Generalization of Eagon-Reiner theorem and h-vectors of graded rings

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Abstract

We generalize the Eagon-Reiner Theorem as follows: $\text{reg } I_\Delta - \text{indeg } I_\Delta = \dim k[\Delta^*] - \text{ depth } k[\Delta^*]$. As an application, we give (i) a necessary and sufficient condition for a sequence of integers to be the h-vector of a homogeneous k-algebra $R = k[x_1, x_2, \ldots, x_n]/I$ with $\text{reg } I - \text{indeg } I \leq c$ for a fixed $c \geq 0$, and (ii) an upper bound for multiplicities, which improves one of the Herzog-Srinivasan inequalities.

Introduction

Recenty Alexander duality theorem plays an important role in the study on a minimal free resolution of Stanley-Reisner rings. (See [Br-He$_2$], [Te-Hi$_1$], [Te-Hi$_2$], for example.) In particular, Eagon and Reiner introduced Alexander dual complexes and proved the following interesting theorem:

THEOREM 0.1([Ea-Re]). *Let k be a field. and let Δ be a simplicial complex and Δ^* its Alexander dual complex. Then $k[\Delta]$ has a linear resolution if and only if $k[\Delta^*]$ is Cohen-Macaulay.*

The above result is a starting point of this article. We generalize it in the following way.

THEOREM 0.2. *Let k be a field. Let Δ be a $(d-1)$-dimensional complex on the vertex set $[n]$. Suppose $d \leq n-2$. Then*

$$\text{reg } I_\Delta - \text{indeg } I_\Delta = \dim k[\Delta^*] - \text{ depth } k[\Delta^*].$$

Note that Theorem 0.2 corresponds to Theorem 0.1 in the case that either side of the equality is 0.

On the other hand, it is one of important problems to characterize the h-vectors of a good class of homogeneous k-algebras (i.e., noetherian graded k-algebras gererated by elements in degree one and degree 0 part is k) for a field k. This kind of a problem was originated in Macaulay's work (see Theorem 1.1 in §1), and developed by a lot of mathematicians in algebraic, geometric, and/or combinatoric methods. See, for example, [St_1] and [St_3] to survey this topic. In this article we give a necessary and sufficient condition for a sequence of integers to be the h-vector of a homogeneous ring $R = k[x_1, x_2, \ldots, x_n]/I$ with reg I − indeg $I \le c$ for a fixed $c \ge 0$, as an application of the above theorem using the Gröbner basis theory.

As another application, we give some upper bound for the multiplicities of homogeneous k-algebras. In [He-Sr] Herzog and Srinivasan give a conjecture for the upper bound for multiplicities as follows:

Conjecture 0.3 ([He-Sr, Conjecture 2]). Let R be a homogeneous k-algebra of codimension h_1. Then

$$\mathrm{e}(R) \le \frac{\prod_{i=1}^{h_1} \mathrm{reg}_i(R)}{h_1!},$$

where $\mathrm{reg}_i(R) := \max\{j \mid \beta_{i,j}(R) \neq 0\}$.

And among other things, they proved following inequality:

Theorem 0.4([He-Sr, Corollary 3.8]). *Let $R = k[x_1, x_2, \ldots, x_n]/I$ be a homogeneous k-algebra. Then*

$$\mathrm{e}(R) \le \binom{\mathrm{reg}\ I + n - \mathrm{depth\ R} - 1}{n - \mathrm{depth\ R}}.$$

We obtain a bound as follows:

Theorem 0.5. *Let $R = A/I$ be a homogeneous k-algebra of codimension $h_1 \ge 2$. Then*

$$\mathrm{e}(R) \le \binom{\mathrm{reg}\ I + h_1 - 1}{h_1}.$$

Theorem 0.5 improves Theorem 0.4 for the non-Cohen-Macaulay rings. As a corollary we obtain some partial affirmative result on Conjecture 0.3.

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§1. Preliminaries

We first fix notation. Let **N**(resp. **Z**) denote the set of nonnegative integers (resp. integers). Let $| S |$ denote the cardinality of a set S.

