Europ. J. Combinatorics (1993) **14**, 251–258

A Monotonicity Property of h-vectors and h^*-vectors

RICHARD P. STANLEY

Dedicated to Bernt Lindström on his 60th birthday

If Δ is a Cohen–Macaulay simplicial complex of dimension $d-1$ and Δ' is a Cohen–Macaulay subcomplex of Δ of dimension $e-1$, such that no $e+1$ vertices of Δ' form a face of Δ, then we show that $h(\Delta') \leqslant h(\Delta)$, where h denotes the h-vector. In particular, $h(\Delta') \leqslant h(\Delta)$ if Δ and Δ' are Cohen–Macaulay of the same dimension. Using similar techniques we obtain a class of Gorenstein complexes Δ, the h-vector of which is unimodal. Most of these results were obtained earlier by Kalai in a somewhat more complicated way. We then use our methods to give an analogous monotonicity property of Ehrhart polynomials of lattice polytopes (and more general objects). Our results on Ehrhart polynomials may be regarded as 'lattice analogues' of the well known monotonicity results concerning intrinsic volumes or quermassintegrals.

1. Introduction

Let Δ be a finite (abstract) simplicial complex of dimension $d-1$ with f_i i-dimensional faces (or i-faces, for short). (For undefined terminology, see, e.g. [12, Ch. II] or [13]). The *f-vector* of Δ is given by $f(\Delta) = (f_0, \ldots, f_{d-1})$, with the understanding that $f_{-1} = 1$ unless $\Delta = \varnothing$. It is often more convenient to deal not with the f-vector itself, but rather with the *h-vector* $h(\Delta) = (h_0, \ldots, h_d)$, defined by

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}. \tag{1}$$

(Sometimes we write $h(\Delta) = (h_0, \ldots, h_d)$ even though $\dim \Delta = e-1 < d-1$. In this case, of course, we have $h_{e+1} = \cdots = h_d = 0$.) For instance, if Δ is a Cohen–Maculay complex [12, Ch. II, Def. 3.1] then $h_i \geqslant 0$ (and in fact $h(\Delta)$ can be completely characterized in an elegant way). In Section 2 we will investigate properties of the f-vector (or h-vector) of certain subcomplexes of Δ. Most of our results here have also been obtained by Kalai [7], although our proofs are somewhat simpler. Our primary reason for including Section 2 is that we use similar techniques in Section 3 to prove a result for which Kalai's methods seem inapplicable. Section 3 is devoted primarily to answering a question raised by Hibi and the present author concerning the Ehrhart polynomial of a convex lattice polytope (or more generally, certain lattice polyhedral complexes): namely, if $\mathcal{Q} \subseteq \mathcal{P}$ are two such polytopes, then $h^*(\mathcal{Q}) \leqslant h^*(\mathcal{P})$, where h^* denotes the h^*-vector. I am grateful to T. Hibi for pointing out an error in the original formulation of Theorems 2.1 and 3.3.

2. Subcomplexes of Cohen–Macaulay Complexes

Let Δ be a (finite) Cohen–Macaulay $(d-1)$-dimensional simplicial complex over an infinite field K. Let $K[\Delta]$ denote the face ring (or Stanley–Reisner) ring of Δ over K (see, e.g., [12, Ch. II] for definitions). Since K is infinite, there exists a homogeneous system of parameters (h.s.o.p.) $\theta_1, \ldots, \theta_d$ of degree one (so $\theta_1, \ldots, \theta_d \in K[\Delta]_1$). Let $R = R_\Delta = K[\Delta]/(\theta_1, \ldots, \theta_d)$. R inherits a grading $R = R_0 \oplus R_1 \oplus \cdots$ from $K[\Delta]$, and

0195-6698/93/030251 + 08 \$08.00/0

the statement that Δ is Cohen–Macaulay is equivalent to the formula

$$\dim_K R_i = h_i, \qquad \text{for all } i,$$

where $h(\Delta) = (h_0, \ldots, h_d)$. Thus $R_i = 0$ for all $i > d$, so $R = R_0 \oplus R_1 \oplus \cdots \oplus R_d$.

