Counting Network Flows *

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1 Introduction.

Consider a network G with n nodes and m arcs, with integer-valued capacity and excess functions $c : arcs(G) \to \mathbf{Z}_{\geq 0}$ and $b : nodes(G) \to \mathbf{Z}$. A *flow* is a function $f : arcs(G) \to \mathbf{Z}_{\geq 0}$ so that, for any node x, the sum of flow values in outgoing arcs minus the sum of values in incoming arcs equals $b(x)$, and $0 \leq f(i,j) \leq c(i,j)$. Denote by $\phi_G(b,c) = \phi_G(b_1,\ldots,b_n,c_1,\ldots,c_m)$ the number of flows. We present an algebraic algorithm whose output allows for instantaneous evaluation of the function $(b,c) \mapsto \phi_G(b,c)$. It produces a piecewise polynomial representation of $\phi_G(b,c)$. Applications range from statistics [6] to representation theory [9].

The set of all flows is a convex polytope, the *flow polytope*, which is defined by the node-arc incidence matrix A_G of the network G, as follows:

$$A_G \cdot x \;=\; b, \quad 0 \leq x \leq c$$

The rows and columns of A_G are indexed by the nodes and arcs respectively: each column has precisely two non-zero entries -1 and $+1$, encoding the incidence relation in G. The vectors b and c contain the excesses and capacities. All vertices of the flow polytope are integral because A_G is totally unimodular. It is convenient to rewrite the above system as follows:

$$\begin{bmatrix} A_G & 0 \\ I & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}, \quad x, y \geq 0.$$

The new enlarged matrix is denoted $\widehat{A}_G$ and called the *extended network matrix.* It is also totally unimodular. We are interested in the number $\phi_G(b,c)$ of lattice points in the flow polytope. Clearly, $\phi_G(b,c)$ is zero unless the right hand side $[b,c]^T$ lies in the *extended network cone* of G, which is the cone $pos(\widehat{A}_G)$ spanned by positive linear combinations of the columns of the extended network matrix

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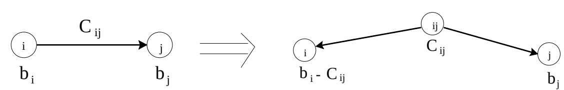
$\widehat{A_G}$ of G. This cone has a natural polyhedral decomposition, called the *chamber complex*, which is defined as the common refinement of all triangulations of $pos(\widehat{A}_G)$ with rays taken from the columns $\widehat{A}_G$. See [1, 4] for details on triangulations and chamber complexes.

The function $\phi_G(b, c)$ is the *vector partition function* for the extended network matrix. Our starting point is the following result that follows from the theory of vector partition functions; see [3] and specifically [14, Remark 2 on page 305].

Theorem 1.1 *The vector partition function $\phi_G(b, c)$ is a piecewise polynomial function of degree $m-n+1$ in the variables $b_1, \ldots, b_n, c_1, \ldots, c_m$. The domains of polynomiality are precisely the maximal cones in the chamber complex of $\widehat{A}_G$. Similarly, if G is acyclic and has no specified capacities, then $\phi_G(b) := \phi_G(b, \infty)$ is the vector partition function for the network matrix A_G and is represented as a piecewise polynomial function on the chamber complex of A_G.*

Any extended network matrix $\widehat{A}_G$ can be transformed by elementary row operations into an ordinary network matrix $A_{\widehat{G}}$ for a larger acyclic network $\widehat{G}$:

Lemma 1.2 *Given a network G with n nodes and m arcs, with capacity and excess functions c, b, there is an acyclic uncapacitated network $\widehat{G}$ with $n+m$ nodes, $2m$ arcs, and excess function $\widehat{b}$ (a linear combination of b,c) such that the flows in both networks are in bijection. The network $\widehat{G}$ is obtained from G by replacing each arc by two new arcs as illustrated in the figure below.*



We shall therefore restrict our discussion to the second case of Theorem 1.1, that is, we assume that G is acyclic and we consider the function $\phi_G(b)$ which counts the number of non-negative integer solutions x of the equation $A_G \cdot x = b$.