We recall some notation on simplicial complexes and Stanley-Reisner rings according to [St$_1$]. We refer the reader to, e.g., [Br–He], [Hi], [Ho] and [St$_1$] for the detailed information about combinatorial and algebraic background.

A *simplicial complex* Δ on the *vertex set* $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$ such that (i) $\{i\} \in \Delta$ for every $1 \leq i \leq n$ and (ii) $F \in \Delta$, $G \subset F \Rightarrow G \in \Delta$. Each element F of Δ is called a *face* of Δ. We call $F \in \Delta$ an *i-face* if $| F |= i + 1$ We set $d = \max\{| F || F \in \Delta\}$ and define the *dimension* of Δ to be $\dim \Delta = d - 1$.

Let $f_i = f_i(\Delta)$, $0 \leq i \leq d - 1$, denote the number of i-faces in Δ. We define $f_{-1} = 1$. We call $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ the *f-vector* of Δ. Define the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of Δ by

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}.$$

If F is a face of Δ, then we define a subcomplex $\text{link}_\Delta F$ as follows:

$$\text{link}_\Delta F = \{G \in \Delta \mid F \cap G = \emptyset, F \cup G \in \Delta\}.$$

Let $\tilde{H}_i(\Delta; k)$ denote the i-th *reduced simplicial homology group* of Δ with the coefficient field k.

Let $A = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n-variables over a field k. Define I_Δ to be the ideal of A which is generated by square-free monomials $x_{i_1}x_{i_2}\cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, with $\{i_1, i_2, \ldots, i_r\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the *Stanley-Reisner ring* of Δ over k.

Next we summarize basic facts on the Hilbert series. Let k be a field and R a homogeneous k-algebra. We means a homogeneous k-algebra R by a noetherian graded ring $R = \bigoplus_{i \geq 0} R_i$ generated by R_1 with $R_0 = k$. In this case R can be written as a quotient algebra $k[x_1, x_2, \ldots, x_n]/I$, where $\deg x_i = 1$. In this article we always use the representatation A/I with $A = k[x_1, x_2, \ldots, x_n]$ a polynomial ring and with $I_1 = (0)$.

Let M be a graded R-module with $\dim_k M_i < \infty$ for all $i \in \mathbf{Z}$, where $\dim_k M_i$ denotes the dimension of M_i as a k-vector space.

The *Hilbert series* of M is defined by

$$F(M,t) = \sum_{i\in\mathbf{Z}} (\dim_k M_i)t^i.$$

It is well known that the Hilbert series $F(R,t)$ of R can be written in the form

$$F(R,t) = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^{\dim R}},$$

where $h_0(=1)$, $h_1, \ldots, h_s$ are integers with $\mathrm{e}(R) := h_0 + h_1 + \cdots + h_s \geq 1$. The vector $h(R) = (h_0, h_1, \ldots, h_s)$ is called the *h-vector* of R and the number $\mathrm{e}(R)$ the multiplicity of R.

Let f and i be positive integers. Then f can be uniquely written in the form

$$f = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geq j \geq 1$. Define

$$\begin{aligned} f^{<i>} &= \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i} + \cdots + \binom{n_j+1}{j+1}, \\ 0^{<i>} &= 0. \end{aligned}$$

Theorem 1.1(Macaulay, Stanley [St$_3$, Theorem 2.2]). *Let $h = (h_i)_{i\geq 0}$ be a sequence of integers. Then the following conditions are equivalent:*
(1) There exists a homogeneous k-algebra R with $F(R,t) = \sum_{i\geq 0}(\dim_k R_i)t^i$.
(2) $h_0 = 1$ and $0 \leq h_{i+1} \leq h_i^{<i>}$ for $i \geq 1$.

We say that a sequence $h = (h_i)_{i\geq 0}$ of integers is an *O-sequence* if it satisfies the equivalent conditions in Theorem 1.1.

For a finite sequence $(h_0, h_1, \ldots, h_s)$, we identify it with the infinite sequence $(h_0, h_1, \ldots, h_s, 0, 0, \ldots)$.