THEOREM 2.1. *Let Δ' be a* subcomplex *of Δ, i.e. a simplicial complex which is a subset of Δ. Let $e - 1 = \dim \Delta' \leq \dim \Delta = d - 1$. Assume that Δ and Δ' are Cohen–Macaulay, and that no set of $e + 1$ vertices of Δ' forms a face of Δ. (This last condition automatically holds if $d = e$.) Then $h(\Delta') \leq h(\Delta)$ (co-ordinate-wise $\leq$).*

PROOF. Let the vertices of Δ be $x_1, \ldots, x_n$. If $F \in \Delta$, write $x^F = \prod_{x_i \in F} x_i \in K[\Delta]$. Let I be the ideal of $K[\Delta]$ generated by all monomials x^F, where $F \notin \Delta'$. Clearly, $K[\Delta'] \simeq K[\Delta]/I$.

Now recall the condition [10, Remark on p. 150; 3, p. 66] (see [16, Prop. 4.2] for a proof) for a sequence $\theta_1, \ldots, \theta_d$ of homogeneous elements of $K[\Delta]$ of degree one to be an h.s.o.p.: namely, the restriction of $\theta_1, \ldots, \theta_d$ to each face F of Δ must span a vector space of dimension $\#F$. From this it follows easily that when K is infinite (as we are assuming) we can find an h.s.o.p. $\theta_1, \ldots, \theta_d$ of $K[\Delta]$ of degree one with the following property: each of $\theta_{e+1}, \ldots, \theta_d$ is a linear combination of vertices not contained in Δ'. In other words, the images of $\theta_{e+1}, \ldots, \theta_d$ in $K[\Delta']$ are all zero. Let us call such an h.s.o.p. *special.*

Identify the special h.s.o.p. $\theta = \{\theta_1, \ldots, \theta_d\} \subset K[\Delta]_1$ with its image in $K[\Delta']$ (or, alternatively, think of $K[\Delta']$ as a $K[\Delta]$-module). Since $\theta_1, \ldots, \theta_d$ is special, it follows that $\theta_1, \ldots, \theta_e$ is an h.s.o.p. for $K[\Delta']$, and that

$$K[\Delta']/(\theta_1, \ldots, \theta_d) = K[\Delta']/(\theta_1, \ldots, \theta_e).$$

Now we have a degree-preserving surjection

$$R := K[\Delta]/(\theta) \xrightarrow{f} R' := K[\Delta']/(\theta) \simeq R/I.$$

Since Δ and Δ' are Cohen–Macaulay we have $\dim_K R_i = h_i(\Delta)$ and $\dim_K R_i' = h_i(\Delta')$. The surjection $f: R_i \to R_i'$ shows that $h_i(\Delta) \geq h_i(\Delta')$, as desired. □

NOTE. Theorem 2.1, in the special case that $\dim \Delta = \dim \Delta'$, was also proved by G. Kalai (unpublished) as part of his theory of algebraic shifting. Adin [1, Thm 6.5] has also proved this special case, using the same method as ours.

Now suppose that Δ is a nonacyclic Gorenstein complex over the field K. (See [12, Ch. II, Thm 5.1] for some characterizations of such complexes. One characterization is that Δ is Cohen–Macaulay over K and an orientable pseudomanifold over K.) Let $\dim \Delta = d - 1$. The Dehn–Sommerville equations for Δ assert that $h_i = h_{d-i}$ for all i, where $h(\Delta) = (h_0, \ldots, h_d)$. Δ is said to satisfy the *Generalized Lower Bound Conjecture* (GLBC) if $h(\Delta)$ is unimodal, i.e. $h_0 \leq h_1 \leq \cdots \leq h_{[d/2]}$ (so $h_{[d/2]} \geq h_{[d/2]+1} \geq \cdots \geq h_d$). It was shown in [11] (see [3] and [13] for a survey) that if Δ is the boundary complex of a simplicial convex polytope, then Δ satisfies the GLBC. In fact, a complete characterization, known as the *g-condition* or *McMullen's conditions* of such Δ was obtained. We will establish that certain nonacyclic Gorenstein complexes, more general than boundary complexes of simplicial polytopes, also satisfy the GLBC (but we are unable to show that they satisfy the g-condition). A very similar result was earlier obtained by Kalai [7, §8] using algebraic shifting.

