TOPOLOGICAL EFFECTS ON MINIMUM WEIGHT STEINER TRIANGULATIONS

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Abstract. Let $mwt(\mathcal{X})$ denote the sum of the Euclidean edge lengths of a minimum weight triangulation of a point set $\mathcal{X} \in \mathbb{R}^2$. We investigate a curious property of some n-point sets $\mathcal{X}$, which allow for an $(n+1)^{\text{st}}$ point P (called a Steiner point) to give $mwt(\mathcal{X} \cup \{P\}) < mwt(\mathcal{X})$. We call the regions of the plane where such a P reduces the length of the minimum weight triangulation *Steiner reducing regions.* We demonstrate by example that these Steiner reducing regions may have many disconnected components or fail to be simply connected. By examining randomly generated point sets, we show that the surprising topology of these Steiner reducing regions is more common than one might expect.

1. Introduction

We are concerned with a long-standing classical problem in computational geometry: that of finding a minimum weight triangulation of a point set. We can describe this task as a game of connect-the-dots. You are given a collection of dots on a sheet of paper, and you are told to draw as many straight line segments as possible, with one caveat: no new edge may be drawn which crosses one that you have already drawn. Once no more edges can be drawn, you should see a collection

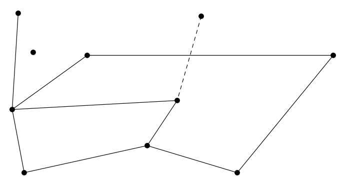

Figure 1. Mid-game, we notice that the dashed line can't be legally drawn, because it crosses an edge already drawn.

of triangles connecting the dots. Your score for this game is the sum of the lengths of all the edges you drew, and as in golf, the lowest score wins. Can you develop a strategy that will lead you *quickly* to the "best" drawing, one with a lower score than any other possible drawing? Will your strategy give you perpetual victory over your opponents if applied to *any* point set that you are given? These are questions which have stumped mathematicians and computer scientists alike for over thirty years. Indeed, of all the problems of unknown computational complexity collected in [12], this problem is one of the few that remains yet unclassified.

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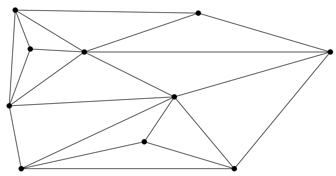

FIGURE 2. The left triangulation is longer than the one at right.

Triangulations are very useful objects in the realm of applied computational geometry. By breaking space into smaller regions, they are useful for graphical rendering, numerical estimation of volume integrals, and many other types of mathematical modelling. For a thorough mathematical treatment of triangulations, consult [8]. Formally, a *triangulation* of a point set $\mathcal{X} \in \mathbb{R}^2$ is an inclusion-maximal set $\mathbb{T}$ of non-intersecting straight line segments connecting pairs of points in $\mathcal{X}$. A triangulation is specified by its *combinatorial type*: a listing of either its edges or its point-empty triangles. The *length* or *weight* of a triangulation of $\mathcal{X}$ is the sum of the Euclidean lengths of the edges used, so a *minimum weight triangulation* of X is a triangulation which has length less than or equal to the length of every other triangulation of $\mathcal{X}$. We note that such a triangulation is not necessarily unique. We denote the length of a minimum weight triangulation of $\mathcal{X}$ by $mwt(\mathcal{X})$, and we denote the set of edges used by $\mathrm{MWT}(\mathcal{X})$.

The problem of efficiently finding an optimal triangulation of a point set has been of interest to mathematicians and computer scientists for some time. Different measures of optimality for triangulations have given rise to useful applications and encouraged algorithm development. These algorithms are are deemed *efficient* if they require a number of steps which is relatively small compared to the possibility of examining all or most of the candidate objects involved. For a survey of optimization with regard to triangulations, see [4] or [8].

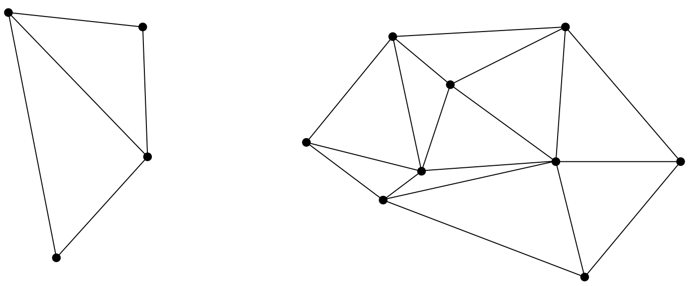

FIGURE 3. The minimum weight triangulation of a quadrilateral uses the shorter of its two diagonals. For larger point sets, it is much harder to find the minimum weight triangulation.

Much progress has been made towards special instances of the minimum weight triangulation problem. For example, there are polynomial-time algorithms for determining the minimum weight triangulation of special classes of point sets, such as polygonal domains [13, 15]. Certain edges and progressively larger subsets of edges have been proven to belong to the minimum weight triangulation. These include the shortest edge [13], all mutual nearest-neighbor edges [22], and two different sets of edges known as the β-skeleton [6, 14] and the LMT-skeleton [1, 9]. Additional work has been done to create and evaluate different methods of finding the exact minimum weight triangulation and also approximating the minimum weight triangulation of point sets in $\mathbb{R}^2$ [2, 10, 16, 17, 18, 19, 20] and higher dimensions [3, 5, 7]. One common method of approximation allows for the addition of a small number of new points, called *Steiner points*, to the input set before triangulating. If the number of Steiner points is small, they do not greatly affect the time or space required for computation. Indeed, adding a collection of points to our input set before triangulating can create various desirable properties, such as triangulations that avoid small angles, make certain computations easier, or approximate triangulations which are difficult to calculate.

