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A Lower Bound Theorem for Ehrhart Polynomials of Convex Polytopes

TAKAYUKI HIBI

Department of Mathematics, Faculty of Science, Hokkaido University, Kita-ku, Sapporo 060, Japan

Let $\mathscr{P} \subset \mathbb{R}^N$ be an *integral* convex polytope; i.e., a convex polytope any of whose vertices has integer coordinates of dimension d, and let $\partial\mathscr{P}$ denote the boundary of $\mathscr{P}$. Given a positive integer n we write $i(\mathscr{P}, n)$ for the number of those rational points $(\alpha_1, \alpha_2, ..., \alpha_N)$ in $\mathscr{P}$ such that each $n\alpha_i$ is an integer. In other words,

$$i(\mathscr{P}, n) = \#(n\mathscr{P} \cap \mathbb{Z}^N).$$

Here $n\mathscr{P} := \{n\alpha;\ \alpha \in \mathscr{P}\}$ and $\#(X)$ is the cardinality of a finite set X. The systematic study of $i(\mathscr{P}, n)$ originated in the work of Ehrhart (cf. [Ehr]), who established that the function $i(\mathscr{P}, n)$ possesses the following fundamental properties:

(0.1) $i(\mathscr{P}, n)$ is a polynomial in n of degree d. (Thus $i(\mathscr{P}, n)$ can be defined for *every* integer n.)

(0.2) $i(\mathscr{P}, 0) = 1$.

(0.3) ("loi de réciprocité") $(-1)^d\, i(\mathscr{P}, -n) = \#(n(\mathscr{P} - \partial\mathscr{P}) \cap \mathbb{Z}^N)$ for every integer $n > 0$.

We say that $i(\mathscr{P}, n)$ is the *Ehrhart polynomial* of $\mathscr{P}$. See, e.g., [Sta_3, pp. 235–241; and H_5] for an introduction to Ehrhart polynomials.

We define the sequence $\delta_0, \delta_1, \delta_2, ...$ of integers by the formula

$$(1-\lambda)^{d+1}\left[1 + \sum_{n=1}^{\infty} i(\mathscr{P}, n)\, \lambda^n\right] = \sum_{i=0}^{\infty} \delta_i \lambda^i. \qquad (1)$$

Then, the basic facts (0.1) and (0.2) on $i(\mathscr{P}, n)$ together with a fundamental result on generating functions, e.g., [Sta_3, Corollary 4.3.1], guarantee that $\delta_i = 0$ for every $i > d$. We say that the sequence $\delta(\mathscr{P}) := (\delta_0, \delta_1, ..., \delta_d)$ which appears in Eq. (1) is the *δ-vector* of $\mathscr{P}$. Thus $\delta_0 = 1$ and $\delta_1 = \#(\mathscr{P} \cap \mathbb{Z}^N) - (d+1)$. On the other hand, it follows easily from (0.3) that $\delta_d = \#((\mathscr{P} - \partial\mathscr{P}) \cap \mathbb{Z}^N)$. Moreover, each δ_i is non-negative [Sta_1].

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Now, our result in this paper is

(1.1) THEOREM. *Let $\mathscr{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d with the δ-vector $\delta(\mathscr{P}) = (\delta_0, \delta_1, ..., \delta_d)$ and suppose that $(\mathscr{P} - \partial\mathscr{P}) \cap \mathbb{Z}^N$ is non-empty; i.e., $\delta_d \neq 0$. Then we have the inequality $\delta_1 \leqslant \delta_i$ for every $1 \leqslant i < d$.*

The proof of Theorem 1.1 relies on the following two well-known facts. See, e.g., [Sta$_2$] for fundamental definitions and results concerning f-vectors and h-vectors of triangulations of balls and spheres.

