TOPOLOGICAL EFFECTS ON MINIMUM WEIGHT STEINER TRIANGULATIONS

CYNTHIA M. TRAUB
WASHINGTON UNIVERSITY IN SAINT LOUIS

Abstract. Let $mwt(\mathcal{X})$ denote the sum of the Euclidean edge lengths of a minimum weight triangulation of a point set $\mathcal{X} \in \mathbb{R}^2$. We investigate the conditions under which an n-point set $\mathcal{X}$ will allow an $(n+1)^{\text{st}}$ point P (called a Steiner point) to give $mwt(\mathcal{X} \cup \{P\}) < mwt(\mathcal{X})$. We call the regions of the plane where such a P reduces the length of the minimum weight triangulation *Steiner reducing regions.* We demonstrate by example that these Steiner reducing regions may have many disconnected components or fail to be simply connected. By examining randomly generated point sets, we show that the surprising topology of these Steiner reducing regions is more common than one might expect.

1. Introduction

We are concerned with a long-standing classical problem in computational geometry: that of finding a minimum weight triangulation of a point set. Formally, a *triangulation* of a point set $\mathcal{X} \subset \mathbb{R}^2$ is an inclusion-maximal set of non-intersecting straight line segments connecting pairs of points in $\mathcal{X}$. A triangulation is specified by its *combinatorial type*: a listing of either its edges or its point-empty triangles. The *length* or *weight* of a triangulation of $\mathcal{X}$ is the sum of the Euclidean lengths of the edges used, so a *minimum weight triangulation* of X is a triangulation which has length less than or equal to the length of every other triangulation of $\mathcal{X}$. We note that such a triangulation is not necessarily unique. We denote the length of a minimum weight triangulation of $\mathcal{X}$ by $mwt(\mathcal{X})$, and we denote its set of edges by MWT($\mathcal{X}$). For a thorough mathematical treatment of triangulations, consult [8].

Different measures of optimality for triangulations have given rise to useful applications and algorithms. For a survey of optimization with regard to triangulations, see [4] or [8]. Of all the problems of unknown computational complexity collected in [12], the minimum weight triangulation problem is one of the few that remains yet unclassified. There are polynomial-time algorithms for determining the minimum weight triangulation of special classes of point sets, such as polygonal domains [13],[15]. Certain edges and progressively larger subsets of edges have been proven to belong to the minimum weight triangulation. These include the shortest edge [13], all mutual nearest-neighbor edges [22], and two different sets of edges known as the β-skeleton [6],[14] and the LMT-skeleton [1],[9]. Additional work has been done to create and evaluate different methods of finding the exact minimum weight triangulation and also approximating the minimum weight triangulation of point sets in $\mathbb{R}^2$ [2], [10], [16], [17], [18], [19], [20] and higher dimensions [3], [5], [7].

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One method of triangulation approximation allows for the addition of a small number of new points, called *Steiner points*, to the input set before triangulating. If the number of Steiner points is small, they do not greatly affect the time or space required for computation. A potentially surprising effect of adding Steiner points is that the new point set can have a minimum weight triangulation with length less than that of the original point set. We say that a point set $\mathcal{X}$ is *Steiner reducible* if there exists a point $P = (x, y) \in \mathbb{R}^2 - \mathcal{X}$ such that

$$\mathrm{MWT}\big(\mathcal{X} \cup \{P\}\big) < \mathrm{MWT}(\mathcal{X}).$$

Such a point P is said to *reduce* the length of the triangulation, and we refer to P as a *Steiner reducing point.* For a point set $\mathcal{X}$, we are concerned with the region of the plane consisting of all reducing points, which we refer to as the *Steiner reducing region.* We emphasize that in this paper we are considering the effects of adding *one* Steiner point to our original input set. Eppstein has shown in [11] that the simultaneous addition of n Steiner points can reduce the length of the minimum weight triangulation by a factor of $\Omega(n)$. It is not known if there exist point sets $\mathcal{Y}$ for which one Steiner point cannot reduce $\mathrm{MWT}(\mathcal{Y})$, but the addition of $k \geq 2$ Steiner points $P_1, \ldots, P_k$ will give the reduction

$$\mathrm{MWT}(\mathcal{Y} \cup \{P_i\}_{i=1}^{k}) < \mathrm{MWT}(\mathcal{Y}).$$

It is surprising that our new point set contains one or more points than the original set, but can be triangulated with a shorter total edge length! So, given a point

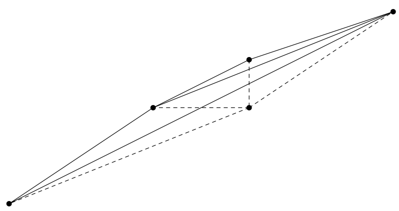

FIGURE 1. A fifth point added to this quadrilateral reduces the length of the minimum weight triangulation of the new larger point set!

set, in what regions of the plane can one add a Steiner reducing point to reduce the length of the minimum weight triangulation? That is, what do these Steiner reducing regions look like? The results are somewhat unexpected. In Figure **??** we see that a set with as few as five points can have a Steiner reducing region with two disconnected components, and as the number of points increases, so does the complexity of the topology. Here are our main results:

Theorem 1. *There exists an 18-point set that admits a connected Steiner reducing region whose first homology group has rank at least 13.*

Theorem 2. *There exists a 15-point set that admits a Steiner reducing region with 20 disconnected components.*

We also demonstrate that the existence of Steiner reducing regions is relatively common in random point sets. This is intriguing, since it is easy to artificially

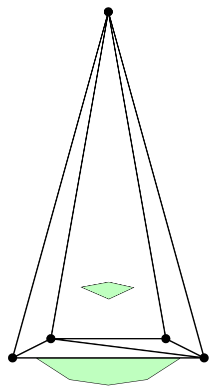

FIGURE 2. This 5-point set has a Steiner reducing region with 2 components.

create many disconnected components in the Steiner reducing region by scattering small examples of point sets with Steiner reducing regions at large distance from one another. The performance of random point sets seems to indicate that multiple Steiner reducing regions can live peacefully in close proximity to one another without forcing the points of the set to be clustered far apart from one another. The random point sets tested also indicate that it is much more likely for a Steiner reduction to occur exterior to the convex hull of our input set.