We consider $k[\Delta]$ as the graded algebra $k[\Delta] = \bigoplus_{i\geq 0} k[\Delta]_i$ with $\deg x_j = 1$ for $1 \leq j \leq n$. The Hilbert series $F(k[\Delta],t)$ of a Stanley-Reisner ring $k[\Delta]$ can be written as follows:

$$F(k[\Delta],t) = 1 + \sum_{i=1}^{d} \frac{f_{i-1}t^i}{(1-t)^i}$$

$$= \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d},$$

where $\dim \Delta = d-1$, $(f_0, f_1, \ldots, f_{d-1})$ is the f-vector of Δ, and $(h_0, h_1, \ldots, h_d)$ is the h-vector of Δ.

THEOREM 1.2 (Hochster's formula on the local cohomology modules (cf. [St$_1$, Theorem 4.1])).

$$F(H^i_{\mathfrak{m}}(k[\Delta]),\ t) = \sum_{\sigma\in\Delta} \dim_k \tilde{H}_{i-|F|-1}(\mathrm{link}_\Delta F; k) \left(\frac{t^{-1}}{1-t^{-1}}\right)^{|F|}.$$

where $H^i_{\mathfrak{m}}(k[\Delta])$ denote the i-th local cohomology module of $k[\Delta]$ with respect to the graded maximal ideal $\mathfrak{m}$.

Let A be the polynomial ring $k[x_1, x_2, \ldots, x_n]$ for a field k. Let M be a finitely generated graded A-module and let

$$0 \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{h,j}(M)} \longrightarrow \cdots \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{0,j}(M)} \longrightarrow M \longrightarrow 0$$

be a graded minimal free resolution of M over A. We call $\beta_i(M) = \sum_{j\in\mathbf{Z}} \beta_{i,j}(M)$ the i-th Betti number of M over A. We sometimes denote $\beta_i^A(M)$ for $\beta_i(M)$ to emphasize the base ring A. We define a *Castelnuovo-Mumford regularity* reg M of M by

$$\mathrm{reg}\ M = \max\ \{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

We define an *initial degree* indeg M of M by

$$\mathrm{indeg}\ M = \min\ \{i \mid M_i \neq 0\} = \min\ \{j \mid \beta_{0,j}(M) \neq 0\}.$$

THEOREM 1.3(Hochster's formula on the Betti numbers[Hoc, Theorem 5.1]).

$$\beta_{i,j}(k[\Delta]) = \sum_{F\subset[n],\ |F|=j} \dim_k \tilde{H}_{j-i-1}(\Delta_F; k),$$

where

$$\Delta_F = \{G \in \Delta \mid G \subset F\}.$$

Finally we quote some result on Gröbner basis we use later. See [Ei, Chapter 15] for complete explanation.

Let A be the polynomial ring $k[x_1, x_2, \dots, x_n]$ for a field k. Let I be a homogeneous ideal in A. We denote Gin (I) to be a *generic initial ideal* of I with respect to the reverse lexicographic order. It is well known that $h(A/\text{Gin }(I)) = h(A/I)$ and, in particular, $\text{e}(A/\text{Gin }(I)) = \text{e}(A/I)$.
Further we have:

THEOREM 1.4([**Ba-St**]).

$$\text{depth } A/\text{Gin }(I) = \text{depth } A/I$$

and

$$\text{reg Gin }(I) = \text{reg } I.$$

§2. Alexander duality and some generalization of the Eagon-Reiner theorem

First we recall the definition of Alexandr dual complexes.

Definition ([Ea-Re]). For a simplicial complex Δ on the vertex set $[n]$, we define an *Alexander dual complex* Δ^* as follows:

$$\Delta^* = \{F \subset [n] : [n] \setminus F \notin \Delta\}.$$

If $\dim \Delta \leq n-3$, then Δ^* is also a simplicial complex on the vertex set $[n]$.

In the rest of the paper we always assume $\dim k[\Delta] = d$ and $\dim k[\Delta^*] = d^*$ for a fixed field k.
Now we give some generalization of the Eagon-Reiner theorem.

