Second, third and fourth Betti numbers of Stanley-Reisner rings

Naoki Terai
Department of Mathematics
Nagano National College of Technology
Nagano 381, Japan

Takayuki Hibi
Department of Mathematics
Hokkaido University
Sapporo 060, Japan

Abstract

We study the Betti numbers which appear in a minimal free resolution of the Stanley–Reisner ring $k[\Delta] = A/I_\Delta$ of a simplicial complex Δ over a field k. The Alexander duality theorem of topology enables us to give a short proof to the fact that the second Betti number of $k[\Delta]$ does not depend on the base field k. Moreover, when the ideal I_Δ is generated by square-free monomials of degree two, we show that the third and fourth Betti numbers are independent of k. Some concrete examples of simplicial complexes Δ for which every Betti number of $k[\Delta]$ is independent of k are also discussed.

Introduction

Let $A = k[x_1, x_2, \ldots, x_v]$ denote the polynomial ring in v-variables over a field k, which will be considered to be the graded algebra $A = \bigoplus_{n\geq 0} A_n$ over k with the standard grading, i.e., each $\deg x_i = 1$. Let $\mathbf{Z}$ (resp. $\mathbf{Q}$) denote the set of integers (resp. rational numbers). We write $A(j)$, $j \in \mathbf{Z}$, for the graded module $A(j) = \bigoplus_{n\in\mathbf{Z}}[A(j)]_n$ over A with $[A(j)]_n := A_{n+j}$. Let I be an ideal of A generated by homogeneous polynomials and R the quotient algebra A/I. When R is regarded as a graded module over A with the quotient grading, it has a graded *finite free resolution*

$$0 \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{hj}} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{1j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} R \longrightarrow 0; \qquad (1)$$

where each $\bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{ij}}$, $1 \leq i \leq h$, is a graded free module of rank $0 \neq \sum_{j\in\mathbf{Z}} \beta_{ij} < \infty$, and where every φ_i is degree-preserving. Moreover, there exists a unique such resolution which minimizes each β_{i_j}; such a resolution is called *minimal*. If a finite free resolution (1) is minimal, then the *homological dimension* $\mathrm{hd}_A(R)$ of R over A is the non-negative integer h and $\beta_i = \beta_i^A(R) := \sum_{j\in\mathbf{Z}} \beta_{i_j}$ is called the i-th *Betti number* of R over A.

In this paper, we study the Betti numbers of $R = A/I$ over A when an ideal I is generated by square-free monomials, i.e., R is the Stanley-Reisner ring $k[\Delta] = A/I_\Delta$ associated with a simplicial complex Δ ([Sta$_1$], [Rei]). Even though $\mathrm{hd}_A(k[\Delta])$ may depend on the base field k, (with a fixed field k) the integer $v - \mathrm{hd}_A(k[\Delta])$ is topological [Mun], i.e., it depends only on the geometric realization of Δ. Since the first Betti number $\beta_1^A(k[\Delta])$ is equal to the minimal number of generators of the ideal I_Δ, $\beta_1^A(k[\Delta])$ is independent of the base field k. However, in general, $\beta_i^A(k[\Delta])$ may depend on k. It is known, e.g., [Bru–Her$_2$] that the second Betti number $\beta_2^A(k[\Delta])$ does not depend on the base field k. We give a short proof of this result by using the Alexander duality theorem of topology. Moreover, when the ideal I_Δ is generated by square-free monomials of degree two (e.g., Δ is the order complex of a finite partially ordered set), we show that both the third and fourth Betti numbers of $k[\Delta]$ over A are independent of k. On the other hand, it would be of interest to find a natural class of simplicial complexes Δ for which all Betti numbers $\beta_i^A(k[\Delta])$ are independent of k. We show that, for example, if the geometric realization of Δ is either a 3-sphere or a 3-ball, then all Betti numbers of $k[\Delta]$ are independent of k.

§1. Simplicial complexes and Hochster's formula

We first recall some notation on simplicial complexes and Hochster's topological formula on Betti numbers of Stanley-Reisner rings. We refer the reader to, e.g., [Bru–Her$_1$], [H$_1$], [Hoc] and [Sta$_1$] for the detailed information about combinatorial and algebraic background.

(1.1) A *simplicial complex* Δ on the *vertex set* $V = \{x_1, x_2, \ldots, x_v\}$ is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for every $1 \le i \le v$ and (ii) $\sigma \in \Delta$, $\tau \subset \sigma \Rightarrow \tau \in \sigma$. Each element σ of Δ is called a *face* of Δ. Let $\sharp(\sigma)$ denote the cardinality of a finite set σ. We set $d = \max\{\sharp(\sigma) \mid \sigma \in \Delta\}$ and define the *dimension* of Δ to be $\dim \Delta = d - 1$.

