References

[1] P Erdős: Solved and unsolved problems in combinatorics and combinatorial number theory, *Congressus Numerantum,* **32** (1981), 49–62.

[2] P Erdős and A. Gyárfás: A variant of the classical ramsey problem, *Combinatorica*, **17** (1997), 459–467.

Dhruv Mubayi

School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332-0160
mubayi@math.gatech.edu

Compare to Orevkov

I think we should include Orevkov.

COMBINATORICA
Bolyai Society – Springer-Verlag

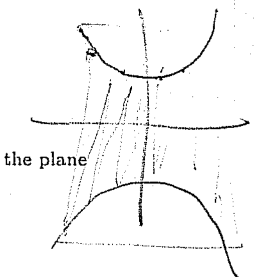
Combinatorica 18 (2) (1998) 297–299

NOTE

ON THE NUMBER OF TRIANGULATIONS OF PLANAR POINT SETS

RAIMUND SEIDEL*

Received November 1, 1995



We show that the number of straight edge triangulations of any set of n points in the plane is at most $2^{12.245113n-\Theta(\log n)}$.

Let S be any set of $n>2$ points in the plane and let E be the set of $\binom{n}{2}$ straight line segments joining points in S. A *(straight edge) triangulation of S* is a maximal subset T of E so that no two edges in T intersect except possibly at common endpoints. We are interested in $g(n)$, the maximum number of different triangulations of any planar set S with n points.

It is well known that if S is in convex position the number of its triangulations is given by the Catalan number $C_{n-2}=\binom{2(n-2)}{n-2}/(n-1)$. Using this it is easy to come up with a set of $2n$ points [2] that has $(C_{n-2})^2\binom{2n-2}{n-1}$ triangulations: place n points each on the curves $y=x^2+1$ and $y=-x^2-1$ with x-coordinates between -1 and $+1$. Thus $g(n)\geq 2^{3n-\Theta(\log n)}$. ?

In this note we are interested in upper bounds for $g(n)$. In the more general context of crossing-free subgraphs Ajtai, Chvátal, Newborn, and Szemerédi [1] proved $\approx 2^{17n}$ that $g(n)\leq 2^{O(n)}$ must hold. Smith [4] proved a bound of $g(n)\leq 173000^n$. In this note we present a rather simple argument showing that $g(n)\leq 2^{12.245113n-\Theta(\log n)}$.

Let S now be a fixed set of $n>2$ points. Without loss of generality we assume Due to Pach that no three points of S are collinear. The idea of our proof is to show how one can encode any triangulation T of S in a binary string $w(T)$ so that given S and string

Mathematics Subject Classification (1991):

* Supported by NSF Presidential Young Investigator award CCR-9058440.

$w(T)$ the triangulation T can be reconstructed. If every possible encoding string has length $f(n)$, then clearly S cannot have more than $2^{f(n)}$ different triangulations.

Our method was inspired by the planar point location method of Kirkpatrick [3]. Let T be a triangulation of S. If $n=3$ there is only one possible triangulation of S, which can be encoded by the empty string. So assume $n > 3$. Since T forms a planar graph its vertices can be 4-colored, which implies that there is an independent set I of vertices of size $\lceil n/4 \rceil$. Remove I along with the incident edges from the triangulation T and retriangulate the "holes" thus created. This results in a triangulation T' of the set $S'=S\setminus I$. Let $E' \subset T'$ be the set of edges created by retriangulating, i.e. $E'=T'\setminus T$. Our encoding of T will then be given recursively through

$$w(T) = \rho(S')\rho(E')w(T'),$$

i.e. the concatenation of a binary string $\rho(S')$ encoding S' as a subset of S, a binary string $\rho(E')$ encoding E' as a subset of T', and the recursive binary encoding of the triangulation T' of S'.