THEOREM 2.1. *Let k be a field. Let Δ be a $(d-1)$-dimensional complex on the vertex set $[n]$. Suppose $d \leq n-2$. Then*

$$\text{reg } I_\Delta - \text{indeg } I_\Delta = \dim k[\Delta^*] - \text{ depth } k[\Delta^*].$$

Proof. Put depth $k[\Delta^*] = p^*$. By Hochster's formula on the local cohomology modules, we have

$$F(H^l_{\mathfrak{m}}(k[\Delta^*]),\ t) = \sum_{F \in \Delta^*} \dim_k \tilde{H}_{l-|F|-1}(\text{link}_{\Delta^*} F; k) \left(\frac{t^{-1}}{1-t^{-1}}\right)^{|F|}.$$

Hence if $l < p^*$, then $\tilde{H}_{l-|F|-1}(\mathrm{link}_{\Delta^*} F; k) = (0)$ for all $F \in \Delta^*$. By the proof in [Ea-Re, Proposition 1], we have $\tilde{H}_{n-l-2}(\Delta_F; k) = (0)$ for all $F \subset [n]$. By Hochster's formula on the Betti numbers this means that $\beta_{i,i+n-l-1}(k[\Delta]) = 0$ for $i \geq 1$. Hence

$$\beta_{i,i+n}(I_\Delta) = \beta_{i,i+n-1}(I_\Delta) = \cdots = \beta_{i,i+n-p^*+1}(I_\Delta) = 0$$

for $i \geq 0$. Similarly, since $\tilde{H}_{n-p^*-2}(\Delta_{[n]\setminus F}; k) \cong \tilde{H}_{p^*-|F|-1}(\mathrm{link}_{\Delta^*} F; k) \neq (0)$ for some $F \in \Delta$, we have $\beta_{i,i+n-p^*}(I_\Delta) \neq 0$ for some $i \geq 0$. Hence $\mathrm{reg}\ I_\Delta = n - p^*$. By the definition of the Alexander dual complex we have $\mathrm{indeg}\ I_\Delta = n - d^*$. Therefore, we have $\mathrm{reg}\ I_\Delta - \mathrm{indeg}\ I_\Delta = d^* - p^*$. Q.E.D.

Let $h = (h_0, h_1, \ldots, h_s)$ be a finite O-seqence. Put $p := \min\{i \geq 1 \mid h_i < \binom{h_1+i-1}{i}\}$. We define the *dual sequence* $h^* = (h_i^*)_{i \geq 0}$ by

$$\sum_{i \geq 0} h_i^* t^i = \frac{1 - t^{h_1}(h_0 + h_1(1-t) + \cdots + h_s(1-t)^s)}{(1-t)^p}.$$

Lemma 2.2. *$h_i^* = 0$ for $i > h_1 + s$.*

Proof. We have

$$\begin{aligned}
\sum_{i \geq 0} h_i^* t^i &= \frac{1 - t^{h_1}(h_0 + h_1(1-t) + \cdots + h_s(1-t)^s)}{(1-t)^p} \\
&= \frac{1 - t^{h_1} \sum_{i=0}^{s} h_i (\sum_{j=0}^{i} (-1)^j \binom{i}{j} t^j)}{(1-t)^p} \\
&= \frac{1 - \sum_{j=0}^{s} (-1)^j (\sum_{i=j}^{s} h_i \binom{i}{j}) t^{h_1+j}}{(1-t)^p}.
\end{aligned}$$

For $l > h_1 + s$, we have

$$\begin{aligned}
h_i^* &= \binom{p+l-1}{l} - \sum_{(j+h_1)+m=l} ((-1)^j \sum_{i=j}^{s} h_i \binom{i}{j}) \binom{p+m-1}{m} \\
&= \binom{p+l-1}{l} - \sum_{j=0}^{l-h_1} ((-1)^j \sum_{i=j}^{s} h_i \binom{i}{j}) \binom{p+l-h_1-j-1}{l-h_1-j-1} \\
&= \binom{p+l-1}{l} - \sum_{i=0}^{s} h_i \sum_{j=0}^{i} (-1)^j \binom{i}{j} \binom{p+l-h_1-j-1}{l-h_1-j-1}
\end{aligned}$$