Given a subset W of V, the *restriction* of Δ to W is the subcomplex

$$\Delta_W = \{\sigma \in \Delta \mid \sigma \subset W\}$$

of Δ. In particular, $\Delta_V = \Delta$ and $\Delta_\emptyset = \{\emptyset\}$. On the other hand, if σ is a face of Δ, then we define the subcomplexes $\mathrm{link}_\Delta(\sigma)$ and $\mathrm{star}_\Delta(\sigma)$ to be

$$\mathrm{link}_\Delta(\sigma) = \{\tau \in \Delta \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\};$$

$$\mathrm{star}_\Delta(\sigma) = \{\tau \in \Delta \mid \sigma \cup \tau \in \Delta\}.$$

Thus, in particular, $\mathrm{link}_\Delta(\emptyset) = \mathrm{star}_\Delta(\emptyset) = \Delta$.

Let $\tilde{H}_i(\Delta; k)$ denote the i-th *reduced simplicial homology group* of Δ with the coefficient field k. Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \neq \{\emptyset\}$ and

$$\tilde{H}_i(\{\emptyset\}; k) = \begin{cases} 0 & (i \geq 0) \\ k & (i = -1). \end{cases}$$

(1.2) Let $A = k[x_1, x_2, \ldots, x_v]$ be the polynomial ring in v-variables over a field k. Here, we identify each $x_i \in V$ with the indeterminate x_i of A. Define I_Δ to be the ideal of A which is generated by square-free monomials $x_{i_1}x_{i_2}\cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq v$, with $\{x_{i_1}, x_{i_2}, \cdots, x_{i_r}\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the *Stanley-Reisner ring* of Δ over k. In what follows, we consider A to be the graded algebra $A = \bigoplus_{n \geq 0} A_n$ with the standard grading, i.e., each $\deg x_i = 1$, and may regard $k[\Delta] = \bigoplus_{n \geq 0}(k[\Delta])_n$ as a graded module over A with the quotient grading.

(1.3) Let $h = \mathrm{hd}_A(k[\Delta])$ denote the homological dimension of $k[\Delta]$ over A and consider a graded minimal free resolution

$$0 \longrightarrow \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{h_j}} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{1_j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \longrightarrow 0 \quad (2)$$

of $k[\Delta]$ over A. It is known that $v - d \leq h \leq v$. Hochster's formula [Hoc, Theorem (5.1)] guarantees that

$$\beta_{i_j} = \sum_{W \subset V,\ \sharp(W) = j} \dim_k \tilde{H}_{j-i-1}(\Delta_W; k). \quad (3)$$

Thus, in particular,

$$\beta_i^A(k[\Delta]) = \sum_{W \subset V} \dim_k \tilde{H}_{\sharp(W)-i-1}(\Delta_W; k). \quad (4)$$

Some combinatorial and algebraic applications of Hochster's formula have been studied. Munkres [Mun] proved that $v - \mathrm{hd}_A(k[\Delta])$ depends only on the geometric realization of Δ. Moreover, if Δ is the order complex of a modular lattice, then the last Betti number of $k[\Delta]$ can be computed by means of the Möbius function of the lattice ([H$_2$], [H$_3$]). See also [Bac], [B–H$_1$], [B–H$_2$], [Frö] and [H$_4$] for related topics and results.

§2. Second Betti numbers of Stanley–Reisner rings

It is known, e.g., [Bru–Her$_2$] that the second Betti number of a Stanley-Reisner ring is independent of the base field. By virtue of Hochster's formula together with the Alexander duality theorem of topology, we give a short proof of this result. Let $\mid \Delta \mid$ denote the geometric realization of a simplicial complex Δ.

(2.1) LEMMA. *Let Δ be a simplicial complex on the vertex set V with $\sharp(V) = v$ and k a field. Then $\dim_k \tilde{H}_{v-3}(\Delta; k)$ is independent of k.*

Proof. Let 2^V denote the set of all subsets of V. Thus, the geometric realization X of the simplicial complex $2^V - \{V\}$ is the $(v-2)$-sphere. We may assume that $V \notin \Delta$; in particular, $\mid \Delta \mid$ is a subspace of X. Note that $\widetilde{H}_{v-3}(\mid \Delta \mid; k) \cong \widetilde{H}^{v-3}(\mid \Delta \mid; k)$ since k is a field. Now, the Alexander duality theorem guarantees that $\widetilde{H}^{v-3}(\mid \Delta \mid; k) \cong \tilde{H}_0(X - \mid \Delta \mid; k)$. On the other hand, $\dim_k \tilde{H}_0(X - \mid \Delta \mid; k) + 1$ is equal to the number of connected components of $X - \mid \Delta \mid$. Thus, $\dim_k \widetilde{H}_{v-3}(\Delta; k) = \dim_k \tilde{H}_0(X - \mid \Delta \mid; k)$ is independent of the base field k as required. Q. E. D.