In Section 2 we study the case where G is the complete acyclic network K_n. The function ϕ_{K_n} is the *Kostant partition function* which plays an important role in representation theory. We determine the number of chambers up to $n = 7$ and we explicitly compute all polynomials representing $\phi_{K_n}(b)$ up to $n = 6$. Section 3 provides a complexity analysis of the chamber complex. Our main algorithm is presented in Section 4. It is based on methods from algebraic geometry, specifically, we demonstrate how to effectively compute the Todd cohomology class of a toric manifold defined by a unimodular matrix. Details of our implementation are discussed in Section 5.

The state of the art on unimodular counting is the paper of Mount [10]. Mount reports impressive computational results on counting *contingency tables*, the case when G is a complete bipartite graph. We wish to demonstrate that our Gröbner bases algorithm are competitive to the methods (interpolation, divide-and-conquer) proposed by Mount, and are much easier to implement.

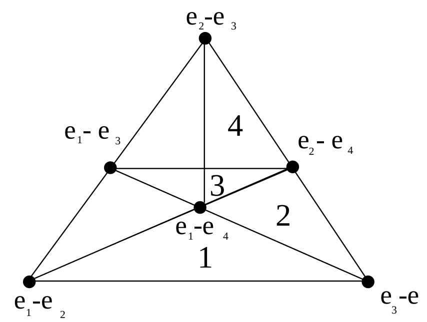


Figure 1: The chamber complex for the tournament K_4

2 Kostant's Partition Function.

Let K_n be the complete network with n nodes and arcs (i, j) for $1 \leq i < j \leq n$. Kirillov [9, question in page 57] posed the problem of determining the number of chambers and of computing the exact polynomials representing Kostant's partition function $\phi_{K_n}(b)$. Combinatorial formulas for specific classes of chambers were found by Postnikov and Stanley (private communication).

We computed all the polynomials representing $\phi_{K_n}(b)$ for $n \leq 6$. There is too much data to display here but the reader can play with these polynomials on-line and obtain specific values of Kostant partitition function. Please visit *www.math.ucdavis.edu/˜ deloera/kostant.html*. As an illustration we present the solution for the $n = 4$. The network cone spanned by the columns of the node-arc incidence matrix of K_n is a three-dimensional triangular cone. The chamber complex is a subdivision of this cone into seven triangular cones. See Figure 1 for a 2-dimensional slice of the chamber complex. The formulas below are given only in terms of b_1, b_2, b_3, in view of $b_4 = -b_1 - b_2 - b_3$. By the symmetry of the example it is enough to give the four polynomials for the indicated chambers in Figure 1 (the number of a chamber in the figure and the polynomial match).

1. If $\min\{b_3, -b_2, b_1 + b_2\} \geq 0$ then $\phi_{K_4}(b) \quad = \quad (b_1 + b_2 + 3)(b_1 + b_2 + 2)(b_1 + b_2 + 1)/6.$
2. If $\min\{b_1, b_2, b_3\} \geq 0$ then $\phi_{K_4}(b) \quad = \quad (b_1 + 1)(b_1 + 2)(b_1 + 3b_2 + 3)/6$
3. If $\min\{b_1, b_2, b_1 + b_3, b_2 + b_3, -b_3\} \geq 0$ then $\phi_{K_4}(b) \quad = \quad 1 + \frac{11}{6}\, b_1 + 2/3\, b_3 + b_2 + 3/2\, b_1\, b_2 + b_1{}^2 + 1/6\, b_1{}^3 + 1/2\, b_1{}^2 b_2 - 1/6\, b_3{}^3 \; -1/2\, b_1\, b_3{}^2 + 1/2\, b_1\, b_3 - 1/2\, b_3{}^2$
4. If $\min\{b_1, b_2 + b_3, -b_1 - b_3\} \geq 0$ then $\phi_{K_4}(b) \quad = \quad (b_1 + 2)(b_1 + 1)(2b_1 + 3b_2 + 3 + 3b_3)$

