the following result.

(2.4) Corollary. *The δ-vector $\delta(\mathscr{P}) = (\delta_0, \delta_1, \dots, \delta_d)$ of an arbitrary convex polytope $\mathscr{P} \in \mathscr{C}^*(d)$ satisfies the inequality $\delta_i \ge \delta_1$ for every $1 \le i < d$.*

In [H10], the above Corollary (2.4) is generalized as follows:

(2.5) Theorem (**[H10]**). *Let $\mathscr{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d with the δ-vector $\delta(\mathscr{P}) = (\delta_0, \delta_1, \dots, \delta_d)$ and suppose that $(\mathscr{P} - \partial\mathscr{P}) \cap \mathbb{Z}^N$ is nonempty, i.e., $\delta_d \ne 0$. Then we have the inequality $\delta_1 \le \delta_i$ for every $1 \le i < d$.*

Fix an integer $d > 1$ and let $\mathscr{Z}^*(d)$ be the set of δ-vectors $\delta(\mathscr{P})$ of $\mathscr{P} \in \mathscr{C}^*(d)$. Also, we write $\mathscr{Z}_t^*(d)$ for the subset of $\mathscr{Z}^*(d)$, which consists of δ-vectors $\delta(\mathscr{P})$ of $\mathscr{P} \in \mathscr{C}^*(d)$ such that $\partial\mathscr{P}$ possesses a compressed triangulation with the vertex set $V = \partial\mathscr{P} \cap \mathbb{Z}^d$. We remark that $\mathscr{Z}^*(d) = \mathscr{Z}_t^*(d)$ if $d \leq 3$; however, $\mathscr{Z}_t^*(d) \neq \mathscr{Z}^*(d)$ for every $d \geq 4$.

It would be of interest to find a combinatorial characterization of the sequences in $\mathscr{Z}^*(d)$ (or $\mathscr{Z}_t^*(d)$).

On the other hand, by virtue of **[Hen, Theorem 3.6]**, the supremum of volumes $\mathrm{vol}(\mathscr{P})$ of $\mathscr{P} \in \mathscr{C}^*(d)$ is bounded. Hence, Lemma (1.1) and equation (4), together, imply the following:

(2.6) Proposition. *The set* $\mathscr{Z}^*(d)$ *is finite for every* $d > 1$.

We do not know the exact values of $\nu(d) := \max\{\mathrm{vol}(\mathscr{P}); \mathscr{P} \in \mathscr{C}^*(d)\}$ and $\mu(d) := \#(\mathscr{Z}^*(d))$ when $d \geq 3$. Note that $\nu(2) = 9/2$ **[Sco]** and $\mu(2) = 7$; however $\nu(d) > (d+1)d/d!$ if $d \geq 3$ (cf. **[Z-P-W]**). Also, see **[Hen, §4]**.

3. Toric varieties

We are now in the position to study toric varieties arising from triangulations of the boundaries of convex polytopes. We refer the reader to, e.g., **[Oda]** and **[Dan]** for basic information on toric varieties.

Suppose that Δ is a triangulation of the boundary $\partial\mathscr{P}$ of $\mathscr{P} \in \mathscr{C}^*(d)$ with the vertex set $V = \partial\mathscr{P} \cap \mathbb{Z}^d$. Then we can construct a simplicial complete fan $\mathscr{F}(\Delta)$ (cf. **[Dan, §5]**) in the $\mathbb{Q}$-vector space $\mathbb{Q}^d$ associated with Δ in the obvious way (**[Sta10, p. 218; Oda, Proposition 2.19]**). In fact, for each face σ of Δ, we define a simplicial convex polyhedral cone $C(\sigma)$ (with apex at the origin) to be the union of all rays whose vertex is the origin and which pass through σ. Then the set $\mathscr{F}(\Delta)$ of all such cones $C(\sigma)$ forms a complete fan. We write $\mathscr{X}(\Delta)$ for the complete toric variety associated with $\mathscr{F}(\Delta)$. Here we should remark that $\mathscr{X}(\Delta)$ is nonsingular **[Oda, Theorem 1.10]** if and only if Δ is compressed (cf. Proposition (2.2)). In general, the toric variety $\mathscr{X}(\Delta)$ is not necessarily projective (even though $\mathscr{X}(\Delta)$ is nonsingular, see **[Oda, p. 84]**). On the other hand, if the toric variety $\mathscr{X}(\Delta)$ is nonsingular and projective, then the δ-vector $\delta(\mathscr{P})$ of $\mathscr{P}$ coincides with the h-vector of some simplicial convex polytope of dimension d (cf. **[Sta10, p. 219]**); thus, in particular, $\delta(\mathscr{P})$ is unimodal, i.e., $\delta_0 \leq \delta_1 \leq \cdots \leq \delta_{[d/2]}$ (**[Sta5]** and **[Sta12, Theorem 20]**). Consult, e.g., **[Sta11]** for further results related with toric varieties and unimodal sequences.