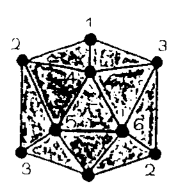
(2.2) THEOREM. *The second Betti number $\beta_2^A(k[\Delta])$ of the Stanley-Reisner ring $k[\Delta] = A/I_\Delta$ of a simplicial complex Δ is independent of the base field k.*

Proof. By virtue of Hochster's formula (4), the second Betti number $\beta_2^A(k[\Delta])$ is equal to $\sum_{W \subset V} \dim_k \tilde{H}_{\sharp(W)-3}(\Delta_W; k)$, which is independent of k by Lemma (2.1) as desired. Q. E. D.

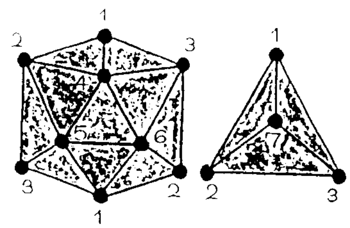
Let Γ be the simplicial complex on the vertex set $V = \{1, 2, 3, 4, 5, 6\}$ drawn below (cf. [Rei]). Thus, $\mid \Gamma \mid$ is the real projective plane. We then have

$$\operatorname{hd}_A(k[\Gamma]) = \begin{cases} 3 & (\operatorname{char}(k) \neq 2) \\ 4 & (\operatorname{char}(k) = 2). \end{cases}$$

We have $\beta_0 = 1, \beta_1 = 10, \beta_2 = 15, \beta_3 = 6$ if $\operatorname{char}(k) \neq 2$, while $\beta_0 = 1, \beta_1 = 10, \beta_2 = 15, \beta_3 = 7, \beta_4 = 1$ if $\operatorname{char}(k) = 2$.



On the other hand, let Δ denote the simplicial complex on the vertex set $V = \{1,2,3,4,5,6,7\}$ drawn below (cf. [Bjö$_2$], [H$_2$]). Then $\mathrm{hd}_A(k[\Delta]) = 4$ for an arbitrary field k. However, we have $\beta_0 = 1, \beta_1 = 13, \beta_2 = 27, \beta_3 = 19, \beta_4 = 4$ if $\mathrm{char}(k) \neq 2$, while $\beta_0 = 1, \beta_1 = 13, \beta_2 = 27, \beta_3 = 20, \beta_4 = 5$ if $\mathrm{char}(k) = 2$.



§3. Ideals I_Δ generated by monomials of degree two

The purpose of this section is to show that the third and fourth Betti numbers of a Stanley-Reisner ring $k[\Delta] = A/I_\Delta$ are independent of the base field k when the ideal I_Δ is generated by square-free monomials of degree two. For example, the ideal I_Δ associated with a simplicial complex Δ is generated by square-free monomials of degree two when, e.g., Δ is the order complex ([Sta$_3$, p.120]) of a finite partially ordered set.

Let Δ (resp. Δ') be a simplicial complex on the vertex set V (resp. V') and suppose that $V \cap V' = \emptyset$. Recall that the *simplicial join* $\Delta * \Delta'$ of Δ and Δ' is the simplicial complex on the vertex set $V \cup V'$ which consists of all subsets of $V \cup V'$ of the form $\sigma \cup \tau$ with $\sigma \in \Delta$ and $\tau \in \Delta'$.