At first we need to argue that the triangulation T of S can be reconstructed from $w(T)$ and S. This can be done as follows: If $|S| = 3$, there is only one triangulation. Otherwise use $\rho(S')$ to determine $S' \subset S$ and recursively use $w(T')$ to determine the triangulation T' of S'. Next remove from T' all edges described by $\rho(E')$. This leaves $\lceil n/4 \rceil$ "holes" in T', one for each point of $I=S\setminus S'$. To finally obtain T add to $T'\setminus E'$ all edges connecting a point in I with a point in S' that do not cross any edge in $T'\setminus E'$.

Finally we need to bound the length of the binary string $w(T)$. This clearly depends on the subset encodings $\rho(S')$ and $\rho(E')$. These are most easily done via characteristic vectors: S can be canonically ordered into $p_1,\ldots,p_n$, say by lexicographic order of the coordinates; then one can use for $\rho(S')$ the binary string s of lenght n with $s_i=1$ iff $p_i \in S'$. Similarly the edges in the triangulation T' can be canonically ordered into $e_1,\ldots,e_m$, where $m=|T'|$, and one uses for $\rho(E')$ the binary string t of length m with $t_i=1$ iff $e_i \in E'$. Note that m depends only on S' since every triangulation of S' has the same number of edges, namely

$$m = 3|S'| - 3 - \#\ \textit{of extreme points of}\ S' \le 9n/4 - 6\,.$$

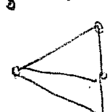
Thus the length $f(n)$ of the string $w(T)$ is bounded by

$$f(n) \le n + 9n/4 - 6 + f(\lfloor 3n/4 \rfloor),$$

which with the boundary condition $f(3)=0$ implies $f(n) \le 13n - \Theta(\log n)$.

This can be slightly improved by using a better encoding for $\rho(S')$. Since S' is a subset of S of size exactly $\lfloor 3n/4 \rfloor$ there are only $b(n)=\binom{n}{\lfloor 3n/4 \rfloor}$ choices for S', and hence S' can be encoded by a string of length $\log_2 b(n)$. Since $\binom{n}{\alpha n} \le 2^{nH(\alpha)}$, where $H(x)=-x\log_2 x-(1-x)\log_2(1-x)$ we therefore get that S' can be encoded by a binary string of length $H(3/4)n \le 0.81128n$.

where did he use general position? ya se! si $u \in \partial P$ we don't leave a noticeable hole.



ok no problem.

Unfortunately this type of improvement cannot be applied to $\rho(E')$ since the size of $E' \subset E$ can be anywhere between 0 and more than half of the size of E.

Thus $f(n)$, the length of the encoding string $w(T)$, satisfies

$$f(n) \le H(3/4)n + 9n/4 - 6 + f(\lfloor 3n/4 \rfloor),$$

which yields the claimed bound of

$$f(n) \le (4 \cdot H(3/4) + 9)n - \Theta(\log n) \le 12.245113n - \Theta(\log n)\,.$$

Note that typically a triangulation T of S will be correctly encoded by many strings $w(T)$. Thus our upper bound for $g(n)$ is certainly not tight. Most likely the true value of $g(n)$ is much closer to the known lower bound of $2^{3n-\Theta(\log n)}$.

References

[1] M. Ajtai, V. Chvátal, M. Newborn, and E. Szemerédi: Crossing-free Subgraphs, *Annals of Discrete Math.*, **12** (1982), 9–12.

[2] A. García, M. Noy, and J. Tejel: Lower bounds for the number of crossing free subgraphs of K_n, *Proc. 7th Canadian Conf. on Computational Geometry*, (1995), 97–102.

[3] D. G. Kirkpatrick: Optimal Search in Planar Subdivisions, *SIAM J. on Computing*, **12** (1983), 28–35.

[4] W. D. Smith: Studies in Computational Geometry motivated by Mesh Generation, Ph.D. Thesis, Princeton Univ. (1989).

Raimund Seidel

FB Informatik
Universität der Saarlandes
Postfach 15 1150
D-66041 Saarbrücken
Germany
seidel@cs.uni-sb.de