Let us take a closer look at the example in Figure 4. This point set admits an exterior Steiner reducing region, and this region intersects the long edge of the convex hull. So in a search for the existence of exterior Steiner reducing regions, it will suffice to check if a Steiner point added to any of the convex hull edges will cause a reduction. Consider the Steiner reducing region for the 4-point set in Figure 1:

Notice that the Steiner reducing region in Figure 4 is neither open nor closed - it includes part of the long quadrilateral edge and the rest of the region is bounded by a curve of the form

$$\varphi = \left\{ Z \middle| \sum_{i=1}^{4} dist(Z, W_i) = L \right\},$$

where the right-hand side of the curve-defining equation represents the lengths of the edges being replaced and the left-hand side represents the new edges used in the triangulation. This curve is a cousin to the circle and the ellipse, for it represents the locus of all points whose summed distance to four fixed points is constant. We note that sets of this type and their properties are described as "n-ellipses" by Sekino in [21], where it is shown that the regions are always convex. The curve φ is the boundary of a 4-ellipse. We emphasize here the fundamental connection between Steiner reducing regions and n-ellipses. For a general input set $\mathcal{X}$, a reduction occurs when new replaces old: the new set of edges connects our Steiner reducing point Z to a subset of input points $\mathcal{F} \subseteq \mathcal{X}$, and the old set $\mathbb{E}$ of replaced edges formed a minimal triangulation of the possibly non-convex polygon formed by the points of $\mathcal{F}$. Let $L = \sum_{e \in \mathbb{E}} length(e)$. The subset of the Steiner reducing region corresponding to the combinatorial type implied by $\mathcal{F}$ will be itself a subset of

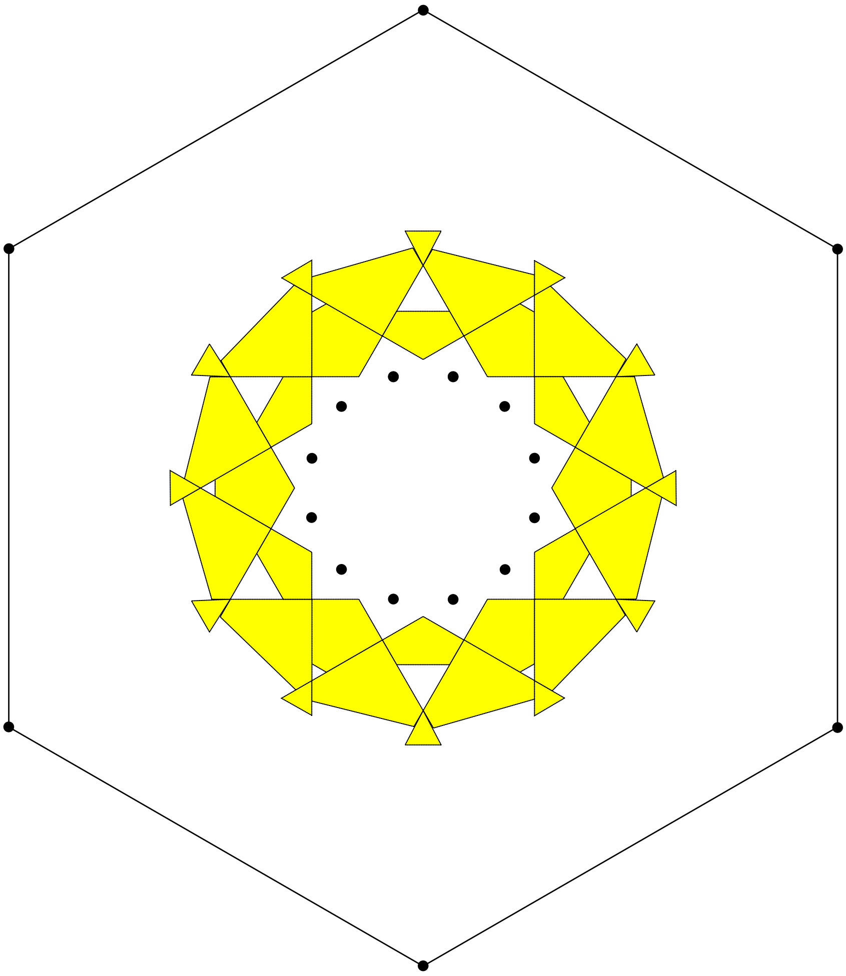

FIGURE 3. The Steiner reducing region with rank 13 first homology.

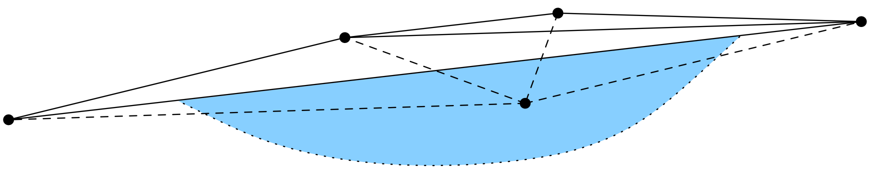

FIGURE 4. An approximation of the Steiner reducing region. [Note to self: this was freehand - need better approx for final draft.]

the k-ellipse $\mathcal{M} = \{(x,y) | \sum_{f\in\mathcal{F}} \text{dist}\big((x,y),f\big) < L\}$, where $k = |\mathcal{F}|$. Notice in particular that if $\mathcal{F}$ has one element, then M will be a circle, and for a set $\mathcal{F}$ with two elements, M will be an ellipse. For values of $n > 2$, Sekino showed that these n-ellipses remain convex, though they may be asymmetric.

2. First homology of Steiner reducing regions

We consider now a point set $\mathcal{P}$ consisting of a regular hexagon G_6 containing a smaller regular 12-gon G_{12}, where specifically,

$$G_6 = \left\{ \left(83 \cos \frac{2\pi(2k-1)}{12}, 83 \sin \frac{2\pi(2k-1)}{12} \right) \middle| j = 1..6 \right\}, \text{ and}$$

$$G_{12} = \left\{ \left(20 \cos \frac{2\pi(2k-1)}{24}, 20 \sin \frac{2\pi(2k-1)}{24} \right) \middle| k = 1..12 \right\}.$$

We label the points of G_6 by $A, \ldots, F$, for values of $j = 1..6$. We similarly label the points of G_{12} by $G, \ldots, R$, for values of $k = 1..12$. Notice that our point set is preserved under the standard group action of D_6, the dihedral group of order 12. We will once again utilize the symmetries of our point set to reduce the number of cases we much consider.

We now establish, for our particular point set, a subset of the minimum weight triangulation that will simplify our task of finding the overall minimal triangulation of $\mathcal{P}$.

Claim 3. *The minimum weight triangulation of $\mathcal{P}$ includes a minimum weight triangulation of the 12-gon formed by the points of G_{12}.*

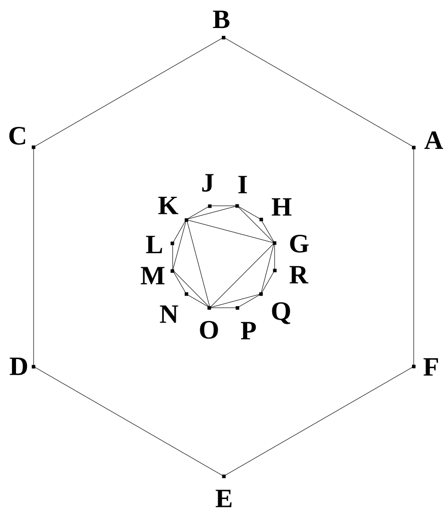

Figure 5. $\mathcal{P}$ with minimally triangulated 12-gon.