It would be of interest to ask if there exists a natural class of convex polytopes $\mathscr{P} \in \mathscr{C}^*(d)$ such that the boundary $\partial\mathscr{P}$ of $\mathscr{P}$ possesses a triangulation Δ with the vertex set $V = \partial\mathscr{P} \cap \mathbb{Z}^d$ for which the corresponding toric variety $\mathscr{X}(\Delta)$ is projective (or nonsingular).

On the other hand, there exists a convex polytope $\mathscr{P} \in \mathscr{C}^*(d)$ and triangulations Δ, Δ' of the boundary $\partial\mathscr{P}$ of $\mathscr{P}$ with the vertex set $V = \partial\mathscr{P} \cap \mathbb{Z}^d$

such that $\mathscr{X}(\Delta)$ is nonsingular (resp. projective), but $\mathscr{X}(\Delta')$ is not nonsingular (resp. projective) if $d \geq 4$ (resp. $d \geq 3$).

Throughout the remainder of this section, we suppose that X is a finite partially ordered set (*poset* for short) with elements $y_1, y_2, \dots, y_d$ labeled so that if $y_i < y_j$ in X, then $i < j$ in $\mathbb{Z}$. A totally ordered subset of X is called a *chain* of X. Set $l := \max\{\#(C); C \text{ is a chain of } X\}$. We say that X is *pure* if every maximal chain of X has the cardinality l. If $\alpha \in X$, then $r(\alpha)$ denotes the greatest integer $m > 0$ for which there exists a chain $\alpha = \beta_1 < \beta_2 < \cdots < \beta_m$ in X.

Now, we write $\mathscr{C}(X)$ for the subset of $\mathbb{R}^d$, which consists of those points $(\alpha_1, \alpha_2, \dots, \alpha_d)$, such that

(i) $0 \leq \alpha_i + r(\alpha_i) \leq l + 1$ for each $1 \leq i \leq d$, and
(ii) $\alpha_i + r(\alpha_i) \leq \alpha_j + r(\alpha_j)$ if $y_i \geq y_j$ in X.

Let $\mathscr{O}(X)$ be the *order polytope* **[Sta7]** associated with X. Then

$$(7) \qquad \mathscr{C}(X) = (l+1)\mathscr{O}(X) - (r(\alpha_1), r(\alpha_2), \dots, r(\alpha_d)).$$

Thus, $\mathscr{C}(X) \subset \mathbb{R}^d$ is an integral convex polytope of dimension d. Moreover, $\mathscr{C}(X)$ is of standard type, i.e., $\mathscr{C}(X) \in \mathscr{C}_0(d)$. We say that $\mathscr{C}(X)$ is the *fat order polytope* associated with X.

(3.1) Lemma. *The fat order polytope $\mathscr{C}(X) \in \mathscr{C}_0(d)$ associated with X is contained in $\mathscr{C}^*(d)$ if and only if X is pure.*

Sketch of Proof. Thanks to **[Sta7]** and equation (7), we know the equations of supporting hyperplanes $\mathscr{H} \subset \mathbb{R}^d$ of $\mathscr{C}(X)$ such that $\mathscr{H} \cap \mathscr{C}(X)$ is a facet of $\mathscr{C}(X)$. In other words, we have information on the vertices of the dual polytope $\mathscr{C}(X)^*$ of $\mathscr{C}(X)$ (cf. **[Grü, p. 47]**). Thus, the required result follows easily from Theorem (1.4). Q.E.D.