LEMMA 2.2. *Let Δ be a nonacyclic Gorenstein complex of dimension $d-1$ with vertex set V over a field K. Let Δ' be a Cohen–Macaulay subcomplex of Δ. Let $\theta_1, \ldots, \theta_d$ be a special* (*as defined in the proof of Theorem* 2.1) *h.s.o.p. of $K[\Delta]$ of degree* 1, *and set $R = K[\Delta]/(\theta_1, \ldots, \theta_d) = R_0 \oplus \cdots \oplus R_d$. Suppose that there exists an element $\omega \in R_1$* (*called a* Lefschetz element) *with the following property*: *for all $0 \leq i \leq [d/2]$, the map $\omega^{d-2i}: R_i \to R_{d-i}$ given by multiplication by ω^{d-2i} is a bijection. Then the h-vector $h(\Delta') = (h_0, \ldots, h_d)$ satisfies*:

$$h_i \geq h_{d-i}, \qquad \text{for } 0 \leq i \leq [d/2],$$

$$h_{[d/2]} \geq h_{[d/2]+1} \geq \cdots \geq h_d.$$

PROOF. The condition on ω implies that $\omega: R_i \to R_{i+1}$ is surjective for $i \geq [d/2]$. Hence if $S = R/\omega R = S_0 \oplus S_1 \oplus \cdots$, then $S_i = 0$ for $i > [d/2]$. As in the proof of Theorem 2.1, we have a surjective degree-preserving map

$$f: S \to K[\Delta']/(\theta_1, \ldots, \theta_d, \omega) := S'.$$

Thus $S_i' = 0$ for $i > [d/2]$. If we set $R' = K[\Delta']/(\theta_1, \ldots, \theta_d)$, then $\dim_K R_i' = h_i(\Delta')$, since $\theta_1, \ldots, \theta_d$ is special. Since $S_i' = 0$ for $i > [d/2]$, it follows that $\omega: R_i' \to R_{i+1}'$ is surjective for $i \geq [d/2]$. Thus $h_i(\Delta') \geq h_{i+1}(\Delta')$ for $i \geq [d/2]$.

Now fix $i \geq [d/2]$, and set $T = R/\omega^{d-2i}R = T_0 \oplus T_1 \oplus \cdots$. Since $\omega^{d-2i}: R_i \to R_{d-i}$ is surjective, we have $T_{d-i} = 0$. As in the previous paragraph, we have a surjective degree-preserving map

$$g: T \to K[\Delta']/(\theta_1, \ldots, \theta_d, \omega^{d-2i}) := T'.$$

Thus $T_{d-i}' = 0$; so the map $\omega^{d-2i}: R_i' \to R_{d-i}'$ (with R' as above) is surjective. Hence $h_i(\Delta') \geq h_{d-i}(\Delta')$. □

NOTE. It follows from [11] that if Δ is the boundary complex of a simplicial d-polytope, then Δ satisfies all the hypotheses of Lemma 2.2. It remains open at present whether a Lefschetz element ω exists for arbitrary triangulations of spheres (or, more generally, nonacyclic Gorenstein complexes) for a suitable choice (or possibly every choice) of h.s.o.p. $\theta_1, \ldots, \theta_d$ of degree one when char $K = 0$. (When char $K \neq 0$ then ω does not exist for certain choices of Δ and $\theta_1, \ldots, \theta_d$.)

Recall that a $(d-1)$-dimensional *pseudomanifold with boundary* is a pure $(d-1)$-dimensional simplicial complex Δ such that every $(d-2)$-face is contained in exactly one or two facets, and which satisfies a certain connectivity property which is automatic when Δ is Cohen–Macaulay [4, Prop. 11.7]. The *boundary* $\partial\Delta$ of Δ is the subcomplex of Δ generated by all $(d-2)$-faces contained in exactly one facet. A result equivalent to the following lemma, in the case when Δ triangulates a ball, appears in [9, Thm 2].