(3.1) LEMMA. *Let Δ be a simplicial complex on the vertex V with $\sharp(V) = v$ and suppose that the ideal I_Δ is generated by square-free monomials of degree two. Then $\tilde{H}_n(\Delta; k) = 0$ if $v < 2(n+1)$. Moreover, if $v = 2(n+1)$, then $\tilde{H}_n(\Delta; k) \neq 0$ if and only if Δ is the simplicial join of $n+1$ copies of the 0-sphere $\mathrm{S}^0 (= \bullet \;\; \bullet)$.*

Proof. We first show that $\tilde{H}_n(\Delta; k) = 0$ if $v < 2(n+1)$. Suppose that $I_\Delta \neq (0)$ and $x, y \in V$ with $xy \in I_\Delta$. We set $\Delta_1 = \mathrm{star}_\Delta(\{x\})$ and $\Delta_2 = \Delta_{V-\{x\}}$. Then $\Delta_1 \cup \Delta_2 = \Delta$ and $\Delta_1 \cap \Delta_2 = \mathrm{link}_\Delta(\{x\})$. Note that the ideals $I_{\Delta_1}, I_{\Delta_2}, I_{\Delta_1 \cap \Delta_2}$ are generated by square-free monomials of degree two. On the other hand, since $\{y\} \notin \Delta_1$, $\{x\} \notin \Delta_2$ and $\{x\}, \{y\} \notin \Delta_1 \cap \Delta_2$, we may assume that $\tilde{H}_n(\Delta_1; k) = 0$, $\tilde{H}_n(\Delta_2; k) = 0$ and $\tilde{H}_{n-1}(\Delta_1 \cap \Delta_2; k) =$

0. Hence, thanks to the reduced Mayer-Vietoris exact sequence, we have $\tilde{H}_n(\Delta; k) = 0$ as desired.

Secondly, let us assume $v = 2(n+1)$. If Δ is the simplicial join of $n+1$ copies of $\mathbf{S}^0$, then the geometric realization of Δ is the n-sphere $\mathbf{S}^n$. Thus $\tilde{H}_n(\Delta; k) \neq 0$. On the other hand, suppose that $\tilde{H}_n(\Delta; k) \neq 0$. Then $I_\Delta \neq (0)$. Let $xy \in I_\Delta$ and $\Delta_1 = \mathrm{star}_\Delta(\{x\})$, $\Delta_2 = \Delta_{V-\{x\}}$ as above. Since $\tilde{H}_n(\Delta_1; k) = 0$, $\tilde{H}_n(\Delta_2; k) = 0$ and $\tilde{H}_n(\Delta; k) \neq 0$, the reduced Mayer-Vietoris exact sequence guarantees that $\tilde{H}_{n-1}(\mathrm{link}_\Delta(\{x\}); k) \neq 0$. Let v' be the number of vertices of $\mathrm{link}_\Delta(\{x\})$. Then $v' \leq v - 2 = 2n$ since $\{x\}, \{y\} \notin \mathrm{link}_\Delta(\{x\})$, while $v' \geq 2n$ since $\tilde{H}_{n-1}(\mathrm{link}_\Delta(\{x\}); k) \neq 0$. Hence $v' = 2n$. Thus, we may assume that $\mathrm{link}_\Delta(\{x\})$ is the simplicial join of n copies of $\mathbf{S}^0$. Let $z \in V$ be an arbitrary vertex of Δ with $z \neq x$ and $z \neq y$. Then, since $v' = 2n$, $\{z\} \in \mathrm{link}_\Delta(\{x\})$. Hence, there exists an element $w \in V - \{x, y\}$ such that $zw \in I_{\mathrm{link}_\Delta(\{x\})}$. Since I_Δ is generated by square-free monomials of degree two, we have $zw \in I_\Delta$. Consequently, for an arbitrary element $\alpha \in V$, there exists a unique element $\beta \in V$ such that $\{\alpha, \beta\} \notin \Delta$. Hence, Δ is the simplicial join of $n+1$ copies of the 0-sphere $\mathbf{S}^0$ as required.

Q. E. D.

(3.2) COROLLARY. *Suppose that the ideal I_Δ is generated by square-free monomials of degree two and that a finite free resolution* (2) *of $k[\Delta] = A/I_\Delta$ over A is minimal. Then, $\beta_{i_j} = 0$ for all i and j with $j > 2i$.*

Proof. By Lemma (3.1), we have $\tilde{H}_{\sharp(W)-i-1}(\Delta_W; k) = 0$ if $\sharp(W) < 2(\sharp(W) - i)$, i.e., $\sharp(W) > 2i$. Hence, thanks to Hochster's formula (3), $\beta_{i_j} = 0$ for all i and j with $j > 2i$.

Q. E. D.

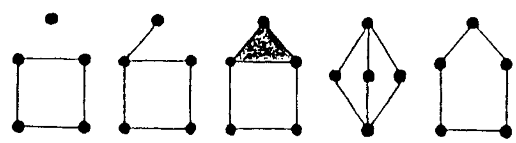
Taylor [Tay] constructed an explicit (not nesessarily minimal) finite free resolution of $k[\Delta] = A/I_\Delta$ over A. The above Corollary (3.2) also follows immediately from Taylor resolutions.