We now state a combinatorial result on triangulations of the boundary $\partial\mathscr{C}(X)$ of the fat order polytope $\mathscr{C}(X)$ when X is pure.

In **[H8]**, we prove that certain (complete and simplicial) toric varieties arising from canonical triangulations **[Sta7]** of order polytopes are projective. Thus, combining **[H8]** with **[T-E, Chap. III, §2]**, we obtain the following:

(3.2) Theorem. *When X is pure, there exists a triangulation Δ of the boundary $\partial\mathscr{C}(X)$ of $\mathscr{C}(X)$ with the vertex set $V = \partial\mathscr{C}(X) \cap \mathbb{Z}^d$ such that the complete toric variety $\mathscr{X}(\Delta)$ is nonsingular and projective.*

Our next work is to compute the δ-vector $\delta(\mathscr{C}(X))$ of $\mathscr{C}(X)$. Let $w_i = w_i(X)$, $0 \leq i < d$, be the number of permutations $\pi = c_1 c_2 \cdots c_d$ of $1, 2, \dots, d$ with the properties that (i) if $y_{c_p} < y_{c_q}$ in X, then $p < q$ (i.e., π is a *linear extension* of X) and (ii) $\#(\{r; c_r > c_{r+1}\})$, the number of *descents* of π, is equal to i. Thus, in particular, $w_0 = 1$. Set $s := \max\{i; w_i \neq 0\}$. Then we easily see the equality $s = d - l$. We say that the sequence $w(X) := (w_0, w_1, \dots, w_s)$ is the *w-vector* of X. Consult, e.g., **[Sta1, Sta7; Sta8, Chap. 4, §5; H3, H4]** for the combinatorial background of w-vectors of finite posets.

In general, if $\varphi(\lambda) = \sum_{i \ge 0} u_i \lambda^i \in \mathbb{R}[\lambda]$ and $\xi > 0$ is an integer, then we write $[\varphi(\lambda)]^{(\xi)}$ for $\sum_{i \ge 0} u_{i\xi} \lambda^i$.

(3.3) PROPOSITION. *Let $w(X) = (w_0, w_1, \dots, w_s)$ be the w-vector of X and $\delta(\mathscr{C}(X)) = (\delta_0, \delta_1, \dots, \delta_d)$ the δ-vector of $\mathscr{C}(X) \in \mathscr{C}_0(d)$. Then we have the equality*

$$(8) \qquad \sum_{i=0}^{d} \delta_i \lambda^i = \left[(1 + \lambda + \lambda^2 + \cdots + \lambda^l)^{d+1} \sum_{j=0}^{s} w_j \lambda^j \right]^{(l+1)}.$$

We refer the reader to, e.g., **[H2, §4]** for some information on equation (8). As an immediate consequence of Theorem (3.2) with Proposition (3.3), we obtain a class of unimodal sequences in our theory of δ-vectors.

(3.4) COROLLARY. *Suppose that X is pure with the w-vector $w(X) = (w_0, w_1, \dots, w_s)$. Then the combinatorial sequence $(\delta_0, \delta_1, \dots \delta_d)$ defined by equation (8) is (symmetric and) unimodal.*

It is conjectured that the w-vector $(w_0, w_1, \dots, w_s)$ of an arbitrary finite poset is unimodal, i.e., $w_0 \le w_1 \le \cdots \le w_j \ge \cdots \ge w_s$ for some $0 \le j \le s$. Consult **[Sta12, pp. 505–506]** for further information. On the other hand, thanks to **[H8]** and **[Sta15, Lemma 2.2]**, we easily prove the inequalities $w_{[(d+1)/2]} \ge w_{[(d+1)/2]+1} \ge \cdots \ge w_s$.