(1.2) LEMMA (cf. [B-M]). *Let $\mathscr{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d with the δ-vector $\delta(\mathscr{P}) = (\delta_0, \delta_1, ..., \delta_d)$. Also, let Δ be a triangulation of $\mathscr{P}$ with the vertex set $\mathscr{P} \cap \mathbb{Z}^N$ whose h-vector is $h(\Delta) = (h_0, h_1, ..., h_d, h_{d+1})$. (Thus, $h_1 = \delta_1$, $h_d = \delta_d$ [Sta$_2$, pp. 80–81] and $h_{d+1} = 0$ [Sta$_2$, p. 67].) Then $h_i \leqslant \delta_i$ for every $0 \leqslant i \leqslant d$.*

(1.3) LEMMA ([Bar$_1$, Bar$_2$]). *Let $h(\Delta) = (h_0, h_1, ..., h_d)$ be the h-vector of a triangulation of the boundary $\partial\mathscr{P}$ of a convex polytope $\mathscr{P}$ of dimension d. (Thus, in particular, $h_i = h_{d-i}$ for each $0 \leqslant i \leqslant d$ [Sta$_2$, p. 77].) Then $h_1 \leqslant h_i$ for every $1 \leqslant i < d$.*

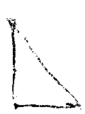
We are now in the position to give a proof of Theorem 1.1. Let $\mathscr{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d and suppose that $(\mathscr{P} - \partial\mathscr{P}) \cap \mathbb{Z}^N$ is non-empty, say $(\mathscr{P} - \partial\mathscr{P}) \cap \mathbb{Z}^N = \{v_1, v_2, ..., v_l\}$. First, we take any triangulation $\Delta(0)$ of the boundary $\partial\mathscr{P}$ of $\mathscr{P}$ with the vertex set $\partial\mathscr{P} \cap \mathbb{Z}^N$, and then we construct a triangulation $\Delta(j)$ of $\mathscr{P}$ with the vertex set $(\partial\mathscr{P} \cap \mathbb{Z}^N) \cup \{v_1, ..., v_j\}$ for each $1 \leqslant j \leqslant l$ in the following way:

(i) Define $\Delta(1)$ to be the triangulation of $\mathscr{P}$ which consists of those simplices $\sigma \subset \mathscr{P}$ such that σ is the convex hull of $\tau \cup \{v_1\}$ in $\mathbb{R}^N$ for some $\tau \in \Delta(0)$; i.e., $\Delta(1)$ is the cone over $\Delta(0)$ with apex v_1.

(ii) If $\Delta(j)$ is constructed and $1 \leqslant j < l$, then let $\tau(j) \in \Delta(j)$ be the smallest face which contains v_{j+1} and write τ' for the subdivision of $\tau(j)$ which is the cone over the boundary $\partial\tau(j)$ of $\tau(j)$ with apex v_{j+1}. Also, let $\mathrm{link}_{\Delta(j)}(\tau(j))$ (resp., $\mathrm{star}_{\Delta(j)}(\tau(j))$) be the link [Sta$_2$, p. 70] (resp., star [Sta$_2$, p. 72]) of $\tau(j)$ in $\Delta(j)$. We then define $\Delta(j+1)$ to be the triangulation of $\mathscr{P}$ which consists of those simplices $\sigma \subset \mathscr{P}$ such that σ is either (a) the convex hull of $\zeta \cup \xi$ in $\mathbb{R}^N$ for some $\zeta \in \tau'$ and $\xi \in \mathrm{link}_{\Delta(j)}(\tau(j))$ or (b) $\sigma \in \Delta(j) - \mathrm{star}_{\Delta(j)}(\tau(j))$.

Triangulation

We now investigate the relation between the h-vector of $\Delta(j)$ and that of $\Delta(j+1)$ for each $0 \leqslant j < l$. We write $(h_0^{(j)}, h_1^{(j)}, h_2^{(j)}, ...)$ for the h-vector $h(\Delta(j))$ of $\Delta(j)$. Since $\Delta(1)$ is the cone over $\Delta(0)$ with apex v_1, we know that $h_i^{(0)} = h_i^{(1)}$ for each $0 \leqslant i \leqslant d$. Thus, by virtue of Lemma 1.3, we have