Proof: We note that if all edges of the convex hull of G_{12} are present in the minimum weight triangulation, then our claim must hold, for the interior of the 12-gon will be triangulated minimally. Assume that some edge of the 12-gon is not present. There are two types of edges in the convex hull of the 12-gon: those symmetric to GH (edges IJ, KL, MN, OP, and QR) and those symmetric to HI (edges JK, LM, NO, PQ, and RG).

Assume towards a contradiction that edge GH is not in the minimum weight triangulation. Then there must be some edge that passes between G and H. There are three such possible edges, up to symmetry: AM, BR and AD. Assume AM is in the minimum weight triangulation. Then it must belong to two triangles. Visibility constraints then require that $\triangle AHM$ will then be in the minimum weight triangulation, and also one of $\triangle AGM$, $\triangle AMN$. Now, if $\triangle AGM$ is in the triangulation as shown in Figure 6, then $AGHM$ will use diagonal AM instead of the shorter GH, a contradiction. Likewise, the use of $\triangle AMN$ forces $AHMN$ to

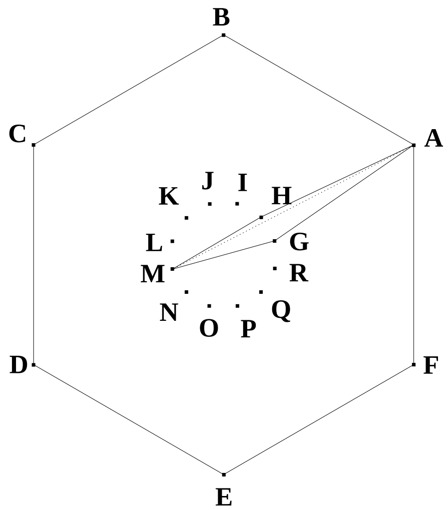

FIGURE 6. Edge AM does not belong to the minimum weight triangulation of $\mathcal{P}$.

use diagonal AM instead of the shorter HN. Thus AM does not belong to the minimum weight triangulation. Now assume that BR is in the minimum weight triangulation. Then $\triangle BGR$ is forced to belong to the triangulation, as is $\triangle BHR$. This means that $BGRH$ uses BR instead of the shorter GH, a contradiction. (See Figure 7.)

Lastly, assume that AD is in the minimum weight triangulation. This forces triangles which in turn give two possible quadrilaterals (up to symmetry) which would be triangulated by AD in the minimum weight triangulation: $AHDG$ and $AHDN$. Note that AD is longer than GH and HN, the other diagonals of those 4-gons. This implies that AD does not belong to any minimal triangulation.

It follows that edge GH must belong to the minimum weight triangulation of $\mathcal{P}$, and by symmetry, so must edges IJ, KL, MN, OP, and QR.

Now we assume, also towards a contradiction, that edge HI is not in the minimum weight triangulation. Then there must be a segment that passes between H and I. The only two possible such edges are AL and

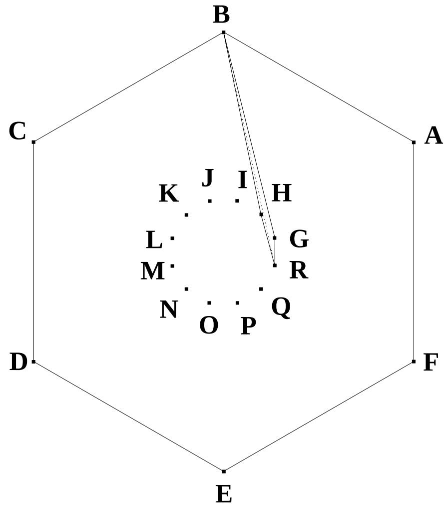

FIGURE 7. Edge BR does not belong to the minimum weight triangulation of $\mathcal{P}$.

BQ, which are symmetric to one another. Assume then, that AL is in the minimum weight triangulation. This forces the inclusion of $\triangle AIL$ in the triangulation, as well as forcing $\triangle AHL$. Then $AILH$ uses AL and not the shorter HI. It follows that edges HI, JK, LM, NO, PQ, and RG are in the minimum weight triangulation of $\mathcal{P}$. We have established that the edges in the convex hull of G_{12} are also edges of the minimum weight triangulation of $\mathcal{P}$. ∎

We now note that the following sets of segments are orbits under the action of D_6, and therefore define equivalence classes based on length.

$$\Gamma := \{AG, AH, BI, BJ, CK, CL, DM, DN, EO, EP, FQ, FR\}$$

$$\Phi := \{AR, AI, BH, BK, CJ, CM, DL, DO, EN, EQ, FP, FG\}$$

$$\Psi := \{AQ, AJ, BG, BL, CI, CN, DK, DP, EM, ER, FO, FH\}$$

All segments in Γ have length

$$\sqrt{20^2 + 83^2 - 2 \cdot 20 \cdot 83 \cos\left(\frac{2\pi}{12} - \frac{2\pi}{24}\right)} = \sqrt{7289 - 3320 \cos\left(\frac{\pi}{12}\right)} \approx 63.8915,$$

segments in Φ have length

$$\sqrt{20^2 + 83^2 - 2 \cdot 20 \cdot 83 \cos\left(\frac{2\pi \cdot 5}{24} - \frac{2\pi}{12}\right)} = \sqrt{7289 - 3320 \cos\left(\frac{\pi}{4}\right)} \approx 70.2951,$$

and segments in Ψ have length

$$\sqrt{20^2 + 83^2 - 2 \cdot 20 \cdot 83 \cos\left(\frac{2\pi \cdot 7}{24} - \frac{2\pi}{12}\right)} = \sqrt{7289 - 3320 \cos\left(\frac{5\pi}{12}\right)} \approx 80.1855.$$

Claim 4. *A minimal triangulation of $\mathcal{P}$ includes all edges in the set Γ and one edge each from the following six pairs of edges:* (AI, BH), (BK, CJ), (CM, DL), (DO, EN), (EQ, FP), (FG, AR).

Proof: Other potential edges in a triangulation of $\mathcal{P}$ are: AC (or one of the symmetric edges BD, CE, DF, AE, BF) and AQ (or one of the symmetric edges from set Ψ). If we can show that none of these two equivalence classes of edges are used, then our above claim about the structure of the minimal triangulation will be true. Our proofs will continue to be structured to look for contradictions of the form of a quadrilateral which uses the long diagonal instead of the short diagonal.

Assume that edge AC is in a minimum weight triangulation of point set $\mathcal{P}$. Then AC forms a triangle also with one of I, J. WLOG, assume $\triangle ACI$ is in this triangulation of $\mathcal{P}$. (Note that $\triangle ACI$ is symmetric to $\triangle ACJ$.) Then $ABCI$ is triangulated with AC instead of the shorter diagonal BI, a contradiction. It follows that neither AC nor any edges symmetric to AC belong to the minimum weight triangulation of $\mathcal{P}$.