(3.5) EXAMPLE. Let $d = 4$ and suppose that $X = \{y_1, y_2, y_3, y_4\}$ is the pure poset with the partial order $y_1 < y_3$, $y_2 < y_3$, and $y_2 < y_4$. Then $l = 2$, $s = 2$, and $w(X) = (1, 3, 1)$. Hence, thanks to equation (8), the δ-vector of the fat order polytope $\mathscr{C}(X) \in \mathscr{C}^*(4)$ is $\delta(\mathscr{C}(X)) = (1, 80, 245, 80, 1)$. On the other hand, the vertices of $\mathscr{C}(X)$ are $(-2, -2, -1, -1)$, $(1, -2, -1, -1)$, $(-2, 1, -1, -1)$, $(1, 1, -1, -1)$, $(-2, 1, -1, 2)$, $(1, 1, 2, -1)$, $(1, 1, -1, 2)$, and $(1, 1, 2, 2)$. Moreover, the vertices of the dual polytope $\mathscr{C}(X)^*$ of $\mathscr{C}(X)$ are $\beta_1 = (1, 0, 0, 0)$, $\beta_2 = (0, 1, 0, 0)$, $\beta_3 = (0, 0, -1, 0)$, $\beta_4 = (0, 0, 0, -1)$, $\beta_5 = (-1, 0, 1, 0)$, $\beta_6 = (0, -1, 1, 0)$, and $\beta_7 = (0, -1, 0, 1)$. Since we know the vertices and the facets of $\mathscr{C}(X)^*$, it is possible by routine computation to determine if there exists a compressed triangulation of $\partial(\mathscr{C}(X)^*)$ with the vertex set $V = \partial(\mathscr{C}(X)^*) \cap \mathbb{Z}^4$. In fact, the triangulation Δ of $\partial(\mathscr{C}(X)^*)$ whose facets are 2345, 1234, 2357, 1237, 3457, 4567, 1347, 1467, 1245, 1456, 1257, and 1567 is a compressed triangulation of $\partial(\mathscr{C}(X)^*)$ with the vertex set $V = \partial(\mathscr{C}(X)^*) \cap \mathbb{Z}^4$. Here, for example, the notation 2345 means the simplex in $\mathbb{R}^4$ with the vertex set $\{\beta_2, \beta_3, \beta_4, \beta_5\}$. Note that $\mathrm{vol}(\mathscr{C}(X)^*) = 12/4!$ and $\delta(\mathscr{C}(X)^*) = (1, 3, 4, 3, 1)$.

It is not difficult to see that there exists a compressed triangulation of $\partial(\mathscr{C}(X)^*)$ with the vertex set $V = \partial(\mathscr{C}(X)^*) \cap \mathbb{Z}^d$ for *every* pure poset X with $\#(X) = d$. Also, when X is pure, does there exist a nice formula like equation (8) to compute $\delta(\mathscr{C}(X)^*)$?

We conclude this section with another example of combinatorial sequences contained in $\mathscr{Z}_t^*(d)$ which is related with nonsingular Fano toric varieties.

(3.6) EXAMPLE. Suppose that a convex polytope $\mathscr{P} \in \mathscr{C}^*(d)$ is simplicial. Then, thanks to Proposition (2.2), we easily see that $h(\mathscr{P}) = \delta(\mathscr{P})$ if and only if $\mathscr{P}$ is a Fano polyhedron (polytope) in the sense of **[V-K, p. 223]**. In **[V-K]** Voskresenskij and Klyachko give a complete classification of centrally symmetric Fano polytopes. (Especially, see **[V-K, p. 234]** on the description of the Fano polytope for the del Pezzo variety associated with the root system of type A of an even rank.) Thus, in particular, we should say that the δ-vectors $\delta(\mathscr{P})$ arising from centrally symmetric simplicial convex polytopes $\mathscr{P} \in \mathscr{C}^*(d)$ with $h(\mathscr{P}) = \delta(\mathscr{P})$ are already known. Is it possible to find a combinatorial characterization of the δ-vectors $\delta(\mathscr{P})$ of simplicial convex polytopes $\mathscr{P} \in \mathscr{C}^*(d)$ with $h(\mathscr{P}) = \delta(\mathscr{P})$?