$h_1^{(1)} \leqslant h_i^{(1)}$ for every $1 \leqslant i < d$. Now, let $j \geqslant 1$ and suppose that $h_1^{(j)} \leqslant h_i^{(j)}$ for every $1 \leqslant i < d$. Let v be a vertex of $\tau(j)$ and set $\Delta^* = \mathrm{star}_{\Delta(j+1)}(\{v_{j+1}\})$. Then we easily see

$$f_i(\Delta(j+1)) = f_i(\Delta(j)) + f_{i-1}(\mathrm{link}_{\Delta^*}(\{v\})) \tag{2}$$

for every $1 \leqslant i \leqslant d$. Here, e.g., $(f_0(\Delta(j)), f_1(\Delta(j)), f_2(\Delta(j)), ...)$ is the f-vector of Δ_j; i.e., $f_i(\Delta(j))$ is the number of i-dimensional simplices of $\Delta(j)$. On the other hand, we set

$$\rho := \mathrm{link}_{\Delta(j+1)}(\{v\}) \cap \mathrm{link}_{\Delta(j+1)}(\{v_{j+1}\}).$$

Then ρ is a triangulation of a $(d-2)$-sphere and $\mathrm{link}_{\Delta^*}(\{v\})$ is the cone over ρ with apex v_{j+1}. Let $h(\rho) = (h_0, h_1, ..., h_{d-1})$ be the h-vector of ρ. It follows from Eq. (2) that

$$h_i^{(j+1)} = h_i^{(j)} + h_{i-1}$$

for each $1 \leqslant i \leqslant d$. Hence $h_1^{(j+1)} \leqslant h_i^{(j+1)}$ for every $1 \leqslant i < d$ because of each $h_i \geqslant h_0$ $(=1)$.

Thanks to Lemma 1.2, the δ-vector $\delta(\mathscr{P}) = (\delta_0, \delta_1, ..., \delta_d)$ of $\mathscr{P}$ satisfies $h_i(l) \leqslant \delta_i$ for every $0 \leqslant i \leqslant d$. On the other hand, we know $h_i^{(l)} \geqslant h_1^{(l)}$ $(=\delta_1)$ for each $1 \leqslant i < d$, thus we have the inequality $\delta_1 \leqslant \delta_i$ for every $1 \leqslant i < d$ as required. Q.E.D.

(1.4) *Remark.* (a) When $\tau(j) \in \Delta(j)$ is a facet (maximal face) of $\Delta(j)$, then $h(\rho) = (1, 1, ..., 1)$. This fact immediately shows that if $(h_0, h_1, ..., h_d, 0)$ is the h-vector of a triangulation of a d-ball, then $(h_0, h_1 + 1, h_2 + 1, ..., h_d + 1, 0)$ is also the h-vector of a triangulation of a d-ball. Thus, in particular, given positive integers d and n, there exists a triangulation Δ of a d-ball with the h-vector $h(\Delta) = (1, n, 1, ..., 1, 0) \in \mathbb{Z}^{d+2}$.

(b) Let Δ be a triangulation of a d-ball and suppose that each facet possesses a vertex contained in the interior $\Delta - \partial\Delta$ of Δ. Then the h-vector $h(\Delta) = (h_0, h_1, ..., h_d, 0)$ of Δ satisfies the linear inequality

$$h_0 + h_1 + \cdots + h_i \leqslant h_d + h_{d-1} + \cdots + h_{d-i}$$

for every $0 \leqslant i \leqslant [d/2]$.

(c) A technique similar to what was done in the proof of Theorem 1.1 enables us to obtain the linear inequalities

$$\delta_{d-1} + \delta_{d-2} + \cdots + \delta_{d-i} \leqslant \delta_2 + \delta_3 + \cdots + \delta_i + \delta_{i+1},$$

$0 \leqslant i \leqslant [(d-1)/2]$, for the δ-vector $\delta(\mathscr{P}) = (\delta_0, \delta_1, ..., \delta_d)$ of an arbitrary integral convex polytope $\mathscr{P} \subset \mathbb{R}^N$ of dimension d. In particular, $\delta_{d-1} \leqslant \delta_2$;

however, [Sta_5, Ex. 3.4] shows that unfortunately we cannot expect $\delta_{d+1-i} \leqslant \delta_i$ when $2 < i$.

It would, of course, be of great interest to find a combinatorial characterization of the δ-vectors of integral convex polytopes.

We refer the reader to [Sta_4, (3.4); Sta_4, (4.1); Sta_5, (3.3)] for further information about δ-vectors ($=h^*$-vectors). Also, see [H_1, H_2, H_3, and H_4].

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