(3.3) LEMMA. *Let Δ be a simplicial complex on the vertex set V with $\sharp(V) = 7$. Suppose that I_Δ is generated by square-free monomials of degree two and that $\tilde{H}_2(\Delta; k) \neq 0$. Then, one of the following conditions* (i) *and* (ii) *is satisfied:*

(i) *Δ is the simplicial join of the cycle of length* 5 *and* 0*-sphere* $\mathbf{S}^0$;
(ii) *there exists $x \in V$ such that* $\Delta_{V-\{x\}} = \mathbf{S}^0 * \mathbf{S}^0 * \mathbf{S}^0$.

Proof. Suppose that there exists no $x \in V$ with $\Delta_{V-\{x\}} = \mathbf{S}^0 * \mathbf{S}^0 * \mathbf{S}^0$. Let $x \in V$ and set $\Delta_1 = \mathrm{star}_\Delta(\{x\})$, $\Delta_2 = \Delta_{V-\{x\}}$. Then $\Delta = \Delta_1 \cup \Delta_2$ and

$\mathrm{link}_\Delta(\{x\}) = \Delta_1 \cap \Delta_2$. Since Δ_1 is contractible, we have $\tilde{H}_2(\Delta_1; k) = 0$. On the other hand, since $\Delta_2 \neq \mathbf{S}^0 * \mathbf{S}^0 * \mathbf{S}^0$, we have $\tilde{H}_2(\Delta_2; k) = 0$ by Lemma (3.1). Thus, thanks to the reduced Mayer-Vietoris exact sequence, we have $\tilde{H}_1(\mathrm{link}_\Delta(\{x\}); k) \neq 0$ since $\tilde{H}_2(\Delta; k) \neq 0$. Let V' denote the vertex set of $\mathrm{link}_\Delta(\{x\})$. Then $\sharp(V') \geq 4$ by Lemma (3.1). Moreover, again by Lemma (3.1), if $\sharp(V') = 4$, then $\mathrm{link}_\Delta(\{x\})$ is the cycle of length 4. If the number of vertices of $\mathrm{link}_\Delta(\{y\})$ is equal to 4 for every $y \in V$, then $\dim \Delta = 2$ and the number of faces σ of Δ with $\sharp(\sigma) = 3$ is $(4 \times \sharp(V)) \div 3 = \frac{28}{3}$, a contradiction. Hence, there exists $z \in V$ such that the number of vertices of $\mathrm{link}_\Delta(\{z\})$ is greater than or equal to 5. If the number of vetrices of $\mathrm{link}_\Delta(\{z\})$ is equal to 6, then $\Delta = \mathrm{star}_\Delta(\{z\})$ and, therefore, Δ is contractible, which contradicts $\tilde{H}_2(\Delta; k) \neq 0$. Thus, the number of vertices of $\mathrm{link}_\Delta(\{z\})$ is equal to 5. Since $\tilde{H}_1(\mathrm{link}_\Delta(\{z\}); k) \neq 0$ and the ideal $I_{\mathrm{link}_\Delta(\{z\})}$ is generated by square-free monomials of degree two, it follows easily that $\mathrm{link}_\Delta(\{z\})$ is one of the following figures:



(a) (b) (c) (d) (e)

If $\mathrm{link}_\Delta(\{z\})$ is one of the above figures (a), (b), (c) and (d), and if $\tilde{H}_2(\Delta; k) \neq 0$, then there exists $x \in V$ with $\Delta_{V-\{x\}} = \mathbf{S}^0 * \mathbf{S}^0 * \mathbf{S}^0$ (the routine details should be omitted). On the other hand, if $\mathrm{link}_\Delta(\{z\})$ is the graph of figure (e) and if $\tilde{H}_2(\Delta; k) \neq 0$, then Δ is the simplicial join of the cycle of length 5 and 0-sphere $\mathbf{S}^0$ as required.

Q. E. D.

We are now in the position to state the main result of this section.

