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NOTE

DUAL POLYTOPES OF RATIONAL CONVEX POLYTOPES*)

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Let $\mathcal{P} \subset \mathbb{R}^d$ be a rational convex polytope with $\dim \mathcal{P} = d$ such that the origin of $\mathbb{R}^d$ is contained in the interior $\mathcal{P} - \partial\mathcal{P}$ of $\mathcal{P}$. In this paper, from a viewpoint of enumeration of certain rational points in $\mathcal{P}$ (which originated in Ehrhart's work), a necessary and sufficient condition for the dual polytope $\mathcal{P}^{\text{dual}}$ of $\mathcal{P}$ to be integral is presented.

Introduction

A convex polytope $\mathcal{P} \subset \mathbb{R}^N$ is called *rational* (resp. *integral*) if each vertex of $\mathcal{P}$ has rational (resp. integer) coordinates. We write $\partial\mathcal{P}$ for the boundary of $\mathcal{P}$. Let $d = \dim \mathcal{P}$.

Suppose that $N = d$ and that the origin $(0, \ldots, 0)$ of $\mathbb{R}^d$ is contained in the interior $\mathcal{P} - \partial\mathcal{P}$ of $\mathcal{P}$. Then the *dual polytope* $\mathcal{P}^{\text{dual}} \subset \mathbb{R}^d$ is defined by

$$\mathcal{P}^{\text{dual}} := \{(x_1, \ldots, x_d) \in \mathbb{R}^d; \quad \sum_{i=1}^{d} x_i y_i \leq 1 \quad \text{for any} \quad (y_1, \ldots, y_d) \in \mathcal{P}\}.$$

Note that $\dim \mathcal{P}^{\text{dual}} = d$, $(0, \ldots, 0) \in \mathcal{P}^{\text{dual}} - \partial\mathcal{P}^{\text{dual}}$ and $(\mathcal{P}^{\text{dual}})^{\text{dual}} = \mathcal{P}$. Also, if $\mathcal{F} \subset \mathcal{P}$ is an i-face (i.e., $\dim \mathcal{F} = i$), $0 \leq i < d$, then

$$\mathcal{F}^{\wedge} := \{(x_1, \ldots, x_d) \in \mathcal{P}^{\text{dual}}; \quad \sum_{i=1}^{d} x_i y_i = 1 \quad \text{for any} \quad (y_1, \ldots, y_d) \in \mathcal{F}\}$$

is a $(d-1-i)$-face of $\mathcal{P}^{\text{dual}}$. Moreover, any $(d-1-i)$-face of $\mathcal{P}^{\text{dual}}$ is of the form $\mathcal{F}^{\wedge}$ for a unique i-face $\mathcal{F}$ of $\mathcal{P}$. See [1, pp. 46–48]. Thus, if $\mathcal{P}$ is rational then $\mathcal{P}^{\text{dual}}$ is also rational. However, $\mathcal{P}^{\text{dual}}$ is not necessarily integral even though $\mathcal{P}$ is integral. So, it is natural to ask when the dual polytope of an integral convex polytope turns out to be integral.

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The purpose of this paper is to present a necessary and sufficient condition for the dual polytope $\mathcal{P}^{\text{dual}}$ of a rational convex polytope $\mathcal{P}$ to be integral (cf. Theorem (2.1)). In particular, we see that the dual polytope $\mathcal{P}^{\text{dual}}$ of an integral convex polytope $\mathcal{P}$ is integral if and only if the Ehrhart polynomial $i(\mathcal{P}, n)$ of $\mathcal{P}$ satisfies $i(\mathcal{P}, -n-1) = (-1)^d i(\mathcal{P}, n)$ (cf. Corollary (2.2)).

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Throughout this paper we write $\#(X)$ for the cardinality of a finite set X.

1. Ehrhart polynomials

We summarize fundamental facts concerning the Ehrhart function of a rational convex polytope from [3, pp. 235–241].

(1.1) Let $\mathcal{P} \subset \mathbb{R}^N$ be an arbitrary rational convex polytope with $\dim \mathcal{P} = d$. Given a positive integer n, write $i(\mathcal{P}, n)$ for the number of rational points $(x_1, \ldots, x_N) \in \mathcal{P}$ with each $nx_i \in \mathbb{Z}$, and set $i(\mathcal{P}, 0) = 1$. In other words,

$$i(\mathcal{P}, n) = \#(n\mathcal{P} \cap \mathbb{Z}^N),$$

where $n\mathcal{P} := \{n\mathbf{x}; \mathbf{x} \in \mathcal{P}\}$. Also, let $\omega(\mathcal{P}, \lambda)$ be the generating function

$$\omega(\mathcal{P}, \lambda) = \sum_{n=0}^{\infty} i(\mathcal{P}, n)\lambda^n$$

of the sequence $\{i(\mathcal{P}, n)\}_{n=0}^{\infty}$. Then $\omega(\mathcal{P}, \lambda)$ is a rational function in the variable λ.