LEMMA 2.3. *Let Δ be a $(d-1)$-dimensional Cohen–Macaulay pseudomanifold with nonempty boundary. Let $h(\Delta) = (h_0, h_1, \ldots, h_d)$. Then*

$$h(\partial\Delta) = (h_0 - h_d,\ h_0 + h_1 - h_d - h_{d-1},\ h_0 + h_1 + h_2 - h_d - h_{d-1} - h_{d-2}, \ldots, h_0 - h_d).$$

PROOF. Since Δ is Cohen–Macaulay and a pseudomanifold with boundary, it follows easily that

$$\tilde{\chi}(\mathrm{lk}_\Delta F) = \begin{cases} 0, & \text{if } F \in \partial\Delta \\ (-1)^{d-1-|F|}, & \text{if } F \in \Delta - \partial\Delta, \end{cases}$$

where $\tilde{\chi}$ denotes the reduced Euler characteristic and lk_Δ the link in Δ. Hence, by the reasoning used to prove [8, Prop. 1.1] or [12, Ch. II, Cor. 7.2], we have

$$\sum_{F\in\Delta-\partial\Delta}(x-1)^{d-|F|}=h_0+h_1x+\cdots+h_dx^d.$$

Comparing with (1) shows that

$$\begin{aligned}\sum_{i=0}^{d-1}h_i(\partial\Delta)x^{d-1-i}&=\sum_{i=0}^{d-1}f_{i-1}(\partial\Delta)(x-1)^{d-1-i}\\&=\frac{1}{x-1}\left[\sum_{F\in\Delta}(x-1)^{d-|F|}-\sum_{F\in\Delta-\partial\Delta}(x-1)^{d-|F|}\right]\\&=\frac{1}{x-1}\left[\sum_{i=0}^{d}(h_i(\Delta)-h_{d-i}(\Delta))x^{d-i}\right]\\&=h_0-h_d+(h_0+h_1-h_d-h_{d-1})x+\cdots+(h_0-h_d)x^{d-1},\end{aligned}$$

as desired. □

COROLLARY 2.4. *Let Δ be a nonacyclic Gorenstein complex of dimension $d-1$ (over a field K) possessing a Lefschetz element ω (as defined in Lemma 2.1). Let Δ' be a proper Cohen–Macaulay subcomplex of Δ of dimension $d-1$. It is easy to see that Δ' is a pseudomanifold with non-empty (since Δ' is proper) boundary, so $\partial\Delta'$ is defined. Then $\partial\Delta'$ satisfies the GLBC, i.e. if $h(\partial\Delta)=(h'_0, h'_1, \ldots, h'_{d-1})$ then $h'_i=h'_{d-1-i}$ and $h'_0\leq h'_1\leq\cdots\leq h'_{[(d-1)/2]}$.*

PROOF. Let $h(\Delta)=(h_0, h_1, \ldots, h_d)$. Since $\dim\Delta=\dim\Delta'$, there exists a special h.s.o.p. of $K[\Delta]$ of degree 1. Thus, by Lemma 2.2, we have $h_i\geq h_{d-i}$ for $0\leq i\leq[d/2]$. By Lemma 2.3, we have $h'_i=h_0+h_1+\cdots+h_i-h_d-h_{d-1}-\cdots-h_{d-i}$. Thus $h'_i=h'_{d-1-i}$, and if $i\leq[(d-1)/2]$ then $h'_i-h'_{i-1}=h_i-h_{d-i}\geq 0$. □

The simplest instance of a pair (Δ, Δ') satisfying the conditions of Corollary 2.4 occurs when Δ is the boundary complex of a simplicial d-polytope and Δ' is a $(d-1)$-ball, so that $\partial\Delta'$ is actually a $(d-2)$-sphere. In this situation, Corollary 2.4 was proved by Kalai [7, §8], using his theory of algebraic shifting.

3. EHRHART POLYNOMIALS OF SUBCOMPLEXES

Let L be a lattice in $\mathbf{R}^m$. A *lattice polytope* or *L-polytope* $\mathcal{P}$ is a convex polytope in $\mathbf{R}^m$ with vertices in L. Given such a polytope $\mathcal{P}$, define a function $i(\mathcal{P}, n)$ for integral $n\geq 0$ by:

$$\begin{aligned}i(\mathcal{P}, n)&=\#(n\mathcal{P}\cap L),\qquad \text{if } n\geq 1;\\ i(\mathcal{P}, 0)&=1.\end{aligned}$$

Here $n\mathcal{P}=\{n\alpha\colon \alpha\in\mathcal{P}\}$. Then $i(\mathcal{P}, n)$ is known to be a polynomial function of n, called the *Ehrhart polynomial* of $\mathcal{P}$. For basic information and references concerning Ehrhart polynomials, see [14, pp. 235–241]. (In this reference it is assumed that $L=\mathbf{Z}^m$, but the results and proofs are the same for any lattice.)