(3.4) THEOREM. *Let Δ be a simplicial complex and suppose that the ideal I_Δ is generated by square-free monomials of degree two. Then, both the third Betti number $\beta_3^A(k[\Delta])$ and the fourth Betti number $\beta_4^A(k[\Delta])$ of $k[\Delta] = A/I_\Delta$ over A are independent of the base field k.*

Proof. First, we study the third Betti number $\beta_3^A(k[\Delta])$ of $k[\Delta]$ over A. Let V be the vertex set of Δ. Thanks to Proposition (3.2), what we must prove is that β_{3j} is independent of the base field k for every $j \leq 6$. Thus, by virtue of Hochster's formula (3), what we must prove is that

$\dim \tilde{H}_{\sharp(W)-4}(\Delta_W; k)$ is independent of k for every $W \subset V$ with $\sharp(W) \leq 6$. If $\sharp(W) = 5$, then $\tilde{H}_i(\Delta_W; k) = 0$ for every $i \geq 2$ by Lemma (3.1). Thus, since the reduced Euler characteristic $\tilde{\chi}(\Delta)$ and $\dim_k \tilde{H}_0(\Delta_W; k)$ are independent of k, it follows from Euler-Poincaré formula that $\dim \tilde{H}_1(\Delta_W; k)$ is independent of k. On the other hand, if $\sharp(W) = 6$, then $\dim \tilde{H}_2(\Delta_W; k) = 0$ unless Δ_W is the simplicial join of three copies of the 0-sphere by Lemma (3.1). Moreover, if Δ_W is the simplicial join of three copies of the 0-sphere, then $\dim \tilde{H}_2(\Delta_W; k) = 1$ for an arbitrary field k.

Secondly, we show that the fourth Betti number $\beta_4^A(k[\Delta])$ of $k[\Delta]$ over A is independent of the base field k. We must prove that $\dim \tilde{H}_{\sharp(W)-5}(\Delta_W; k)$ is independent of k for every $W \subset V$ with $\sharp(W) \leq 8$. If either $\sharp(W) = 6$ or $\sharp(W) = 8$, then we can show that $\dim \tilde{H}_{\sharp(W)-5}(\Delta_W; k)$ is independent of k by the similar technique with Lemma (3.1) as above. Let $\sharp(W) = 7$ and suppose that $\tilde{H}_2(\Delta_W; k) \neq 0$. Then, by Lemma (3.3), we easily see that Δ_W has the homotopy type of one of the following spaces: (i) the 2-sphere; (ii) the disjoint union of the 2-sphere and a single point; (iii) the space $X \cup Y$, where X is the 2-sphere and Y is either the 1-sphere or the 2-sphere, such that $X \cap Y$ consists of a single point. Hence, $\dim_k \tilde{H}_2(\Delta_W; k)$ is independent of the base field k as desired.

Q. E. D.

§4. Finite free resolutions of the n-sphere

In general, it is possible to define the Stanley-Reisner ring $\mathbf{Z}[\Delta] = A/I_\Delta$ of Δ over the commutative ring $\mathbf{Z}$. However, a minimal free resolution of $\mathbf{Z}[\Delta]$ over the polynimial ring $A = \mathbf{Z}[x_1, x_2, \ldots, x_v]$ does not necessarily exist. On the other hand, there exists a minimal free resolution of $\mathbf{Z}[\Delta]$ over A if and only if all Betti numbers $\beta_i^A(k[\Delta])$ are independent of the base field k (see, e.g., [H–K]). Thus, it might be of interest to find a natural class of simplicial complexes Δ for which all Betti numbers $\beta_i^A(k[\Delta])$ are independent of k. The main purpose of this section is to show that if $|\Delta|$ is the n-sphere $\mathbf{S}^n$ (or the n-ball $\mathbf{B}^n$) with $n \leq 3$, then all Betti numbers $\beta_i^A(k[\Delta])$ of $k[\Delta]$ are independent of k. Moreover, we construct a shellable simplicial complex Δ with $|\Delta| = \mathbf{S}^4$ such that some Betti number $\beta_i^A(k[\Delta])$ does depend on the base field k.

(4.1) PROPOSITION. (a) *Let Δ be a simplicial complex and suppose that the geometric realization $|\Delta|$ of Δ is a connected 3-manifold without boundary. Then, all Betti numbers $\beta_i^A(k[\Delta])$ are independent of the base field k if $|\Delta|$ is orientable and $\tilde{H}_1(\Delta; \mathbf{Z}) = 0$.*

(b) *Let Δ be a simplicial complex such that $\mid \Delta \mid$ is a connected 2-manifold without boundary. Then, all Betti numbers $\beta_i^A(k[\Delta])$ are independent of the base field k if and only if $\mid \Delta \mid$ is orientable.*

Proof. By virtue of Hochster's formula, in order for all Betti numbers $\beta_i^A(k[\Delta])$ to be independent of the base field k, it is necessary and sufficient that $\dim_k \tilde{H}_j(\Delta_W; k)$ is independent of k for every subset W of the vertex set V and for each integer $j \geq -1$.