Similarly, assume AQ is in a minimum weight triangulation of $\mathcal{P}$. This edge must belong to two triangles. The only two possible such triangles are $\triangle AQR$ and $\triangle AFQ$. (Note the use of $\triangle AEQ$ would imply the use of edge AE, which is symmetric to AC and therefore not in any minimum weight triangulation by the above argument.) This means $AFQR$ uses diagonal AQ and not the shorter FR. It follows that neither AQ nor any edges symmetric to AQ belong to the minimum weight triangulation of $\mathcal{P}$.

We have therefore established that one minimum weight triangulation of $\mathcal{P}$ uses the following edge set between the convex hulls of G_6 and G_{12}:

$$\Omega := \Gamma \cup \{AI, BK, CM, DO, EQ, FG\}. \quad \blacksquare$$

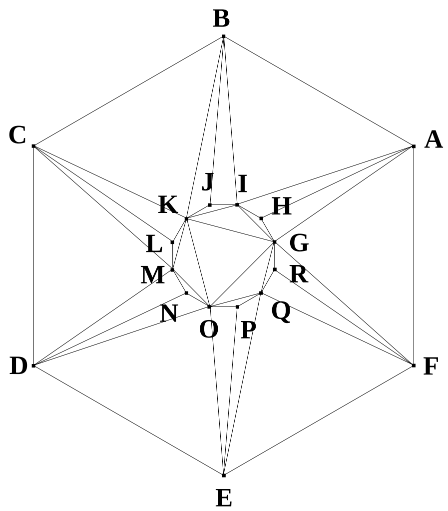

FIGURE 8. $\mathcal{P}$ triangulated minimally with the edges in Ω.

We now seek to establish several convex regions, the union of which will be a connected planar region that is not simply connected. There are five regions, up to symmetry, which we must consider. These regions are bounded by lines extended from the edges of the interior 12-gon. The chambers of this line arrangement define regions of visibility for our new point Z that is to be added. We have established that the 12-gon which is the convex hull of G_{12} is included in all minimum weight triangulations of $\mathcal{P}$.

(CONJ: A similar proof will show that the convex hull of G_{12} will belong to the minimum weight Steiner triangulation of $\mathcal{P}$, provided the Steiner point we add is not on an edge of the 12-gon.)

Since edge-crossing is disallowed by the definition of a triangulation, our Steiner point Z can only be connected to points that do not require those segments to intersect $\text{conv}(G_{12})$. We say that a point of G_{12} that does not require the ray to Z to intersect the 12-gon is said to be *visible* to Z.

When we refer to triangles in our triangulation, we mean triangles that contain no point from our original set. We will sometimes speak of "visibility constraints" or claim that certain results are forced "by visibility." This should be taken to mean that all other choices of triangles would either contain points from our set or would intersect some edge which must belong to the triangulation. As shown in Figure 9 below, if AC belongs to our triangulation, the we say that visibility constraints imply that either $\triangle ACI$ or $\triangle ACJ$ must belong to our triangulation. Moreover, those two cases are the same, up to symmetry: reflecting along the line $\overline{BE}$ will fix AC and map $\triangle ACI$ to $\triangle ACJ$.

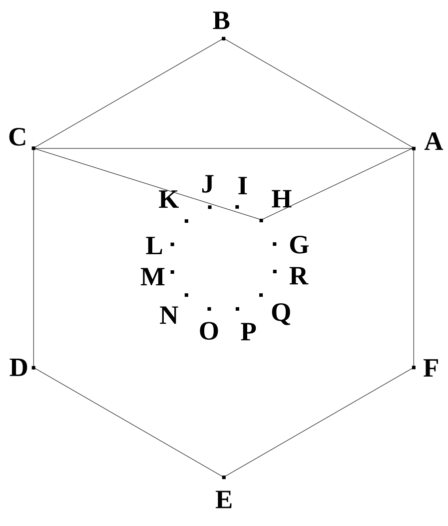

FIGURE 9. Triangle $\triangle ACH$ is disallowed by visibility, since point I is in its interior.

We define region 1 to be the bounded chamber formed by lines HI, GH, KL, and JK. Let

$$\begin{aligned} a &= HI \cap JK \approx (0, 22.30710), \\ b &= GH \cap JK \approx (7.07107, 26.38958), \\ c &= GH \cap (y = 30.675) \approx (4.59688, 30.675), \\ d &= (y = 30.675) \cap JK \approx (-4.59688, 30.675), \text{ and} \\ e &= HI \cap K \approx (-7.07107, 26.38958). \end{aligned}$$

We claim that the interior of the convex hull of $\{a, b, c, d, e\}$ is a reducing region when a new point Z is connected to points in $\mathcal{A} := \{A, B, C, H, I, J, K\}$. The edges $ZA, ZB, ZC, ZH, ZI, ZJ, ZK$ will replace edges AI, BI, BJ, BK from the original triangulation, which have a summed length of 268.374. Let $d_{\mathcal{A}}(Z)$ be the sum over points $P \in \mathcal{A}$ of the distance from P to Z. Then we have $d_{\mathcal{A}}(a) = 254.103$, $d_{\mathcal{A}}(b) = 264.081$, $d_{\mathcal{A}}(c) = 268.349$, $d_{\mathcal{A}}(d) = 268.349$, and $d_{\mathcal{A}}(e) = 264.081$. Since all five of the above values are less than 268.374, any point added within the convex hull of $\{a, b, c, d, e\}$ will indeed reduce the length of the minimum weight triangulation.

We define region 2 to be the bounded chamber formed by lines GH, IJ, GR, and JK. Let

$$\begin{aligned} f &= JK \cap (y = -0.58307x + 41.77457) \approx (16.77621, 31.99285), \\ g &= GR \cap (y = -0.58307x + 41.77457) \approx (19.31852, 30.51051), \\ h &= GH \cap JK \approx (7.07107, 26.38958), \\ i &= GH \cap IJ \approx (11.15355, 19.31852), \text{ and} \\ j &= GR \cap IJ \approx (19.31852, 19.31852). \end{aligned}$$

We claim that the convex hull of $\{f, g, h, i, j\}$ is a reducing region when a new point Z is connected to points in $\mathcal{B} := \{A, B, G, H, I, J\}$. The edges ZA, ZB, ZG, ZH, ZI, ZJ will replace edges AH, AI, BI from the original triangulation, which have a summed length of 198.079. Let $d_{\mathcal{B}}(Z)$ be the sum over points $P \in \mathcal{B}$ of the distance from P to Z.

Then we have

$$\begin{aligned} d_{\mathcal{B}}(f) &= 197.124, \\ d_{\mathcal{B}}(g) &= 197.097, \\ d_{\mathcal{B}}(h) &= 183.697, \\ d_{\mathcal{B}}(i) &= 173.916, \text{ and} \\ d_{\mathcal{B}}(j) &= 183.697. \end{aligned}$$

Since all five of the above values are less than 198.079, any point added within the convex hull of $\{f, g, h, i, j\}$ will indeed reduce the length of the minimum weight triangulation.