If $\mathcal{P}$ is a $(d-1)$-dimensional L-polytope in $\mathbf{R}^m$, then $i(\mathcal{P}, n)$ has degree $d-1$. By standard properties of generating functions [14, Ch. 4.3], there exist integers $h_0^*, \ldots, h_d^*$ such that

$$\sum_{n\geq 0}i(\mathcal{P}, n)x^n=\frac{h_0^*+h_1^*x+\cdots+h_d^*x^d}{(1-x)^d}. \tag{2}$$

We call $h^*(\mathcal{P}) := (h_0^*, \ldots, h_d^*)$ the *h^*-vector* of $\mathcal{P}$. (In fact, we have $h_d^* = 0$, but for a generalization to be discussed below this need not be the case.) In a discussion with T. Hibi the following question arose: If $\mathcal{Q}$ is an L-polytope contained in $\mathcal{P}$, then is $h^*(\mathcal{Q}) \leqslant h^*(\mathcal{P})$ (i.e. $h_i^*(\mathcal{Q}) \leqslant h_i^*(\mathcal{P})$ for all i)? We will give an affirmative answer to this question with an argument analogous to that used to prove Theorem 2.1.

First let us mention some work related to the monotonicity of h^*-vectors. A well known result of Hadwiger [6] asserts that every continuous and additive functional on convex bodies in $\mathbf{R}^m$ which is invariant under rigid motions is a linear combination of $m+1$ functionals $V_0, \ldots, V_m$ known as *intrinsic volumes* (or, with a different normalization, the *quermassintegrals* of Minkowski). The intrinsic volumes have the property of *monotonicity*, i.e. if $X' \subseteq X$ then $V_j(X') \leqslant V_j(X)$. From this it follows that the V_j's are also *non-negative*, i.e. $V_j(X) \geqslant 0$. An analogous result was proved by Betke and Kneser [2] for enumerating lattice points rather than computing volumes. Namely, every additive and unimodular invariant functional on L-polytopes (where L is a fixed lattice in $\mathbf{R}^m$) is a linear combination of the $m+1$ functionals $G_{L,0}, \ldots, G_{L,m}$ defined by

$$i(\mathcal{P}, n) = \sum_{j=0}^{m} G_{L,j}(\mathcal{P}) n^j.$$

In other words, the $G_{L,j}$'s are just the coefficients of the Ehrhart polynomial. But the $G_{L,j}$'s have the defect that they are neither monotone nor non-negative. On the other hand, $h_0^*, \ldots, h_m^*$ is a different basis for the vector space A spanned by $G_{L,0}, \ldots, G_{L,m}$, the elements h_j^* of which are monotone (and hence non-negative). (Non-negativity had been proved earlier by the present author and then by Betke and McMullen.) However, the elements h_j^*, unlike $G_{L,j}$, lack the property of being *homogeneous*, i.e. it is false that for $n \geqslant 0$, $h_j^*(n\mathcal{P}) = n^k h_j^*(\mathcal{P})$ for some fixed $k \geqslant 0$. Since the $G_{L,j}$'s are the unique (up to scalar multiplication) homogeneous basis of A, it follows that no basis can be both homogeneous and monotone (or even non-negative). Contrast this situation with the intrinsic volumes, which *are* both homogeneous and monotone. For some further background information, see the interesting survey in [5].

Our argument which establishes the monotonicity of the h^*-vector of L-polytopes actually applies to a more general situation, which we now discuss. (We could work in the even greater generality of Yuzvinsky's theory of Cohen–Macaulay rings of sections [17], thereby unifying Theorems 2.1 and 3.3, but for the sake of simplicity we will not do so.) An *L-polyhedral complex* Γ in $\mathbf{R}^M$ is a finite collection of L-polytopes in $\mathbf{R}^m$ satisfying: (a) if $\mathcal{P} \in \Gamma$ then every face of $\mathcal{P}$ is in Γ; and (b) if $\mathcal{P}, \mathcal{Q} \in \Gamma$ then $\mathcal{P} \cap \mathcal{Q}$ is a common face (possibly empty) of $\mathcal{P}$ and $\mathcal{Q}$. The *body* $|\Gamma|$ of Γ is defined by