REFERENCES

[Bar1] D. W. Barnette, *The minimal number of vertices of a simple polytope*, Israel J. Math. **10** (1971), 121–125.

[Bar2] ——, *A proof of the lower bound conjecture for convex polytopes*, Pacific J. Math. **46** (1973), 349–354.

[B-M] U. Betke and P. McMullen, *Lattice points in lattice polytopes*, Monatsh. Math. **99** (1985), 253–265.

[Bil] L. J. Billera, *The algebra of continuous piecewise polynomials*, Adv. in Math. **76** (1989), 170–183.

[B-L] L. J. Billera and C. Lee, *A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes*, J. Combin. Theory Ser A. **31** (1981), 237–255.

[B-R] L. J. Billera and L. L. Rose, *Modules of piecewise polynomials and their freeness*, Math. Z. (to appear).

[Dan] V. I. Danilov, *The geometry of toric varieties*, Russian Math. Surveys **33** (1978), 97–154.

[Ehr1] E. Ehrhart, *Polynômes arithmétiques et Méthode des Polyèdres en Combinatoire*, Birkhaüser, Basel and Stuttgart, 1977.

[Ehr2] ——, *Démonstration de la loi de réciprocité du polyedre rationnel*, C. R. Acad. Sci. Paris Ser. A **265** (1967), 91–94.

[Grü] B. Grünbaum, *Convex polytopes*, John Wiley and Sons, Inc. (Interscience), London, New York, and Sydney, 1967.

[Hen] D. Hensley, *Lattice vertex polytopes with interior lattice points*, Pacific J. Math. **105** (1983), 183–191.

[H1] T. Hibi, *Canonical ideals of Cohen-Macaulay partially ordered sets*, Nagoya Math. J. **112** (1988), 1–24.

[H2] ——, *Flawless O-sequences and Hilbert functions of Cohen-Macaulay integral domains*, J. Pure Appl. Algebra **60** (1989), 245–251.

[H3] ——, *Linear and non-linear inequalities concerning a certain combinatorial sequence which arises from counting the number of chains of a finite distributive lattice*, Discrete Appl. Math. (to appear).

[H4] ——, *Hilbert functions of Cohen-Macaulay integral domains and chain conditions of finite partially ordered sets*, J. Pure Appl. Algebra **72** (1991), 265–273.

[H5] ——, *Some results on Ehrhart polynomials of convex polytopes*, Discrete Math. **83** (1990), 119–121.

[H6] ——, *Dual polytopes of rational convex polytopes*, Combinatorica (to appear).

[H7] ——, *A combinatorial self-reciprocity theorem for Ehrhart quasi-polynomials of rational convex polytopes*, European J. Combin. (to appear).

[H8] ——, *Toric varieties arising from canonical triangulations of poset polytopes are projective* (submitted).

[H9] ——, *The Ehrhart polynomial of a convex polytope* (in preparation).

[H10] ——, *A lower bound theorem for Ehrhart polynomials of convex polytopes*, Adv. in Math. (to appear).

[Hoc1] M. Hochster, *Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes*, Ann. of Math. **96** (1972), 318–337.

[Hoc2] ——, *Cohen-Macaulay rings combinatorics, and simplicial complexes*, Ring Theory II (B. R. McDonald and R. Morris, eds.), Lecture Notes in Pure and Appl. Math., no. 26, Dekker, New York, 1977, pp. 171–223.

[H-K] J. Herzog and E. Kunz, *Der kanonische Modul eines Cohen-Macaulay-Rings*, Lecture Notes in Math., vol. 238, Springer-Verlag, Berlin, Heidelberg, and New York, 1971.

[Ish] M.-N. Ishida, *Torus embeddings and dualizing complexes*, Tôhoku Math. J. **32** (1980), 111–146.

[Mac1] I. G. Macdonald, *The volume of a lattice polyhedron*, Proc. Cambridge Philos. Soc. **59** (1963), 719–726.

[Mac2] ——, *Polynomials associated with finite cell-complexes*, J. London Math. Soc. (2) **4** (1971), 181–192.