We define region 3 to be the bounded chamber formed by lines GH, KL, GR, and JK. Let

$$\begin{aligned} k &= KL \cap (y = -0.24958x + 41.81550) \approx (1.60397, 41.41519), \\ l &= GR \cap (y = -0.24958x + 41.81550) \approx (19.31852, 36.99399), \\ m &= GH \cap JK \approx (7.07107, 26.38958), \\ n &= GH \cap KL \approx (0.00000, 38.63703), \text{ and} \\ o &= GR \cap JK \approx (19.31852, 33.46065). \end{aligned}$$

We claim that the convex hull of $\{k, l, m, n, o\}$ is a reducing region when a new point Z is connected to points in $\mathcal{C} := \{A, B, G, H, I, J, K\}$. That set of edges will replace the following edges from the original triangulation: AH, AI, BI, BJ. The summed length of those four edges is 261.971. Let $d_{\mathcal{C}}(Z)$ be the sum over points $P \in \mathcal{C}$ of the distance from P to Z.

Then we have

$$\begin{aligned} d_{\mathcal{C}}(k) &= 259.236, \\ d_{\mathcal{C}}(l) &= 251.262, \\ d_{\mathcal{C}}(m) &= 208.192, \\ d_{\mathcal{C}}(n) &= 251.505, \text{ and} \\ d_{\mathcal{C}}(o) &= 241.551. \end{aligned}$$

Since all five of the above values are less than 261.971, any point added within the convex hull of $\{k, l, m, n, o\}$ will indeed reduce the length of the minimum weight triangulation.

We define region 4 to be the bounded chamber formed by lines KL, GH and convex hull edges AB, BC. Let

$$\begin{aligned} p &= GH \cap (y = 44.6) \approx (-3.12136, 44.6), \\ q &= KL \cap (y = 44.6) \approx (3.12136, 44.6), \text{ and} \\ r &= GH \cap KL \approx (0.00000, 38.63703). \end{aligned}$$

We claim that the convex hull of $\{p, q, r\}$ is a reducing region when a new point Z is connected to points in $\mathcal{D} := \{A, B, C, G, H, I, J, K, L\}$. That set of edges will replace the following edges from the original triangulation: AH, AI, BI, BJ, BK, CK. The summed length of those six edges is 396.158. Let $d_{\mathcal{D}}(Z)$ be the sum over points $P \in \mathcal{D}$ of the distance from P to Z.

Then we have

$$\begin{aligned} d_{\mathcal{D}}(p) &= 389.779, \\ d_{\mathcal{D}}(q) &= 389.779, \text{ and} \\ d_{\mathcal{D}}(r) &= 362.079. \end{aligned}$$

Since all three of the above values are less than 396.158, any point added within the convex hull of $\{p, q, r\}$ will indeed reduce the length of the minimum weight triangulation.

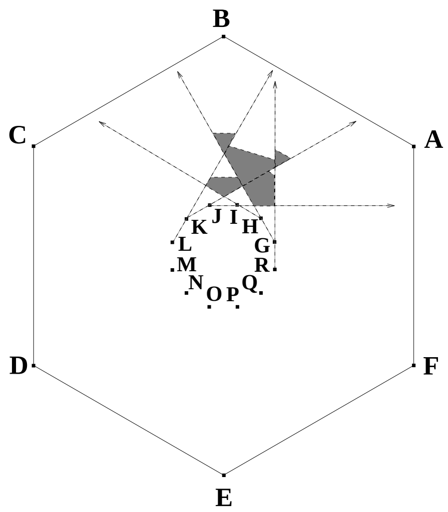

FIGURE 10. Regions 1 through 5 and their locations within $\mathcal{P}$.

We define region 5 to be the bounded chamber formed by lines JK, GR and convex hull edge AB. Let

$$\begin{aligned} s &= GR \cap (y = -0.56463x + 50.38075) \approx (19.31852, 39.47297), \\ t &= JK \cap (y = -0.56463x + 50.38075) \approx (24.58335, 36.50030), \text{ and} \\ u &= JK \cap GR \approx (19.31852, 33.46065). \end{aligned}$$

We claim that the convex hull of $\{s, t, u\}$ is a reducing region when a new point Z is connected to points in $\mathcal{E} := \{A, B, G, H, I, J, K, R\}$. That set of edges will replace the following edges from the original triangulation: AG, AH, AI, BI, BJ. The summed length of those five edges is 325.863. Let $d_{\mathcal{E}}(Z)$ be the sum over points $P \in \mathcal{E}$ of the distance from P to Z.

Then we have

$$\begin{aligned} d_{\mathcal{E}}(s) &= 303.332, \\ d_{\mathcal{E}}(t) &= 303.605, \text{ and} \\ d_{\mathcal{E}}(u) &= 280.188. \end{aligned}$$

Since all three of the above values are less than 325.863, any point added within the convex hull of $\{s, t, u\}$ will indeed reduce the length of the minimum weight triangulation.

We have now established a reducing region that is connected but not simply connected. We now proceed to prove the existence of 13 holes within this reducing region. We will do so by finding points in the interior of the holes that do not reduce, combined with polygonal reducing paths around the holes.

Claim 5. *The point* $X = (0.00000, 35.08709)$ *will not reduce.*

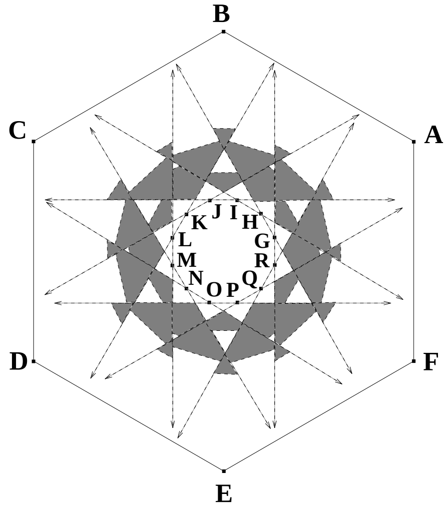

FIGURE 11. The Steiner reducing region of $\mathcal{P}$.

Proof: We first must establish the minimum weight triangulation of $\mathcal{P} \cup \{X\}$, and then we will calculate the length of that triangulation. We claim that the minimum weight triangulation connects X to points A, B, C, H, I, J, K.

Note that the use of edge AC would imply that $\triangle ABC$ and $\triangle ACX$ are both in the minimum weight triangulation, with the latter triangle forced by visibility. Edge BX is shorter than edge AC, a contradiction to minimality. Thus edge AC will not be used in this minimum weight triangulation.

We claim that edge IJ must be in the minimum weight triangulation. Otherwise, an edge from X must cross it, and there is one type of such edge up to symmetry, edge PX. The inclusion of this edge forces triangle $\triangle PIX$ to be in the triangulation, as well as one of $\triangle PJX, \triangle POX$. In the case where $\triangle PJX$ is used, we have $PJXI$ using PX instead of the shorter IJ. In the case where $\triangle POX$ is used, we have $POXI$ using PX instead of the shorter IO. Thus it follows that edge IJ must be included in the new minimum weight triangulation.