$$|\Gamma| = \bigcup_{\mathcal{P} \in \Gamma} \mathcal{P}.$$

A subset $\mathcal{X}$ of $\mathbf{R}^m$ which equals $|\Gamma|$ for some L-polyhedral complex Γ is called an *L-polyhedron*. If $\mathcal{X}$ is an L-polyhedron, then define, for integers $n \geqslant 0$,

$$i(\mathcal{X}, n) = \#(n\mathcal{X} \cap L), \qquad \text{if } n \geqslant 1;$$
$$i(\mathcal{X}, 0) = 1.$$

Then $i(\mathcal{X}, n)$ is a polynomial for $n \geqslant 1$, but not necessarily at $n = 0$. (The 'correct' value of $i(\mathcal{X}, 0)$ which makes it a polynomial for all $n \geqslant 0$ is $\chi(\mathcal{X})$, the Euler characteristic of $\mathcal{X}$.) We call $i(\mathcal{X}, n)$ the *Ehrhart function* of $\mathcal{X}$. As in (2), we define the h^*-vector $h^*(\mathcal{X}) = (h_0^*, \ldots, h_d^*)$ by

$$\sum_{n \geqslant 0} i(\mathcal{X}, n) x^n = \frac{h_0^* + h_1^* x + \cdots + h_d^* x^d}{(1-x)^d},$$

where $d-1=\dim \mathscr{X}$. (We may define $\dim \mathscr{X}$, for instance, as $\max\{\dim \mathscr{P}\colon \mathscr{P}\in\Gamma\}$, where Γ is a polyhedral complex with body $\mathscr{X}$.)

An L-polyhedron $\mathscr{X}$ (or any polyhedron) is said to be *Cohen–Macaulay* if some (equivalently, every) triangulation of $\mathscr{X}$ defines a Cohen–Macaulay simplicial complex. Equivalently, if $\dim \mathscr{X}=d-1$ then

$$\tilde{H}_i(\mathscr{X})=H_i(\mathscr{X},\mathscr{X}-p)=0,\qquad i<d-1,$$

for all $p\in\mathscr{X}$, where $\tilde{H}_i(\mathscr{X})$ denotes the ith reduced homology group of $\mathscr{X}$ and $H_i(\mathscr{X},\mathscr{X}-p)$ the ith local homology group of $\mathscr{X}$ at p (over the ground field K).

Given an L-polyhedral complex Γ in $\mathbf{R}^m$, define a K-algebra $K[\Gamma]$ as follows. A K-basis B_Γ of $K[\Gamma]$ consists of 1 together with all monomials $u^a v^b=u_1^{a_1}\cdots u_m^{a_m}v^b$, where $a\in L$, b is a positive integer and $a/b\in|\Gamma|$. Multiplication of monomials is defined by:

$$u^a v^b\cdot u^c v^d=\begin{cases}u^{a+c}v^{b+d}, & \text{if } a/b\in\mathscr{P} \text{ and } c/d\in\mathscr{P} \text{ for some } \mathscr{P}\in\Gamma;\\ 0, & \text{otherwise.}\end{cases}$$

One easily checks that this indeed defines an algebra with basis B_Γ; the key point is that the *convexity* of the polytopes $\mathscr{P}\in\Gamma$ ensures that multiplication in $K[\Gamma]$ is associative.

Define a grading on $K[\Gamma]$ by setting $\deg u^a v^b=b$. From the definition of $K[\Gamma]$ it follows that the Hilbert function $H(K[\Gamma],n)$ is given by

$$H(K[\Gamma],n)=i(\mathscr{X},n),$$

where $\mathscr{X}=|\Gamma|$. Hence

$$F(K[\Gamma],x):=\sum_{n\geq 0}H(K[\Gamma],n)x^n=\frac{h_0^*+\cdots+h_d^*x^d}{(1-x)^d},$$

where $\dim\Gamma=d-1$, and where $h^*(\mathscr{X})=(h_0^*,\ldots,h_d^*)$. One easily sees that $K[\Gamma]$ is *semi-standard,* i.e. integral over the subalgebra generated by $K[\Delta]_1$. (In fact, $K[\Gamma]$ is integral over the subalgebra generated by $\{u^a v\colon a \text{ is a vertex of some } \mathscr{P}\in\Gamma\}$.) Hence if K is infinite then $K[\Gamma]$ possesses an h.s.o.p. $\theta_1,\ldots,\theta_d$ of degree 1. Moreover, if $K[\Gamma]$ is Cohen–Macaulay, then the ring $R_\Gamma=K[\Gamma]/(\theta_1,\ldots,\theta_d)$ has the grading $R_\Gamma=R_0\oplus\cdots\oplus R_d$, where $\dim_K R_i=h_i^*$. The following result is essentially the same as [15, Lemma 4.6].