(1.2) If $\mathcal{P} \subset \mathbb{R}^N$ is integral then $i(\mathcal{P}, n)$ is a polynomial, called the *Ehrhart polynomial* of $\mathcal{P}$, in n of degree $d = \dim \mathcal{P}$. Moreover, when $N = d$, the leading coefficient of the polynomial $i(\mathcal{P}, n)$ is equal to the volume of $\mathcal{P}$.

(1.3) Suppose that a convex polytope $\mathcal{P} \subset \mathbb{R}^N$ is rational with $\dim \mathcal{P} = d$. Let $j(\mathcal{P}, n)$ be the number of rational points $(x_1, \ldots, x_N)$ contained in the interior $\mathcal{P} - \partial\mathcal{P}$ of $\mathcal{P}$ with each $nx_i \in \mathbb{Z}$, where $n = 1, 2, \ldots$. Then we have the equation

$$\sum_{n=1}^{\infty} j(\mathcal{P}, n)\lambda^n = (-1)^{d+1}\omega(\mathcal{P}, \lambda^{-1})$$

as rational functions in λ. Thus, in particular, if $\mathcal{P}$ is integral, then $j(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n)$.

2. When is $\mathcal{P}^{\text{dual}}$ integral?

We are now in the position to state our main results.

(2.1) Theorem. *Suppose that $\mathcal{P} \subset \mathbb{R}^d$ is a rational convex polytope with $\dim \mathcal{P} = d$ and that the origin of $\mathbb{R}^d$ is contained in the interior $\mathcal{P} - \partial\mathcal{P}$ of $\mathcal{P}$. Then the*

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dual polytope $\mathcal{P}^{\text{dual}}$ of $\mathcal{P}$ is integral if and only if $\omega(\mathcal{P}, \lambda^{-1}) = (-1)^{d+1}\lambda\omega(\mathcal{P}, \lambda$ rational functions in λ.

(2.2) Corollary. *Suppose that $\mathcal{P} \subset \mathbb{R}^d$ is an integral convex polytope with $\dim \mathcal{P}$ and that the origin of $\mathbb{R}^d$ is contained in the interior $\mathcal{P} - \partial\mathcal{P}$ of $\mathcal{P}$. Then a neces and sufficient condition for the dual polytope $\mathcal{P}^{\text{dual}}$ of $\mathcal{P}$ to be integral is that Ehrhart polynomial $i(\mathcal{P}, n)$ of $\mathcal{P}$ satisfies $i(\mathcal{P}, -n-1) = (-1)^d i(\mathcal{P}, n)$.*

(2.3) Example. Let $d = 2$. Given an arbitrary integer $s \geq 1$, write $\mathcal{P}_{[s]}$ for integral convex polytope whose vertices are $(0,1)$, $(0,-1)$, $(1,s)$ and $(-1,-s)$. T the dual polytope $(\mathcal{P}_{[s]})^{\text{dual}}$ of $\mathcal{P}_{[s]}$ is also integral. The Ehrhart polynomial of is $i(\mathcal{P}_{[s]}, n) = 2n^2 + 2n + 1$, thus $i(\mathcal{P}_{[s]}, -n-1) = i(\mathcal{P}_{[s]}, n)$.

3. Key Lemma

(3.1) Lemma. *Suppose that $\mathcal{H} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d; \sum_{i=1}^{d} a_i x_i = b\}$, where eac $b \in \mathbb{Z}$, $b > 1$, and the greatest common divisor of $a_1, \ldots, a_d, b$ is equal to 1, hyperplane in $\mathbb{R}^d$ and that $\mathcal{F} \subset \mathcal{H}$ is an arbitrary rational convex polytope $\dim \mathcal{F} = d - 1$. Then there exist an integer $n > 1$ and a rational number k $n - 1 < k < n$ such that $k\mathcal{F} \cap \mathbb{Z}^d$ is non-empty, where $k\mathcal{F} := \{k\mathbf{x}; \mathbf{x} \in \mathcal{F}\}$.*

Proof. Since $b \geq 2$ and the greatest common divisor of $a_1, \ldots, a_d, b$ is equal some a_i (say a_1) is not divided by b. Let $b/a_1 = q/p$, where p, $q \in \mathbb{Z}$ are relat prime and $q > 1$.