The edge IJ can connect to two possible points, up to symmetry: X and A. If IJ connects to A, then AJ is forced by visibility to connect to X. This implies that the shorter edge XI should have been used instead of AJ. Thus the triangle $\triangle IJX$ is in the minimum weight triangulation.

Edge XI can connect to A, B, or H. If we connect it to A, then we have XAI in the minimum weight triangulation, and edge XA must connect to B. (It cannot connect to C by an earlier comment above.) If XA connects to B, then the shorter edge BI should have been used instead of XA. Thus XAI is not in the minimum weight triangulation. If we connect B to XI, then BI must connect to A or H. Connecting BI to A

implies the use of $\triangle AHI$, which puts us in an interesting position. Now, trapezoid $ABIH$ can be triangulated with either BH or the equal-length AI. If we flip edge AI to BH, then we are back in the above situation of using $\triangle BHI$, which gave us a contradiction. Thus we cannot connect XI to B, so we must attach it to H and include $\triangle HIX$ in the minimum weight triangulation.

Edge HX can connect to B or to A. If it connects to A, then edge XA must connect to B, but we note that $AX > BH$, so we should have used BH instead of AX. Thus triangle $\triangle AHX$ does not belong to the minimum weight triangulation, but $\triangle BHX$ will be in the minimum weight triangulation. Moreover, edge BH belonged to an original minimum weight triangulation.

Edge BX can connect to C, J, or K. If we connect to C and form triangle $\triangle BXC$, then edge XC can connect to J, K, or D. If XC connects to J, then we should have used the shorter BJ instead of XC. If XC connects to K, then we should have used the shorter BK instead of XC. If we connect XC to D, this forces $\triangle XDK$, which implies we should have used the shorter CK as opposed to XD. So we should not use triangle $\triangle BCX$. If we connect BX to J, then we find ourselves considering connecting edge BJ to one of points C or K, which is a case symmetric to our consideration of connecting edge BI to A or H. Recall from arguments above that both of those choices led to contradictions. Thus we are forced to include triangle $\triangle BXK$ in our minimum weight triangulation. Note this also implies that triangle $\triangle JKX$ is in our triangulation.

Now we notice that edges BK and BH are both included in a minimum weight triangulation of our original point set. Therefore our previous work tells us how to triangulate the rest of the point set. We may now consider the length of this new triangulation. We compare the length of the new edges within the non-convex pentagon $BHIJK$ to the length of the edges that originally triangulated $BHIJK$. The new edges are XB, XH, XI, XJ, and XK, and these have a summed length of 154.2164. They replace edges BI and BJ, which have a summed length of 127.78. Therefore the addition of point X does *not* reduce the length of the minimum weight triangulation, as desired. ■

We now note that there will actually be a small neighborhood around point X in which no point will reduce.

Lemma 6. *If a point $p = (x, y)$ in the interior of a visibility region does not reduce the length of the minimum weight triangulation, then there will be a small open neighborhood around that point in which no point will reduce the length of the minimum weight triangulation.*

Proof: Since p is in the interior of the visibility region, there must be a ball $B(p, \delta)$ of radius δ around p, such that all points inside of $B(p, \delta)$ can be connected to the same set of points to which p may be legally connected. An arbitrary point q within $B(p, \delta)$ may or may not give rise to the same combinatorial type of minimum weight triangulation as the addition of p would imply. We know that distance is a continuous function, as is the sum of multiple distance functions. It follows that the length of the minimum weight triangulation cannot change too drastically within $B(p, \delta)$. Specifically, there must exist an $\epsilon \leq \delta$ such that no point within $B(p, \epsilon)$ will reduce the length of the triangulation. ■

The following corollary follows directly from the above lemma, and the fact that X is contained entirely inside a visibility region.

Corollary 7. *There is a non-reducing neighborhood around point X.*

We now work to establish a reducing polygonal path around this hole. We rely on lemma (CONVEXITY LEMMA) to build this path. If we can find two points which reduce, then the segment between them will also reduce.

Claim 8. *The boundary of the triangle formed by points $\alpha = (-6.1021, 28.79429), \beta = (0, 40.61712)$, and $\gamma = (6.1021, 28.79429)$ will reduce.*

Proof: We must show that the points on segments $\alpha\beta, \alpha\gamma$, and $\beta\gamma$ all reduce. We note that segment $\alpha\gamma$ is symmetric to segment $\beta\gamma$, so we only have to work to show that two segments reduce.

For a point on the segment $\alpha\gamma$, we claim that connecting that point to the points of $\mathcal{F} = \{B, C, H, I, J, K, L\}$ will give a reduction in the length of the triangulation. Let $d_{\mathcal{F}}(Z)$ be the sum over points $P \in \mathcal{F}$ of the distance from P to Z. We have $d_{\mathcal{F}}(\alpha) = 214.5609$ and $d_{\mathcal{F}}(\gamma) = 258.501$. We note that connecting our new point (α or γ) to the points of $\mathcal{F}$ replaces the edges CK, BK, BJ, BI and forces edge AI to flip to BH,an edge of equal length. We are replacing edges from our original triangulation that have summed length $(3 \cdot 63.8915) + 70.2951 = 261.9696$. Therefore both α and γ reduce with this combinatorial type of triangulation, and so must all points on the edge $\alpha\gamma$ between them. By symmetry, all points on the edge $\beta\gamma$ will also reduce.

For a point on the segment $\alpha\beta$, we claim that connecting to the points of $\mathcal{A} = \{A, B, C, H, I, J, K\}$ will give a reduction in the length of the triangulation. This will replace edges AI, BI, BJ, BK from the original triangulation, which together have summed length $(2 \cdot 63.8915) + (2 \cdot 70.2951) = 268.3732$. Once again, we let $d_{\mathcal{A}}(Z)$ be the sum over points $P \in \mathcal{A}$ of the distance from P to Z. We see that $d_{\mathcal{A}}(\alpha) = 266.5075$, and by symmetry, $d_{\mathcal{A}}(\beta) = 266.5075$. Note that this is because

$$\begin{aligned}
\text{dist}(A, \alpha) &= \text{dist}(C, \beta), \\
\text{dist}(B, \alpha) &= \text{dist}(B, \beta), \\
\text{dist}(C, \alpha) &= \text{dist}(A, \beta), \\
\text{dist}(H, \alpha) &= \text{dist}(K, \beta), \\
\text{dist}(I, \alpha) &= \text{dist}(J, \beta), \\
\text{dist}(J, \alpha) &= \text{dist}(I, \beta), \text{ and} \\
\text{dist}(K, \alpha) &= \text{dist}(H, \beta).
\end{aligned}$$

It follows that both α and β reduce with this combinatorial type of triangulation, and so must all points on the edge $\alpha\beta$ between them.

