Integration Over a Polyhedron: An Application of the Fourier-Motzkin Elimination Method

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1. INTRODUCTION. Consider the problem of evaluating a double integral over a polyhedron. Suppose we want to convert the double integral to an iterated integral. In order to have simple (i.e., affine) limits of integration, it may be necessary to decompose the region of integration. An example can be seen in Figure 1, where the integration region must be written as the union of A, B and C, assuming we want to integrate with respect to x first. The required decomposition of a two dimensional polyhedron (i.e., a polygon) is easily accomplished by looking at the picture, but in three dimensions this may be a more daunting task, while if more than three variables are involved pictures fail us. The Fourier-Motzkin elimination method for solving a system of linear inequalities can serve as a basis for finding this decomposition.

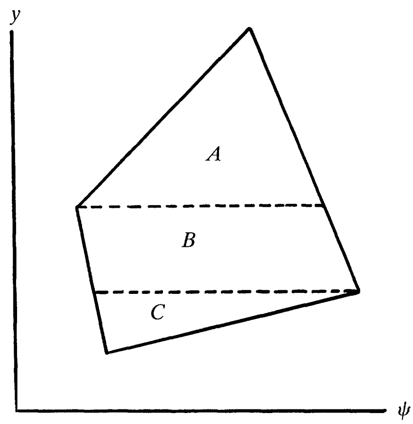

Figure 1. Polygonal domain of integration.

2. THE FOURIER-MOTZKIN ELIMINATION METHOD. The Fourier-Motzkin Elimination method determines whether a given system of linear inequalities and equations is consistent or not and, if it is, enables one to find solutions. It can handle a mixture of strict and non-strict inequalities, although we outline here only the simplest form, dealing with the system $Ax \leq b$. It is much more than a computational technique, since it can serve as the means for establishing the fundamental facts in the theory of linear inequalities, such as the duality theorem of linear programming. This is analogous to the situation in linear algebra, where the Gaussian elimination method serves not only to solve linear systems but to establish key facts about basis, rank and dimension. Full discussion of this method can be found in **[1]** and **[2]**. For our purposes it suffices to describe the computational aspect.

We describe the elimination of x_1. This is done by performing row operations on the augmented matrix (A, b), where A is an $m \times n$ matrix. Here are the steps:

1. Rearrange the rows of (A, b) and multiply rows by positive constants as needed so that the first column becomes a string of 1's followed by a string of -1's followed by a string of zeros. Any of these strings may be empty.
2. For each pair $(1, -1)$ appearing in column 1 construct a new inequality by adding the two rows of (A, b). The resulting inequality is adjoined to the system. Note how this differs from Gaussian elimination, where the number of equations doesn't increase during the elimination process.

The resulting matrix has the form shown in Table 1.

TABLE 1. Eliminating x_1

$$\begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} & b_1 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 1 & a_{k2} & \cdots & a_{kn} & b_k \\ -1 & a_{k+1,2} & \cdots & a_{k+1,n} & b_{k+1} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ -1 & a_{k+r,2} & \cdots & a_{k+r,n} & b_{k+r} \\ 0 & \cdot & \cdots & \cdot & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 & \cdot & \cdots & \cdot & \cdot \end{bmatrix}$$

Suppose $x_2, \ldots, x_n$ are assigned specific values that satisfy all those inequalities not involving x_1 in the system corresponding to Table 1, that is, all inequalities after the first $k + r$. From the way in which the inequalities adjoined to the system were formed it follows that the bounds placed on x_1 by the first $k + r$ inequalities are compatible, so that x_1 can be chosen to solve these inequalities. If both -1 and 1 appear in column 1 of the augmented matrix after elimination then the values that can be assigned to x_1 constitute a closed bounded interval, otherwise a closed halfline. From this observation we can state the following:

Theorem. *Let $\hat{A}x \le \hat{b}$ be the system obtained from $Ax \le b$ by using steps* 1 *and* 2 *above to eliminate x_1. Then*

1. *The systems $Ax \le b$ and $\hat{A}x \le \hat{b}$ have the same set of solutions.*
2. *Let $P = \{x | Ax \le b\}$ and let P^1 denote the set of points $(x_2, x_3, \ldots, x_n)$ that satisfy all the inequalities not involving x_1 in the system $\hat{A}x \le \hat{b}$. Then P^1 is the projection of P onto the hyperplane $x_1 = 0$.*

Now we complete the description of the Fourier-Motzkin elimination method. Having eliminated x_1, we now ignore the rows involving x_1 as well as the first column and eliminate x_2 from the remaining rows, etc. This procedure can terminate in one of two ways:

1. For some k, when we attempt to eliminate x_k we find that all the equations have non-zero x_k coefficients and these are of the same sign, so that the elimination procedure terminates. In this case there exist solutions for arbitrarily chosen values of $x_{k+1}, \ldots, x_n$. If $k = n$ then there are no arbitrary variable values to choose but x_n can have any value on a closed halfline.
2. We eliminate x_n. The result of this elimination is a set of inequalities with all coefficients zero. A solution to $Ax \leq b$ exists if and only if the right hand sides of these inequalities are all nonnegative.