Let $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(m)}$, $m \geq d$, with each $\mathbf{v}^{(i)} \in \mathbb{Q}^d$, be the vertices of $\mathcal{F}$. $\mathbf{v} := \mathbf{v}^{(1)} + \ldots + \mathbf{v}^{(m)}$, $\mathbf{g} := (1/m)\mathbf{v}$ and $\boldsymbol{\alpha} := (mb/a_1, 0, \ldots, 0) \in \mathbb{Q}^d$. positive integer c with $c(\mathbf{v} - \boldsymbol{\alpha}) \in \mathbb{Z}^d$ and define $\boldsymbol{\delta}(= (\delta_1, \ldots, \delta_d)) := c(\mathbf{v} - \boldsymbol{\alpha})$. $a_1\delta_1 + \ldots + a_d\delta_d = 0$. Also, fix a positive integer n_0 such that $k\mathbf{g} - \boldsymbol{\delta} \in k\mathcal{F}$ fo rational number $k \geq n_0$. The existence of such integer n_0 is geometrically ob since $k\mathbf{g}$ is the center of gravity of $k\mathcal{F}$.

Now, since $(b/a_1)\mathbb{Z} \cap \mathbb{Z} = q\mathbb{Z}$ and $q > 1$, there exist integers t and n such that $(n-1)b < a_1 t < nb$. Let $k = a_1 t/b$. Then $(n_0 \leq) n - 1 < k$ Moreover, if $\boldsymbol{\beta} := (t, 0, \ldots, 0) \in \mathbb{Z}^d$ then $\boldsymbol{\beta} = (k/m)\boldsymbol{\alpha}$, thus $\boldsymbol{\beta} + (k/cm)\boldsymbol{\delta} =$ Hence $\boldsymbol{\beta} + \lfloor k/cm \rfloor \boldsymbol{\delta} \in \mathbb{Z}^d \cap k\mathcal{F}$ as required.

4. Proof of Theorem (2.1)

Suppose that $\mathcal{P} \subset \mathbb{R}^d$ is a rational convex polytope with $\dim \mathcal{P} = d$ suc the origin of $\mathbb{R}^d$ is contained in the interior $\mathcal{P} - \partial\mathcal{P}$ of $\mathcal{P}$. Thanks to (1.3), f proof of Theorem (2.1), it is enough to show that the equality $j(\mathcal{P}, n) = i(\mathcal{P},$ holds for an arbitrary positive integer n if and only if the following condition satisfied:

($\star$) If a hyperplane $\mathcal{H} = \{(x_1, \dots, x_d) \in \mathbb{R}^d; \sum_{i=1}^{d} a_i x_i = b\}$, where each a_i, $b \in \mathbb{Z}$, $b > 0$, and the greatest common divisor of $a_1, \dots, a_d, b$ is equal to 1, is a supporting hyperplane of $\mathcal{P}$ such that $\mathcal{H} \cap \mathcal{P}$ is a facet, then $b = 1$.**)

Let $\mathcal{H}_1, \dots, \mathcal{H}_s$ be the supporting hyperplanes of $\mathcal{P}$ such that $\mathcal{F}_j := \mathcal{P} \cap \mathcal{H}_j$ is a facet of $\mathcal{P}$, $1 \leq j \leq s$, and $\mathcal{H}_j = \{(x_1, \dots, x_d) \in \mathbb{R}^d; \sum_{i=1}^{d} a_i^{(j)} x_i = b^{(j)}\}$, where each $a_i^{(j)}$, $b^{(j)} \in \mathbb{Z}$, $b^{(j)} > 0$, and the greatest common divisor of $a_1^{(j)}, \dots, a_d^{(j)}, b^{(j)}$ is equal to 1. Then a point $(\beta_1, \dots, \beta_d) \in \mathbb{Z}^d$ is contained in $n(\mathcal{P} - \partial\mathcal{P})$ (resp. $(n-1)\mathcal{P}$) if and only if $\sum_{i=1}^{d} a_i^{(j)} \beta_i < n b^{(j)}$ (resp. $\sum_{i=1}^{d} a_i^{(j)} \beta_i \leq (n-1) b^{(j)}$) for any $1 \leq j \leq s$. Thus, if each $b^{(j)} = 1$, then $n(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^d = (n-1)\mathcal{P} \cap \mathbb{Z}^d$, i.e., $j(\mathcal{P}, n) = i(\mathcal{P}, n-1)$, for any positive integer n.

On the other hand, suppose that some $b^{(j)} > 1$. Then, apply Lemma (3.1) to $\mathcal{H}_j$ and $\mathcal{F}_j$, and we see that there exist a rational number k and an integer $n \geq 2$ with $n - 1 < k < n$ such that $k\mathcal{F}_j \cap \mathbb{Z}^d$ is non-empty. since $k\mathcal{F}_j \subset n(\mathcal{P} - \partial\mathcal{P})$ and $k\mathcal{F}_j \cap (n-1)\mathcal{P} = \emptyset$, we have $(n-1)\mathcal{P} \cap \mathbb{Z}^d \subsetneqq n(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^d$, thus $i(\mathcal{P}, n-1) \neq j(\mathcal{P}, n)$ as required. ∎

A generalization of Theorem (2.1) is obtained in [2].

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**) Egon Schulte pointed out to the author that the condition ($\star$) is equivalent to the condition that the dual polytope of $\mathcal{P}$ is integral.