Thus the boundary of triangle $\triangle\alpha\beta\gamma$ will reduce as desired. ■

Claim 9. *The point $Y = (18.47521, 32.00000)$ will not reduce.*

Proof: We first must establish the minimum weight triangulation of $\mathcal{P} \cup \{Y\}$, and then we will compare the length of that triangulation to the length of our original triangulation.

We must connect segment AB to a third point, one of C, F, or Y. If we connect to C, then we are subsequently forced to use triangle $\triangle ACY$. This makes the triangulation use edge AC instead of the shorter BY, a contradiction. If we connect to F we get the same type of problem: we are forced to use BF instead of the shorter AY. Thus our minimum weight triangulation must use triangle $\triangle ABY$.

Edge AY can connect to F, G, H, I, or J. If triangle $\triangle AFY$ is included in the triangulation, then edge FY must connect to G or H. In either situation, diagonal YF is longer than each of AG, AH, so $\triangle AFY$ is not included in the triangulation. If we connect AY to H, then AH must connect to either F or G. If we connect to F and $\triangle AFH$ is included in the triangulation, then $\triangle FGH$ will also be included. That will imply that edge FH is used instead of the shorter AG, a contradiction. If instead we connect AH to G, we will get a contradiction from using AH instead of the shorter GY. Thus we cannot connect AY to H in the minimum weight triangulation. We now try to connect AY to I. This will force $\triangle AHI$ to be included in the minimum weight triangulation, which gives a contradiction from using AI instead of the shorter HY. If we connect AY to J, then we force $\triangle AIJ$ to be included in the minimum weight triangulation , which gives a contradiction for using AJ instead of the shorter YI. It follows that we must connect AY to G and use $\triangle AGY$ in our triangulation.

Edge GY is only visible to B and H. If we connect GY to B, this will force $\triangle BGH$ to belong to our triangulation, which in turn forces the longer BG to be used instead of the shorter HY. Therefore we connect GY to H and include $\triangle GHY$ in our triangulation.

We continue around the interior 12-gon and consider edge HI. This edge is visible to points B and Y. If we connect it to B, then we have a contradiction for using BH instead of IY. Thus we must connect it to Y and include $\triangle HIY$ in our minimum weight triangulation.

Now consider edge IJ, which is visible to B, C, and Y. If we connect IJ to B, we get a contradiction for using BI instead of the shorter edge JY. If we connect IJ to C, then we must connect IC to either B or Y. Both of those situations contradict minimality by using IC instead of the shorter respective diagonal. It follows that we must connect IJ to Y, and use $\triangle YIJ$ in our minimum weight triangulation.

Finally, we aim to find the second triangle to which edge YJ belongs. The only two points visible to this edge are B and C. If we connect to C, we will have a contradiction for using CY instead of the shorter diagonal BJ. Thus we include $\triangle BJY$ in our minimum weight triangulation.

Of all the edges we have currently established to be in the minimum weight triangulation, we notice in particular edges BJ and AG. These two edges belonged to our original minimum weight triangulation, so the remaining region to be triangulated is now bounded by edges which were present in our original triangulation. We know how to triangulate that region. Let us now compare the length of the new edges we use to the length of the edges they replaced. We have connected the point Y to the points of $\mathcal{B} = \{A, B, G, H, I, J\}$. We note that $d_{\mathcal{B}}(Y) = 198.912$, which is greater than the summed lengths of edges BI, AH, and AI : 198.079. It follows that point Y does not reduce. ■

Claim 10. *The boundary of the triangle formed by points* $\kappa = (20.40390, 26.69670), \lambda = (13.15766, 31.08258)$, *and* $\mu = (20.40390, 35.27778)$ *will reduce.*

Proof: As above, we must show that the points on segments $\kappa\lambda, \kappa\mu$, and $\lambda\mu$ all reduce. It will suffice to show that, pairwise, the endpoints of those segments will reduce with the same combinatorial type.

For a point on the segment $\kappa\lambda$, we claim that connecting that point to the points of $\mathcal{B} = \{A, B, G, H, I, J\}$ will give a reduction in the length of the triangulation. As before, let $d_{\mathcal{B}}(Z)$ be the sum over points $P \in \mathcal{B}$ of the distance from P to Z. Then we have $d_{\mathcal{B}}(\kappa) = 192.795$ and $d_{\mathcal{B}}(\lambda) = 192.570$. We note that connecting our new point (κ or λ) to the points of $\mathcal{B}$ replaces the edges AH, AI, BI. We are thus replacing edges from our original triangulation that have summed length $(2 \cdot 63.8915) + 70.2951 = 198.079$. Therefore both κ and λ reduce with this combinatorial type of triangulation, and so must all points on the edge $\kappa\lambda$ between them.

For a point on the segment $\kappa\mu$, we claim that connecting that point to the points of $\mathcal{G} = \{A, B, G, H, I, J, R\}$ will give a reduction in the length of the triangulation. We let $d_{\mathcal{G}}(Z)$ be the sum over points $P \in \mathcal{G}$ of the distance from P to Z. Then we have $d_{\mathcal{G}}(\kappa) = 224.4615$ and $d_{\mathcal{G}}(\mu) = 248.5943$. We note that connecting our new point (κ or μ) to the points of $\mathcal{G}$ replaces the edges AG, AH, AI, BI. We are thus replacing edges from our original triangulation that have summed length $(3 \cdot 63.8915) + \cdot 70.2951 = 261.971$. Therefore both κ and μ reduce with this combinatorial type of triangulation, and so must all points on the edge $\kappa\mu$ between them.

For a point on the segment $\lambda\mu$, we claim that a reduction in the length of the triangulation can be obtained by connecting to the points of $\mathcal{H} = \{A, B, G, H, I, J, K\}$. We let $d_{\mathcal{H}}(Z)$ be the sum over points $P \in \mathcal{H}$ of the distance from P to Z. Then we have $d_{\mathcal{H}}(\lambda) = 224.924$ and $d_{\mathcal{H}}(\mu) = 248.6243$. We note that connecting our new point (λ or μ) to the points of $\mathcal{H}$ replaces the edges AH, AI, BI, BJ. We are thus replacing edges from our original triangulation that have summed length $(3 \cdot 63.8915) + 70.2951 = 261.971$. Therefore both λ and μ reduce with this combinatorial type of triangulation, and so must all points on the edge $\lambda\mu$ between them.

Thus the boundary of triangle $\triangle\kappa\lambda\mu$ will reduce as desired. ■

The symmetry of our point set now grants us 12 distinct holes. We now aim for the lucky 13^{th} hole in the center of our configuration.

Claim 11. *A point added in the center of the 12-gon will not reduce.*

Proof: Let $Z = (0, 0)$ be the point in the center of the 12-gon. We now make some claims about which edges will not be included in the triangulation.