(a) Suppose that $\mid \Delta \mid$ is orientable and that $\tilde{H}_1(\Delta; \mathbf{Z}) = 0$. Let $W = V$, i.e., $\Delta_W = \Delta$. Obviously, $\dim_k \tilde{H}_0(\Delta; k) = 0$. Since $\tilde{H}_1(\Delta; \mathbf{Z}) = 0$, it follows that $\dim_k \tilde{H}_1(\Delta; k) = 0$. Moreover, by Poincaré duality, $\dim_k \tilde{H}_3(\Delta; k) = 1$ and $\dim_k \tilde{H}_2(\Delta; k) = 0$. Let W denote an arbitrary non-empty subset of V with $W \neq V$. Since $\mid \Delta \mid$ is orientable, by Alexander duality, we have $(\tilde{H}_2(\Delta_W; k) \cong)\tilde{H}^2(\Delta_W; k) \cong \tilde{H}_0(\mid \Delta \mid - \mid \Delta_W \mid; k)$. Hence, $\dim_k \tilde{H}_2(\Delta_W; k)$ is independent of k. Thus, since $\tilde{H}_3(\Delta_W; \mathbf{Z})$ is torsion-free, it follows that $\dim_k \tilde{H}_3(\Delta_W; k)$ is independent of k. Moreover, since $\tilde{\chi}(\Delta_W)$ is independent of k, $\dim_k \tilde{H}_1(\Delta_W; k)$ is also independent of the base field k.

(b) First, suppose that $\mid \Delta \mid$ is non-orientable. Then $\tilde{H}_2(\Delta; \mathbf{Q}) = 0$. Since $H_2(\Delta; \mathbf{Z}/2\mathbf{Z}) \cong H^0(\Delta; \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$ by Poincaré duality, it follows that $\dim_k \tilde{H}_2(\Delta; k)$ depends on the base field k. On the other hand, let us assume that $\mid \Delta \mid$ is orientable. By Poincaré duality, we have $\dim_k \tilde{H}_2(\Delta; k) = 1$. Moreover, if W is a subset of V with $W \neq V$ and if Δ_W is of dimension two, then Δ_W possesses non-empty boundary. Hence Δ_W has the homotopy type of the geometric realization of a one-dimensional simplicial complex; in particular $\dim_k \tilde{H}_2(\Delta_W; k) = 0$. Consequently, for every subset W of V, $\dim_k \tilde{H}_2(\Delta_W; k)$ is independent of k. Since $\tilde{\chi}(\Delta_W)$ is independent of k, $\dim_k \tilde{H}_1(\Delta_W; k)$ is also independent of k. Q. E. D.

On the other hand, it follows easily that, for a simplicial complex Δ on the vertex set V, all Betti numbers $\beta_i^A(k[\Delta])$ are independent of k if one of the following conditions is satisfied: (i) $\dim \Delta \leq 1$; (ii) Δ is a 2-manifold with non-empty boundary; (iii) $\sharp(V) \leq 5$.

(4.2) THEOREM. *Let Δ be a simplicial complex and suppose that the geometric realization $\mid \Delta \mid$ of Δ is the n-sphere $\mathbf{S}^n$ (or the n-ball $\mathbf{B}^n$) with $n \leq 3$. Then, the Betti number $\beta_i^A(k[\Delta])$ is independent of the base field k for every $i \geq 0$.*

Proof. If $\mid \Delta \mid = \mathbf{S}^n$, then the above Proposition (4.1) guarantees that all Betti numbers $\beta_i^A(k[\Delta])$ are independent of the base field k.

On the other hand, suppose that $\mid \Delta \mid= \mathbf{B}^n$ and define Δ' to be the simplicial complex $\Delta \cup (\partial\Delta * \{$ a single point $\})$. Thus, $\mid \Delta' \mid= \mathbf{S}^n$. Let V denote the vertex set of Δ. Then $\Delta'_V = \Delta$. Hence, it follows that, for every subset W of V and for each integer $j \geq -1$, $\dim_k \tilde{H}_j(\Delta_W; k)$ is independent of the base field k as required.