Return for a moment to the minimum weight triangulation game we described earlier. Imagine now that before you begin to connect the dots, you are given the chance to change your point set. You may not delete any of the given points, but you may now add as many points as you wish before playing the game with the new, possibly larger set of points. Can it possibly benefit your "golf" score to add more points to the mix? Shockingly, yes! We say that a point set $\mathcal{X}$ is *Steiner reducible* if there exists a point $P = (x, y) \in \mathbb{R}^2 - \mathcal{X}$ such that

$$\mathrm{MWT}\big(\mathcal{X} \cup \{P\}\big) < \mathrm{MWT}(\mathcal{X}).$$

Such a point P is said to *reduce* the length of the triangulation, and we refer to P as a *Steiner reducing point.* For a given point set, we are concerned with the region of the plane consisting of all reducing points, which we refer to as the *Steiner reducing region.* We note that in this paper we are considering the effects of adding *one* Steiner point to our original input set. Eppstein has shown in [11] that the simultaneous addition of n Steiner points can reduce the length of the minimum weight triangulation by a factor of $\Omega(n)$. It is not known if there exist point sets $\mathcal{Y}$ for which one Steiner point cannot reduce $\mathrm{MWT}(\mathcal{Y})$, but the addition of $k \geq 2$ Steiner points $P_1, \ldots, P_k$ will give the reduction

$$\mathrm{MWT}(\mathcal{Y} \cup \{P_i\}_{i=1}^{k}) < \mathrm{MWT}(\mathcal{Y}).$$

It is surprising that our new point set contains one or more points than the original set, but can be triangulated with a shorter total edge length! So, given

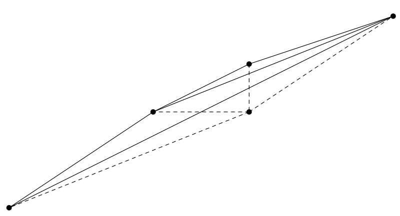

FIGURE 4. A fifth point added to this quadrilateral reduces the length of the minimum weight triangulation of the new larger point set!

a point set, in what regions of the plane can one add a Steiner reducing point to reduce the length of the minimum weight triangulation? The results are somewhat unexpected. A set as small as five points can have a Steiner reducing region with two disconnected components, and as the number of points increases, so does the complexity of the topology. Here are our main results:

⋆ *There exists an 18-point set that admits a connected Steiner reducing region whose first homology group has rank at least 13.*

⋆ *There exists a 15-point set that admits a Steiner reducing region with 20 disconnected components.*

We also demonstrate that the existence of Steiner reducing regions is relatively common in random point sets. This is intriguing, since it is easy to artificially create many disconnected components in the Steiner reducing region by scattering small examples of point sets with Steiner reducing regions at large distance from one another. The performance of random point sets seems to indicate that multiple Steiner reducing regions can live peacefully in close proximity to one another without forcing the points of the set to be clustered far apart from one another. The random point sets tested also indicate that it is much more likely for a Steiner reduction to occur exterior to the convex hull of our input set.

We note here that if a point set admits an exterior Steiner reducing region, then this region will intersect an edge of the convex hull. So in a search for the existence of exterior Steiner reducing regions, it will suffice to check if a Steiner point added to any of the convex hull edges will cause a reduction. Consider the Steiner reducing region for the 4-point set in Figure 4:

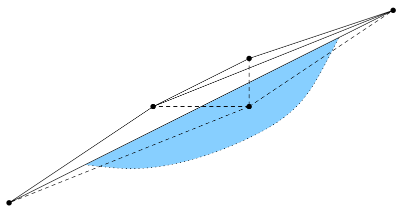

FIGURE 5. An approximation of the Steiner reducing region. [Note to self: this was freehand - need better approx for final draft.]

Notice that the Steiner reducing region for the point set in Figure 5 is neither open nor closed - it includes part of the long quadrilateral edge and the rest of the region is bounded by a curve of the form

$$\varphi = \left\{ Z \middle| \sum_{i=1}^{4} dist(Z, W_i) = L \right\},$$

where the right-hand side of the curve-defining equation represents the lengths of the edges being replaced and the left-hand side represents the new edges used in the triangulation. This curve is a cousin to the circle and the ellipse, for it represents the locus of all points whose summed distance to four fixed points is constant. We note that sets of this type and their properties are described as "n-ellipses" by Sekino in [21]. The curve φ is the boundary of a 4-ellipse. We emphasize here the fundamental connection between Steiner reducing regions and n-ellipses. For a general input set $\mathcal{X}$, a reduction occurs when new replaces old: the new set of edges connects our Steiner reducing point Z to a subset of input points $\mathcal{F} \subseteq \mathcal{X}$, and the old set $\mathbb{E}$ of replaced edges formed a minimal triangulation of the possibly non-convex polygon formed by the points of $\mathcal{F}$. Let $L = \sum_{e \in \mathbb{E}} \mathit{length}(e)$. The subset of the Steiner reducing region corresponding to the combinatorial type implied by $\mathcal{F}$ will be itself a subset of the k-ellipse $\mathcal{M} = \{(x, y) | \sum_{f \in \mathcal{F}} \operatorname{dist}\big((x, y), f\big) < L\}$, where $k = |\mathcal{F}|$. Notice in particular that if $\mathcal{F}$ has one element, then M will be a circle, and for a set $\mathcal{F}$ with two elements, M will be an ellipse. For values of $n > 2$, Sekino showed that these n-ellipses remain convex, though they may be asymmetric.

2. First homology of Steiner reducing regions

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