First, we assume towards a contradiction that edge GL is in the minimum weight triangulation. This implies that one of $\triangle GHL$ or $\triangle GIL$ is included in the minimum triangulation. If $\triangle GHL$ is included, then edge HL

belongs to another triangle - one of $\triangle HIL, \triangle HJL$, or $\triangle HKL$. The combination of $\triangle GHL$ and $\triangle HIL$ gives $GHIL$ triangulated by HL, which is longer than GI. The combination of $\triangle GHL$ and $\triangle HJL$ gives $GHJL$ triangulated by HL, which is longer than GJ. If $\triangle HKL$ is used, then one of $\triangle HIK$ or $\triangle HJK$ is used. These cases are equivalent up to symmetry, so assume $\triangle HIK$ is used. Notice that $HL+HK+IK < GI+IL+IK$ in the triangulation of hexagon $GHIJKL$. It follows that edge GL must not belong to the minimum weight triangulation, nor may any edge symmetric to GL, such as HM, IN, JO, KP, etc.

Now we assume towards a second contradiction that edge GJ is in the minimum weight triangulation. Then one of GI, HJ is also in the minimum weight triangulation, but these cases are symmetric. Without loss of generality, we will say that GI (and therefore $\triangle GIJ$) is in the minimum weight triangulation. Now we look at possible triangles that use edge $GJ : \triangle GJK, \triangle GJL, \triangle GJZ, \triangle GJQ$, and $\triangle GJR$, and we detail the contradictions these triangles create. Quadrilateral $GIJK$ is triangulated by GJ instead of the shorter IK. The combination of $\triangle GIJ$ and $\triangle GJZ$ uses GJ instead of the shorter IZ. Next, we note that $\triangle GJL$ forces $\triangle GLZ$, and then quadrilateral $GJLZ$ uses GL instead of the shorter JZ. Similarly, $\triangle GJQ$ forces $\triangle JQZ$, and then quadrilateral $GJQZ$ uses JQ instead of the shorter GZ. Lastly, if we use $\triangle GJR$, then pentagon $GHIJR$ should be triangulated by HJ and HR instead of the longer pair GI and GJ. It follows that edge GJ is not used in the minimum weight triangulation, nor is any edge symmetric to GJ.

We note that edges GZ, HZ, IZ, etc. have the same length as edges GI, HJ, IK, etc. No shorter edges exist in the interior of the 12-gon. Thus if a triangulation exists which uses only edges of that length, its triangulation length must be minimal. For this example, many such triangulations exist. Two such triangulations are:

$$\begin{aligned} \Lambda : &= \{GI, IK, KM, MO, OQ, GQ, GZ, IZ, KZ, MZ, OZ, QZ\} \text{ and} \\ \Upsilon : &= \{GZ, HZ, IZ, JZ, KZ, LZ, MZ, NZ, OZ, PZ, QZ, RZ\}. \quad \blacksquare \end{aligned}$$

Claim 12. *The boundary of the 12-gon formed by symmetric copies of* $\rho\sigma$, *where* $\rho = (0, 30)$ *and* $\sigma = (14.56088, 25.22019)$ *will reduce.*

Proof: It will suffice to show that edge $\rho\sigma$ reduces, then the reduction of the 12-gon will follow by symmetry. It may be necessary to employ a third point $\tau = (7.28044, 27.61009)$, the midpoint of $\rho\sigma$. The hope is that ρ and τ will both reduce using the connectivity of region 1, and that σ and τ will both reduce using the connectivity of region 2.

Recall that points in region 1 reduced by connecting to $\mathcal{A} = \{A, B, C, H, I, J, K\}$. We have $d_{\mathcal{A}}(\rho) = 264.8235$ and $d_{\mathcal{A}}(\tau) = 266.2503$. The length of edges we replace is 268.374, so the segment $\rho\tau$ reduces.

Now recall that points in region 2 reduced by connecting to $\mathcal{B} := \{A, B, G, H, I, J\}$. We have $d_{\mathcal{B}}(\tau) = 186.0355$. and $d_{\mathcal{B}}(\sigma) = 182.5459$. The length of edges we replace when using this combinatorial type is 198.079, so segment $\tau\sigma$ reduces.

It follows that segment $\rho\sigma$ reduces, and therefore there is a reducing 12-sided closed path around the interior 12 points of our original point set $\mathcal{P}$. ∎

3. Connectivity of Steiner reducing regions

In this chapter we investigate point sets with multiple disconnected Steiner reducing regions. Let the point set $\mathcal{P}$ consist of a regular pentagon G_5 of radius 32 containing a smaller regular 10-gon G_{10} of radius 8. Explicitly, we require that each of the regular n-gons be rotated by an angle of $\frac{\pi}{n}$ from the standard n-gon construction which uses the point $(1,0)$. The coordinates of $\mathcal{P}$ are:

$$G_5 = \left\{ \left(32\cos\frac{2\pi(2j-1)}{10},\ 32\sin\frac{2\pi(2j-1)}{10} \right) \middle| j = 1..5 \right\}$$

$$G_{10} = \left\{ \left(8\cos\frac{2\pi(2k-1)}{20},\ 8\sin\frac{2\pi(2k-1)}{20} \right) \middle| k = 1..10 \right\}$$

We label the points of G_5 by $A,\ldots,E$, for values of $j = 1..5$. We similarly label the points of G_{10} by $F,\ldots,O$, for values of $k = 1..10$. We note that the dihedral group of order 10, D_5, will act on the point set $\mathcal{P}$ and create many symmetries which we shall exploit in the course of our proof. We may claim that certain cases are unique "up to symmetry" - by this we will mean that we are avoiding the consideration of duplicate cases that arise by the action of some element of D_5 which leaves the elements of our hypotheses fixed. We will say that edge ST *is symmetric to* edge UV if both segments are in the same orbit under the action of the dihedral group.

(revise following paragraph to give careful definition of *visibility*)

We will rely heavily on proofs by contradiction when making claims about the structure of a given triangulation. Once we know (or if we assume) that a certain edge is included in the triangulation, then visibility constraints will give a set of possible triangulations that used the specified edge. We will seek to find local contradictions to minimality if possible: for example, pairs of triangles which share an edge that is the long diagonal of the 4-gon formed by their union. If, however, we assume that a certain edge is *not* present, then we know that some edge used in the triangulation must cross that segment. We now establish, for our particular point set, a subset of the minimum weight triangulation that will simplify our task of finding the overall minimal triangulation of $\mathcal{P}$.

Lemma 13. *Any minimum weight triangulation of $\mathcal{P}$ contains a minimum weight triangulation of G_{10}.*

Lemma 14. *Any minimum weight triangulation of $\mathcal{P}$ contains the edges in the set $\{AF, AG, BH, BI, CJ, CK, DL, DM, EN, EO\}$, plus one edge each from the following five pairs of edges: (AH, BG), (BJ, CI), (CL, DK), (DN, EM), and (AO, EF).*

Theorem 15. *The point set $\mathcal{P} = G_5 \cup G_{10}$ described above has 20 disconnected Steiner reducing regions within its convex hull.*

4. Miscellany - INSERT BETTER TITLE HERE

In this section we summarize the rest of our findings.

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