Q. E. D.

(4.3) EXAMPLE. Let Γ denote the simplicial complex on the vertex set $V = \{1, 2, 3, 4, 5, 6\}$, discussed in §2, whose geometric realization $\mid \Gamma \mid$ is the real projective plane. Let Δ denote the simplicial complex which consists of all subsets σ of V with $\sigma \neq V$. Thus, $\mid \Delta \mid$ is the 4-sphere. We consider Γ to be a subcomplex of Δ in the obvious way. Let $\mathrm{Sd}(\Delta)$ denote the barycentric subdivision of Δ. If W is the vertex set of $\mathrm{Sd}(\Gamma)$, then $\sharp(W) = 31$ and $\mathrm{Sd}(\Delta)_W = \mathrm{Sd}(\Gamma)$. Thus, we have

$$\dim_{\mathbf{Z}/2\mathbf{Z}} \tilde{H}_{31-28-1}(\mathrm{Sd}(\Delta)_W; \mathbf{Z}/2\mathbf{Z}) > \dim_{\mathbf{Q}} \tilde{H}_{31-28-1}(\mathrm{Sd}(\Delta)_W; \mathbf{Q});$$

$$\dim_{\mathbf{Z}/2\mathbf{Z}} \tilde{H}_{31-29-1}(\mathrm{Sd}(\Delta)_W; \mathbf{Z}/2\mathbf{Z}) > \dim_{\mathbf{Q}} \tilde{H}_{31-29-1}(\mathrm{Sd}(\Delta)_W; \mathbf{Q}).$$

Hence

$$\beta^A_{28}((\mathbf{Z}/2\mathbf{Z})[\mathrm{Sd}(\Delta)]) > \beta^A_{28}(\mathbf{Q}[\mathrm{Sd}(\Delta)]);$$
$$\beta^A_{29}((\mathbf{Z}/2\mathbf{Z})[\mathrm{Sd}(\Delta)]) > \beta^A_{29}(\mathbf{Q}[\mathrm{Sd}(\Delta)]).$$

Note that $\mathrm{hd}_A(k[\mathrm{Sd}(\Delta)]) = 57$ and $\beta^A_{28}(k[\mathrm{Sd}(\Delta)]) = \beta^A_{29}(k[\mathrm{Sd}(\Delta)])$. Since Δ is the boundary complex of the 5-simplex, it follows that Δ is shellable (defined in, e.g., [B–M]). Hence, thanks to [Bjö$_1$], $\mathrm{Sd}(\Delta)$ is also shellable.

(4.4) EXAMPLE. Let Δ denote the simplicial complex as in Example (4.3) and define Δ' to be $\Delta - \{\{1, 2, 3, 4, 5\}\}$. Then $\mid \Delta' \mid$ is the 4-ball. The similar technique as in Example (4.3) enables us to see that some Betti numbers $\beta^A_i(k[\mathrm{Sd}(\Delta')])$ of the Stanley-Reisner ring $k[\mathrm{Sd}(\Delta')]$ of the barycentric subdivision $\mathrm{Sd}(\Delta')$ of Δ' depend on the base field k. The simplicial complex $\mathrm{Sd}(\Delta')$ is also shellable.

The above Examples (4.3) and (4.4) illustrate the following

(4.5) PROPOSITION. *Fix an integer $n \geq 4$ and let V denote the finite set $\{1, 2, \ldots, n, n+1, n+2\}$. Define Δ_n to be the simplicial complex which consists of all subsets σ of V with $\sigma \neq V$. Moreover, let Δ'_n denote the simplicial complex $\Delta_n - \{\{1, 2, \ldots, n+1\}\}$. Then, there exist integers i and j such that $\beta^A_i(k[\mathrm{Sd}(\Delta_n)])$ and $\beta^A_j(k[\mathrm{Sd}(\Delta'_n)])$ depend on the base field k. Note that both $\mathrm{Sd}(\Delta_n)$ and $\mathrm{Sd}(\Delta'_n)$ are shellable with $\mid \mathrm{Sd}(\Delta_n) \mid= \mathbf{S}^n$ and $\mid \mathrm{Sd}(\Delta'_n) \mid= \mathbf{B}^n$.*

It would, of course, be of interest, for every fixed integer $n \geq 4$, to find an interesting class of simplicial complexes Δ with $| \Delta |= \mathbf{S}^n$ such that all Betti numbers $\beta_i^A(k[\Delta])$ are independent of the base field k.

(4.6) **CONJECTURE.** Let $\mathcal{O}(P)$ be the order polytope [Sta$_2$] associated with a finite partially ordered set P. Let Δ be the canonical triangulation of $\mathcal{O}(P)$ discussed in [Sta$_2$] and $\partial\Delta$ the boundary of Δ. Thus, $| \partial\Delta |= \mathbf{S}^{n-1}$ with $\sharp(P) = n$. Then, all Betti numbers $\beta_i^A(k[\partial\Delta])$ of the Stanley-Reisner ring $k[\partial\Delta]) = A/I_{\partial\Delta}$ are independent of the base field k.

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