[Mc1] P. McMullen, *Valuations and Euler-type relations on certain classes of convex polytopes*, Proc. London Math. Soc. (3) **35** (1977), 113–135.

[Mc2] ——, *Lattice invariant valuations on rational polytopes*, Arch. Math. **31** (1978), 509–516.

[Oda] T. Oda, *Convex bodies and algebraic geometry: An introduction to the theory of toric varieties*, Springer-Verlag, Berlin, Heidelberg, and New York, 1988.

[Rei] G. Reisner, *Cohen-Macaulay quotients of polynomial rings*, Adv. in Math. **21** (1976), 30–49.

[Sco] P. R. Scott, *On convex lattice polygons*, Bull. Austral. Math. Soc. **15** (1976), 395–399.

[Sta1] R. P. Stanley, *Ordered structures and partitions*, Mem. Amer. Math. Soc. **119** (1972).

[Sta2] ——, *The upper bound conjecture and Cohen-Macaulay rings*, Stud. Appl. Math. **54** (1975), 135–142.

[Sta3] ——, *Hilbert functions of graded algebras*, Adv. in Math. **28** (1978), 57–83.

[Sta4] ——, *Decompositions of rational convex polytopes*, Annals of Discrete Math. **6** (1980), 333–342.

[Sta5] ——, *The number of faces of a simplicial convex polytope*, Adv. in Math. **35** (1980), 236–238.

[Sta6] ——, *Combinatorics and Commutative Algebra*, Birkhaüser, Boston, Basel, and Stuttgart, 1983.

[Sta7] ——, *Two poset polytopes*, Discrete Comput. Geom. **1** (1986), 9–23.

[Sta8] ——, *Enumerative Combinatorics*, vol. I, Wadsworth & Brooks/Cole, Monterey, CA, 1986.

[Sta9] ——, *On the number of faces of centrally-symmetric simplicial polytopes*, Graphs Combin. **3** (1987), 55–66.

[Sta10] ——, *The number of faces of simplicial polytopes and spheres*, Discrete Geometry and Convexity, Ann. of New York Acad. Sci., vol. 440, 1985, pp. 212–223.

[Sta11] ——, *Generalized h-vectors, intersection cohomology of toric varieties, and related results*, Commutative Algebra and Combinatorics (M. Nagata and H. Matsumura, eds.), Adv. Stud. Pure Math., vol 11, North-Holland, Amsterdam, 1987, pp. 187–213.

[Sta12] ——, *Log-concave and unimodal sequences in algebra, combinatorics, and geometry*, Graph Theory and its Applications: East and West, Ann. of New York Acad. Sci. vol. 576, 1989, pp. 500–535.

[Sta13] ——, *F-vectors and h-vectors of simplicial posets*, J. Pure Appl. Algebra **71** (1991), 319–331. (in press).

[Sta14] —— 73 (

[Sta15] ——

[Tei] B. T. Not 71-

[T-E] G. K ture

[V-K] V. E USS

[Z-P-W] J. Ele

DEPART
SAPPORO 0

[Sta14] ——, *On the Hilbert function of a graded Cohen-Macaulay domain*, J. Pure Appl. Algebra **73** (1991), 307–314.

[Sta15] ——, *A monotonicity property of h-vectors and h^*-vectors*, preprint, 1991.

[Tei] B. Teissier, *Variétés toriques et polytopes, Séminaire Bourbaki* 1980/81, *Exp.* 565, Lecture Notes in Math. vol. 901, Springer-Verlag, Berlin, Heidelberg, and New York, 1981, pp. 71–84.

[T-E] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal Embeddings* I, Lecture Notes in Math., vol. 339, Springer-Verlag, Berlin, Heidelberg, and New York, 1973.

[V-K] V. E. Voskresenskij and A. A. Klyachko, *Toroidal Fano varieties and root systems*, Math. USSR Izv. **24** (1985), 221–244.

[Z-P-W] J. Zaks, M. A. Perles, and J. M. Wills, *On lattice polytopes having interior lattice points*, Elem. Math. **37** (1982), 44–46.

Department of Mathematics, Faculty of Science, Hokkaido University, Kita-ku, Sapporo 060, Japan