$$
\begin{aligned}
&= \binom{p+l-1}{l} - \sum_{i=0}^{s} h_i \binom{p+l-h_1-i-1}{l-h_1} \quad \text{([Ri, Page8 (5)])} \\
&= \binom{p+l-1}{l} - \sum_{i=0}^{p-1} \binom{h_1+i-1}{i} \binom{p+l-h_1-i-1}{l-h_1} \\
&= \binom{p+l-1}{l} - \binom{p+l-h_1-1+h_1}{l-h_1+h_1} \quad \text{([Ri, Page8 (3b)])} \\
&= \binom{p+l-1}{l} - \binom{p+l-1}{l} \\
&= 0.
\end{aligned}
$$

Q.E.D.

By the above lemma we can define h^* by

$$
\begin{aligned}
& t^{h_1}(h_0 + h_1(1-t) + \cdots + h_s(1-t)^s) \\
= \; & 1 - (1-t)^p(h_0^* + h_1^* t + \cdots + h_{h_1+s}^* t^{h_1+s}).
\end{aligned}
$$

We justify the notation h^* by the following lemma:

Lemma 2.3. Let Δ be a simplicial complex on the vertex set $[n]$. Then we have

$$h^*(\Delta) = h(\Delta^*).$$

Proof. By the definition we have $f_i(\Delta^*) = \binom{n}{i+1} - f_{n-i-2}(\Delta)$. Put $\dim k[\Delta] = d$, $\dim k[\Delta^*] = d^*$, and $\tau = 1 - t$. Then we have

$$
\begin{aligned}
& \frac{h_0(\Delta) + h_1(\Delta)t + \cdots + h_d(\Delta)t^d}{(1-t)^d} \\
= \; & \sum_{i=0}^{d} \frac{f_{i-1}(\Delta)t^i}{(1-t)^i} \\
= \; & \sum_{i=0}^{n} \frac{(\binom{n}{i} - f_{n-i-1}(\Delta^*))t^i}{(1-t)^i} \\
= \; & \sum_{i=0}^{n} \frac{\binom{n}{i} t^i}{(1-t)^i} - \sum_{i=0}^{n} \frac{f_{n-i-1}(\Delta^*)t^i}{(1-t)^i} \\
= \; & \left(1 + \frac{t}{1-t}\right)^n - \left(\frac{t}{1-t}\right)^n \sum_{i=0}^{n} \frac{f_{n-i-1}(\Delta^*)\tau^{n-i}}{(1-\tau)^{n-i}}
\end{aligned}
$$

$$= \frac{1}{(1-t)^n} - \left(\frac{t}{1-t}\right)^n \frac{h_0(\Delta^*) + h_1(\Delta^*)\tau + \cdots + h_{d^*}(\Delta^*)\tau^{d^*}}{(1-\tau)^{d^*}}$$
$$= \frac{1}{(1-t)^n} - \frac{t^{n-d^*}(h_0(\Delta^*) + h_1(\Delta^*)(1-t) + \cdots + h_{d^*}(\Delta^*)(1-t)^{d^*})}{(1-t)^n}.$$

Therefore, since $p = n - d^*$ we have

$$\begin{aligned} &(1-t)^{n-d}(h_0(\Delta) + h_1(\Delta)t + \cdots + h_d(\Delta)t^d) \\ = \; &1 - t^{n-d^*}(h_0(\Delta^*) + h_1(\Delta^*)(1-t) + \cdots + h_{d^*}(\Delta^*)(1-t)^{d^*}) \end{aligned}$$

Q.E.D.

§3. Application to the h-vectors of homogeneous rings

For a sequence $h = (h_i)_{i \geq 0}$ of integers, we define the *partial sum sequence* Sh of h by

$$Sh = (h_0, h_0 + h_1, h_0 + h_1 + h_2, \ldots, \sum_{j=0}^{i} h_j, \ldots).$$

And inductively we define the i-th iterated partial sum sequence $S^i h$ by $S^i h = S(S^{i-1}h)$.

The next proposition is a variation of Stanley.