LEMMA 3.1. *Let Γ be an L-polyhedral complex. If $|\Gamma|$ is a Cohen–Macaulay polyhedron, then the algebra $K[\Gamma]$ defined above is Cohen–Macaulay.* □

COROLLARY 3.2. *Let Γ be an L-polyhedral complex. If $\mathscr{X}:=|\Gamma|$ is a Cohen–Macaulay polyhedron, then $h^*(\mathscr{X})\geq 0$.* □

We come to the main result of this section.

THEOREM 3.3. *Let $\mathscr{X}$ and $\mathscr{Y}$ be Cohen–Macaulay L-polyhedra in $\mathbf{R}^m$, with $\mathscr{Y}\subseteq\mathscr{X}$. Let* $\dim\mathscr{Y}=e-1$, *and suppose that $\mathscr{Y}$ is contained in an affine subspace of $\mathbf{R}^m$ of dimension $e-1$. (This last condition automatically holds if $\mathscr{Y}$ is convex.) Then $h^*(\mathscr{Y})\leq h^*(\mathscr{X})$.*

PROOF. It is easy to see that there exists an L-polyhedral complex Γ with the following two properties: (a) $|\Gamma| = \mathscr{X}$; (b) there is a subcomplex Λ of Γ with $|\Lambda| = \mathscr{Y}$. It is also easy to check that the vector space surjection $f: K[\Gamma] \to K[\Lambda]$ defined by setting $f(1) = 1$ and for $b > 0$,

$$f(u^a v^b) = \begin{cases} u^a v^b, & \text{if } a/b \in \mathscr{Y}, \\ 0, & \text{if } a/b \notin \mathscr{Y} \end{cases},$$

is actually a (degree-preserving) algebra homomorphism. The proofs now proceeds as in that of Theorem 2.1 Let $\dim \mathscr{X} = d - 1$ and $\dim \mathscr{Y} = e - 1$. As in the proof of Theorem 2.1, we can find as h.s.o.p. $\theta = \{\theta_1, \ldots, \theta_d\}$ of degree 1 for $K[\Gamma]$ such that the images of $\theta_{e+1}, \ldots, \theta_d$ in $K[\Lambda]$ are all 0. Hence

$$K[\Lambda]/(\theta_1, \ldots, \theta_d) = K[\Lambda]/(\theta_1, \ldots, \theta_e).$$

Then f induces a surjection $f: R_\Gamma \to R_\Lambda$, where $R_\Gamma = K[\Gamma]/\theta K[\Gamma]$ and $R_\Lambda = K[\Lambda]/\theta K[\Lambda]$. Thus we have a vector space surjection $(R_\Gamma)_i \to (R_\Lambda)_i$. By the Cohen–Macaulay hypothesis, we have $\dim_K (R_\Gamma)_i = h_i^*(\mathscr{X})$ and $\dim_K (R_\Lambda)_i = h_i^*(\mathscr{Y})$, so $h_i^*(\mathscr{Y}) \leq h_i^*(\mathscr{X})$ as desired. □

Suppose now that the integral polyhedron $\mathscr{X}$ is nonacyclic Gorenstein; i.e. any triangulation of $\mathscr{X}$ defines a nonacyclic Gorenstein simplicial complex. Then the h^*-vector $h^*(\mathscr{X}) = (h_0^*, \ldots, h_d^*)$ satisfies the Dehn–Sommerville equations, i.e. $h_i^* = h_{d-i}^*$. It is natural to ask, in analogy with the situation for Corollary 2.4, whether $h^*(\mathscr{X})$ satisfies the GLBC, i.e. $h_0^* \leq h_1^* \leq \cdots \leq h_{[d/2]}^*$. The following example shows that, unfortunately, this question has a negative answer, even when $\mathscr{X}$ is the boundary of a simplex.