Example 1. Consider the system

$$\begin{array}{rrrl} x & +y & +z & \leq 2 \\ 2x & +y & & \leq 1 \\ -x & & & \leq 0 \\ & -y & & \leq 0 \\ & & -z & \leq 0 \end{array} \tag{1}$$

The augmented matrix for this system is M_1 and the result of applying the Fourier-Motzkin elimination method is M_2, where

$$M_1 = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1/2 & 0 & 1/2 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The last two rows of M_2 tell us that a solution exists, which was quite obvious, since $(0,0,0)$ is a solution. Rows 7 and 8 tell us that z must be chosen on the interval $[0,2]$. Take $z = 1$. Then rows 3, 4, and 5 tell us that $y \leq 1$ and $y \geq 0$. Choose $y = 0$. Rows 1, 2, and 3 tell us that $x \leq 1$, $x \leq 1/2$, and $x \geq 0$. Choosing $x = 1/2$, we get the solution $(1/2, 0, 1)$.

Let P denote the solution set for the system of inequalities in this example. The Fourier-Motzkin elimination method tells us that the projection of P onto the z axis is the interval $[0,2]$ and that the projection of P onto the plane $x = 0$ is given by $0 \leq y \leq 1$, $0 \leq z \leq 2$, $y + z \leq 2$. Both these facts can be easily verified graphically.

The maximum value that z can have subject to the inequalities (1) is 2. We've indirectly solved a linear programming problem by the Fourier-Motzkin elimination method. In fact, every linear programming problem can be solved by this method, though perhaps not efficiently.

3. AFFINE LIMITS OF INTEGRATION. Suppose that we want to integrate a function f over a bounded non-empty polyhedron $P = \{x \mid Ax \leq b\}$ in R^n. Applying the Fourier-Motzkin elimination method to the given system of inequalities we get an equivalent system with augmented matrix of the form shown in Table 1.

Define a set of affine functions by

$$A_i^+(x_2,\dots,x_n) = b_i - \sum_{j=2}^{n} a_{ij}x_j, \quad i = 1,\dots,k$$

$$A_i^-(x_2,\dots,x_n) = -b_{k+i} + \sum_{j=2}^{n} a_{k+i,j}x_j, \quad i = 1,\dots,r$$

and let P^1 denote the projection of P onto $x_1 = 0$, so that P^1 is the set of solutions of all the inequalities other than the first $k + r$. Then P may be described as follows: $x \in P$ if and only if $(x_2,\dots,x_n) \in P^1$, $x_1 \le A_i^+$ for $i = 1,\dots,k$ and $x_1 \ge A_i^-$ for $i = 1,\dots,r$. This characterization may be written more simply if we define two more functions M and m by

$$m(x_2,\dots,x_n) = \max\{A_j^-(x_2,\dots,x_n), \quad j = 1,\dots,r\}$$

$$M(x_2,\dots,x_n) = \min\{A_j^+(x_2,\dots,x_n), \quad j = 1,\dots,k\}$$

Then $x \in P$ if and only if $(x_2,\dots,x_n) \in P^1$ and

$$m(x_2,\dots,x_n) \le x_1 \le M(x_2,\dots,x_n)$$

We may write

$$\int_P f(x)\,dx = \int_{P^1}\int_m^M f(x)\,dx_1 dx_2 \cdots dx_n.$$

Our aim is to have only affine functions for the limits of integration. To do this we decompose P^1 into pieces on which m and M are affine. For $i = 1,\dots,k$ and $j = 1,\dots,r$ let P_{ij}^1 be the set of points in P^1 for which $M = A_i^+$ and $m = A_j^-$. Then

$$\int_P f(x)\,dx = \sum_{i,j}\int_{P_{ij}^1}\int_{A_j^-}^{A_i^+} f(x)\,dx_1 dx_2 \cdots dx_n.$$

Now we address the problem of finding P_{ij}^1. To do this we start with the inequalities defining P^1, namely, those after the first $k + r$ inequalities in the system corresponding to Table 1, and adjoin the following inequalities:

$$A_i^+ \le A_p^+, \quad p = 1,\dots,k, \quad p \neq i$$

$$A_j^- \ge A_p^-, \quad p = 1,\dots,r, \quad p \neq j.$$

Taking into account the definitions of A_i^+ and A_j^- the reader may verify that the inequalities defining P_{ij}^1 may be obtained by the following recipe: for $p = 1,\dots,k$, $p \neq i$, replace row p of table 1 by (row p) $-$ (row i) and for $p = k + 1,\dots,k + r$, $p \neq k + j$, replace row p of Table 1 by (row p) $-$ (row $(k + j)$). Note that we find ourselves in the unusual position of *subtracting* two inequalities and not by mistake!