Proposition 3.1(cf. [St$_3$, Corollary 3.11]). *Let k be a field. Let $h = (h_0, h_1, \cdots, h_s)$ be a seqence of integers with $h_0 + h_1 + \cdots + h_s > 0$. We fix an integer $c \geq 0$. Then the following conditions are equivalent:*
(1) There exists a simplicial complex Δ with $\dim k[\Delta] - \operatorname{depth} k[\Delta] \leq c$ such that $h = h(k[\Delta])$.
(2) There exists a homogeneous k-algebra R with $\dim R - \operatorname{depth} R \leq c$ such that $h = h(R)$.
(3) The c-th iterated partial sum sequence $S^c h$ of h is an O-sequence.

Proof. We may assume $| k | = \infty$. Put $\dim R = d$.
(1)$\Rightarrow$ (2). Trivial.

(2)$\Rightarrow$ (3). (A)Case $d-c \leq 0$. The c-th iterated partial sum sequence $S^c h$ of h is the $(c-d)$-th iterated partial sum sequence of $(\dim_k R_i)_{i\geq 0}$. Then $S^c h$ is an O-sequence.

(B)Case $d-c > 0$. We have depth$R \geq d-c$. Let $\{y_1, y_2, \dots y_{d-c}\}$ be a regular sequence in $k[\Delta]_1$. Then the c-th iterated partial sum sequence $S^c h$ of h is $(\dim(R/(y_1, y_2, \dots y_{d-c}))_i)_{i\geq 0}$, which is an O-sequence.

(3)$\Rightarrow$ (1). There exists a monomial ring R (i.e., $R = A/I$, where I is generated by monomials) whose Hilbert function is $S^c h$. Note that $\dim R = c$. Let $k[\Delta]$ be a porlarization of R (See [St-Vo] for the definition and basic properties of the porlarization). Then

$$\begin{aligned} \dim k[\Delta] - \text{depth } k[\Delta] &= \dim R - \text{depth } R \\ &\leq \dim R \\ &= c. \end{aligned}$$

Q.E.D.

We have the following theorem which gives a characterization of h-vector of homogeneous k-algebras $R = A/I$ with reg I − indeg $I \leq c$.

THEOREM 3.2. *Let k be a field. Let $h = (h_0, h_1, \cdots, h_s)$ be an integer sequence with $h_1 \geq 2$, and $h_0 + h_1 + \cdots + h_s > 0$. We fix an integer $c \geq 0$. Then the following conditions are equivalent:*
(1) There exists a homogeneous k-algebra $R = A/I$ with

$$\text{reg } I - \text{indeg } I \leq c$$

such that $h = h(R)$, where A is a polynomial ring and I is a homogeneous ideal with $I_1 = (0)$.
(2) There exists a simplicial complex Δ with

$$\text{reg} I_\Delta - \text{indeg } I_\Delta \leq c$$

such that $h = h(\Delta)$.
(3)The c-th iterated partial sum sequence $S^c(h^*)$ of the dual sequence h^* of h is an O-sequence.

Proof. (1)$\Rightarrow$(2). Let $R = A/I$ be a k-algebra satisfying the conditions in (1). Since we have reg Gin(I) = reg I, we have

$$\text{reg Gin}(I) - \text{indeg Gin}(I) \leq c$$

and $h = h(A/\mathrm{Gin}(I))$. Considering the porlarization, we obtain a Stanley-Reisner ring $k[\Delta]$ satisfying the conditions in (2).

(2)$\Rightarrow$(1). Trivial.

(2)$\Rightarrow$(3). If Δ is a simplicial complex with the conditions in (2), then Theorem 2.1 we have

$$\dim k[\Delta^*] - \text{depth } k[\Delta^*] \leq c.$$

And by Lemma 2.5 we have $h^* = h(k[\Delta^*])$. Hence by Proposition 3.1, the condition (3) holds for h.

(3)$\Rightarrow$(2). If h^* satisfies the condition (3), there exists a simplicial complex Δ such that for its Alexander dual complex Δ^*, $h^* = h(k[\Delta^*])$ and

$$\dim k[\Delta^*] - \text{depth } k[\Delta^*] \leq c.$$

then we have $h = h(k[\Delta])$ and

$$\text{reg} I_\Delta - \text{indeg } I_\Delta \leq c.$$

Q.E.D.