EXAMPLE 3.4. Let $\mathscr{X}$ be the boundary of the 6-simplex in $\boldsymbol{R}^6$ with vertices $(0,0,0,0,0,0)$, $(0,0,0,0,0,1)$, $(0,0,0,0,1,0)$, $(0,0,0,1,0,0)$, $(0,0,1,0,0,0)$, *$(0,3,2,2,2,2)$ and $(1,0,0,0,0,0)$. Then $h^*(\mathscr{X}) = (1, 1, 2, 1, 2, 1, 1)$, which fails to* satisfy the GLBC.

ACKNOWLEDGEMENTS

This research was carried out at MIT, Hebrew University, Universität Augsburg and Tokai University, and supported respectively by NSF grant #DMS-8401376, a grant from the U.S.–Israel Binational Science Foundation (BSF), the Institut für Mathematik der Universität Augsburg, and a fellowship from the Matsumae International Foundation.

REFERENCES

1. R. Adin, Tight bounds on the h-vector of a Cohen–Macaulay complex with a group action, Ph.D. Dissertation (part I), Hebrew University, Jerusalem, 1990.
2. U. Betke and M. Kneser, Zerlegung und Bewertung von Gitterpolytopen, *J. reine angew. Math.*, **99** (1985), 253–265.
3. L. Billera, Polyhedral theory and commutative algebra, in: *Mathematical Programming: The State of the Art*, A. Bachem, M. Grötschel and B. Korte (eds), Springer-Verlag, Berlin, 1983, pp. 55–77.
4. A. Björner, Topological methods, in: *Handbook of Combinatorics*, R. Graham, M. Grötschel, and L. Loväsz, (eds), North-Holland, Amsterdam, to appear.
5. P. Gritzmann and J. M. Wills, Lattice points, in: *Handbook for Convex Geometry*, North-Holland, Amsterdam, to appear.
6. H. Hadwiger, Beweis eines Funktionalsatzes für konvexe Körper, *Abh. Math. Sem. Univ. Hamburg*, **17** (1951), 69–76.

7. G. Kalai, The diameter of graphs of convex polytopes and f-vector theory, preprint.
8. I. G. Macdonald, Polynomial associated with finite cell-complexes, *J. Lond. Math. Soc.* (2), **4** (1971), 181–192.
9. P. McMullen and D. W. Walkup, A generalized lower-bound conjecture for simplicial polytopes, *Mathematika*, **18** (1971), 274–273.
10. R. Stanley, Balanced Cohen–Macaulay complexes, *Trans. Am. Math. Soc.*, **249** (1979), 139–157.
11. R. Stanley, The number of faces of a simplicial convex polytope, *Adv. Math.*, **35** (1980), 236–238.
12. R. Stanley, *Combinatorics and Commutative Algebra*, Progress in Mathematics, vol. 41, Birkhäuser, Boston–Basel–Stuttgart, 1983.
13. R. Stanley, The number of faces of simplicial polytopes and spheres, in: *Discrete Geometry and Convexity*, J. E. Goodman, E. Lutwak, J. Malkevitch, and R. Pollock (eds), *Ann. N.Y. Acad. Sci.*, **440** (1985), 212–223.
14. R. Stanley, *Enumerative Combinatorics, vol. I*, Wadsworth and Brroks/Cole, Pacific Grove, California, 1986.
15. R. Stanley, Generalized h-vectors, intersection homology of toric varieties, and related results, in: *Commutative Algebra and Combinatorics*, M. Nagata and H. Matsumura (eds), Advanced Studies in Pure Mathematics 11, Kinokuniya, Tokyo/North-Holland, Amsterdam, 1987, pp. 187–213.
16. R. Stanley, Subdivisions and local h-vectors, *J. Am. Math. Soc.*, **5** (1992), 805–851.
17. S. Yuzvinsky, Cohen–Macaulay rings of sections, *Adv. Math.*, **63** (1987), 172–195.

Received 16 January 1992 and in revised form 31 August 1992

RICHARD P. STANLEY
Department of Mathematics,
Massachusetts Institute of Technology,
Cambridge, Massachusetts 02139, U.S.A.