4. THE COMPLETE ALGORITHM. By putting together the material in the preceding sections we can construct an algorithm for solving the problem of expressing an integral over a polyhedron $P = \{x \mid Ax \le b\}$ in R^n as the sum of repeated integrals with affine limits. It is convenient to use a rooted tree to describe this algorithm. Each node is a system of inequalities and the nodes at the n'th level correspond to the polyhedra into which P is decomposed to get the integrals with affine limits.

At the root node of the tree is the system $Ax \leq b$, to which the Fourier-Motzkin elimination method has been applied. The nodes at level k correspond to a decomposition of the polyhedron P into polyhedra of a certain form. The system of inequalities at a kth level node must be of the following form: inequalities $2i-1$ and $2i$ tell us that x_i lies between two affine functions of $x_{i+1}, \ldots, x_n$ for $i = 1, \ldots, k$ and the remaining inequalities, which don't involve $x_1, \ldots, x_k$, say that $x_{k+1}, \ldots, x_n$ lie in the projection of P onto $x_1 = x_2 = \cdots = x_k = 0$. To construct the next level of the tree, we find all the children of each node on level k. For each node we have an augmented matrix corresponding to that node's system of inequalities. Let $\mathcal{A}$ denote that matrix with the first $2k$ columns and rows deleted. Apply to $\mathcal{A}$ the procedure described in the preceding section, i.e., for each pair of inequalities that bound x_{k+1} above and below by affine functions of $x_{k+2}, \ldots, x_n$, form another set of inequalities. Apply the Fourier-Motzkin elimination method to this system. If the resulting system has a solution, then, it, together with the first $2k$ inequalities that we temporarily ignored, forms a node at level $k+1$ of the tree. The nth level of the tree gives the desired decomposition of P.

The sub-polyhedra into which P is decomposed by this algorithm are not all disjoint. However their intersections, if not empty, lie in hyperplanes, which themselves may appear as sub-polyhedra. For purposes of integration we may ignore these sub-polyhedra.

It should be noted that this algorithm is not practical for large problems. Even carrying out the Fourier-Motkin elimination method for one system can involve a great many computations. In [**3**, p. 156] there is an example of a system of inequalities with $O(n^3)$ inequalities for which the Fourier-Motzkin elimination method produces a system with more than $2^{n/2}$ inequalities. Our algorithm requires that this process be carried out at each node and furthermore the number of inequalities increases as the algorithm goes from one level of the tree to the next. Nevertheless, the algorithm can be used in some problems where visualization fails.

There are a number of shortcuts one can take in carrying out this algorithm but in the following example we will not take any of these except that obvious redundancies will be deleted from the sets of inequalities.

Example 2. Consider integrating over the polyhedron P defined in Example 1. Let R denote the root of our tree. The set of inequalities associated with R is given by M_2 with the last two rows, which are clearly unnecessary, omitted. R has two children, corresponding to the first two inequalities. Following the recipe in the preceding section these are, after applying the Fourier-Motzkin elimination method to each node and removing obvious redundancies:

$$R_1: \begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -1 \end{bmatrix} \qquad R_2: \begin{bmatrix} 1 & 1/2 & 0 & 1/2 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

R_1 corresponds to $0 \leq x \leq 2 - y - z$, and R_2 corresponds to $0 \leq x \leq (1-y)/2$, with $(y, z) \in P^1$ in both cases. R_1 has 4 children. Those that corresponds to $3 - 2z \le y \le 2 - z$ and $0 \le 2 - z$ give respectively

$$R_{11}: \begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 0 & 0 & \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & -1 & -1 \end{bmatrix} \qquad R_{12}: \begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 0 & 0 & \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

R_{11} gives us the polyhedron $0 \le x \le 2 - y - z$, $3 - 2z \le y \le 2 - z$, $1 \le z \le 3/2$, and R_{12} gives us the polyhedron $0 \le x \le 2 - y - z$, $0 \le y \le 2 - z$, $3/2 \le z \le 2$. Another of the children of R_1 gives the limits $1 \le z \le 1$, which may be omitted for purposes of integration, while the fourth child is empty, as evidenced by the fact that the row $(0\,0\,0\,-1)$ appears in the Fourier-Motzkin form of the augmented matrix. R_2 has 3 children, of which none is empty but one satisfies the condition $1 \le z \le 1$, hence may be ignored for purposes of integration. The final result, which may be verified geometrically, is:

$$\int_P f\,dx\,dy\,dz = \int_0^1 \int_0^1 \int_0^{\frac{1-y}{2}} f\,dx\,dy\,dz + \int_1^{3/2} \int_0^{3-2z} \int_0^{\frac{1-y}{2}} f\,dx\,dy\,dz$$
$$+ \int_1^{3/2} \int_{3-2z}^{2-z} \int_0^{2-y-z} f\,dx\,dy\,dz + \int_{3/2}^2 \int_0^{2-z} \int_0^{2-y-z} f\,dx\,dy\,dz.$$

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