Remark. The inequality reg $I_\Delta -$ indeg $I_\Delta \leq c$ means that at most (indeg I_Δ, indeg $I_\Delta + 1, \ldots,$indeg $I_\Delta + c$)-linear parts appear in the minimal free resolution of I_Δ.

§4. On upper bounds for multiplicities

In this section we give some upper bound for the multiplicities of homogeneous k-algebras. And we deduce some partial affermative result on the Herzog-Srinivasan conjecture.

First we prove the following lemma:

LEMMA 4.1.

$$\mathrm{e}(k[\Delta]) = \beta_{1,h_1}(k[\Delta^*]).$$

Proof. We have

$$\begin{aligned} & h_0(\Delta) + h_1(\Delta)(1-t) + \cdots + h_d(\Delta)(1-t)^d & (1) \\ = & \frac{1 - (1-t)^{n-d^*}(h_0(\Delta^*) + h_1(\Delta^*)t + \cdots + h_{d^*}(\Delta^*)t^{d^*})}{t^{n-d}}, & (2) \end{aligned}$$

by Lemma 2.5. Since indeg $I_{\Delta^*} = n - d = h_1$, we have

$$
\begin{aligned}
& \beta_{1,n-d}(k[\Delta^*]) \\
= \; & (\text{the coefficient of } t^{n-d} \text{ in } -(1-t)^{n-d^*}(h_0(\Delta^*) + h_1(\Delta^*)t + \cdots + h_{d^*}(\Delta^*)t^{d^*})) \\
= \; & (\text{the coefficient of } t^{n-d} \text{ in the numerator in (2)}) \\
= \; & \lim_{t\to 0}(h_0(\Delta) + h_1(\Delta)(1-t) + \cdots + h_d(\Delta)(1-t)^d) \\
= \; & \mathrm{e}(k[\Delta]).
\end{aligned}
$$

Q.E.D.

THEOREM 4.2. *Let $R = A/I$ be a homogeneous k-algebra of codimension $h_1 \geq 2$. Then*

$$\mathrm{e}(R) \leq \binom{\mathrm{reg}\ I + h_1 - 1}{h_1}.$$

Proof. By Theorem 1.4, we have reg Gin(I) = reg I and $h(A/I) = h(A/\mathrm{Gin}(I))$. Considering the porlarization, we obtain a Stanley-Reisner ring $k[\Delta] = B/I_\Delta$ with $\mathrm{e}(A/I) = \mathrm{e}(k[\Delta])$ and reg I = reg I_Δ. To evaluate $\mathrm{e}(k[\Delta])$, we may assume $|\ k\ | = \infty$. Put $p^* =$ depth $k[\Delta^*]$. By Theorem 2.1, we have $d^* - p^* =$ reg $I - (n - d^*)$, where $n =$ embdim $k[\Delta^*]$. Hence reg $I = n - p^*$.

Let $y_1, y_2, \ldots, y_{p^*}$ be a regular sequence in $k[\Delta^*]_1$. By Lemma 4.2 we have

$$
\begin{aligned}
\mathrm{e}(k[\Delta]) &= \beta_{1,h_1}(k[\Delta^*]) \\
&= \beta_{1,h_1}^{B/(y_1,y_2,\ldots,y_{p^*})}(k[\Delta^*]/(y_1, y_2, \ldots, y_{p^*})) \leq \binom{n - p^* + h_1 - 1}{h_1}.
\end{aligned}
$$

Q.E.D.

COROLLARY 4.3 *Let $R = A/I$ be a homogeneous k-algebra of codimension $h_1 \geq 2$ with $\beta_{0,\mathrm{reg}\ I} \neq 0$. Then Conjecture 0.3 holds.*

Proof. Since $\mathrm{reg}_i(R) = \mathrm{reg}\ I + i - 1$ for $1 \leq i \leq h_1$, we have

$$\mathrm{e}(R) \leq \binom{\mathrm{reg}\ I + h_1 - 1}{h_1} = \frac{\prod_{i=1}^{h_1} \mathrm{reg}_i(R)}{h_1!}.$$

Q.E.D.

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