3 Complexity of Chambers

Let $\Gamma(K_n)$ be the chamber complex of the positive cone spanned by the columns of the node-arc incide matrix of the acycli complete graph. We have the following result:

Theorem 3.1 • *$\Gamma(K_n)$ has chambers with at least $2^{\lfloor n/2 \rfloor}$ facets. The integer coordinates of rays grow exponentially in n.*

- *The asymptotic number of chambers for $\Gamma(K_n)$ is is $2^{O(n^2 logn)}$. The exact number of chambers in $\Gamma(K_n)$ for $n \leq 7$ is given by the following table. There exist virtual chambers starting with K_5 (see [4] for the meaning of virtual chambers).*

n	Number of chambers
3	2
4	7
5	48
6	820
7	44288

The first part of the proof relies on two correspondences: Cuts of the digraph K_n are geometrically hyperplanes spanned by subsets of the column vectors of the node-arc incidence matrix. In addition, the simplices supported on vertices of the vector configuration can be read off from trees in the complete graph.

It is well-known that the problem of enumerating flows for is $\#P$-hard. We were also interested on the complexity of finding a chamber containing an specific righthand side b. We have seen that this is hard in the non-unimodular case:

Proposition 3.2 *Let A be an integral $d \times n$ matrix. Let b be a vector in $pos(A)$ and a list F of d-dimensional simplicial cones with rays in the columns of A such that all elements of F contain the vector b. Deciding whether F includes all simplices that contain b (i.e. whether F determines the chamber that contains b) is NP-hard.*

4 Unimodular Counting Using Gröbner bases.

We describe now an algebraic algorithm for solving the following counting problem associated with any unimodular $n \times d$-matrix A: *Determine the number $\phi_A(b)$ of non-negative integer solutions $u \in \mathbf{N}^n$ of the linear equations $A \cdot u = b$.* We assume that the matrix A has rank d and all $d \times d$-minors are -1, 0 or $+1$. We further assume that $pos(A)$ is a pointed cone. Under these hypotheses, the vector partition function ϕ_A exists and is represented by a polynomial of degree $n - d$ on each maximal cone in the chamber complex of A.

Our running example is the following unimodular 3×5-matrix:

$$A \quad = \quad \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}. \tag{1}$$

Up to row operations, this is the network matrix for the acyclic complete graph K_4 minus the edge $(1,4)$. The chamber complex is a subdivision of the positive orthant in $\mathbf{R}^3$ into five triangular cones.

The vector partition function equals

$$\phi_A(a,b,c) \;=\; \begin{cases} bc+b+c+1 & \text{if } a \geq b+c \text{ and } b,c \geq 0 \\ \frac{1}{2}a^2+\frac{3}{2}a+1 & \text{if } min\{b,c\} \geq a \geq 0, \\ ab-\frac{1}{2}b^2+\frac{1}{2}b+a+1 & \text{if } c \geq a \geq b \geq 0 \\ ac-\frac{1}{2}c^2+\frac{1}{2}c+a+1 & \text{if } b \geq a \geq c \\ ab+ac-\frac{1}{2}(a^2+b^2+c^2)+\frac{1}{2}(a+b+c)+1 & \text{if } b+c \geq a \geq max\{b,c\} \end{cases}$$

For our exposition it is more convenient to express the vector partition function as $\psi_A : \mathbf{N}^n \to \mathbf{N}$ where $\psi_A(v)$ is the number of solutions $u \in \mathbf{N}^n$ to the equation $Au = Av$. Clearly, ψ_A and ϕ_A are related by a simple transformation. For instance, in our example we have $\psi_A(a,b,c,d,e) = \phi_A(a+d+e, b+d, c+e)$.

We first characterize the chamber complex in algebraic terms. Let $S = k[x_1,\ldots,x_n]$ be the polynomial ring over a field k which contains the rational numbers. The indeterminates of S index the columns of the matrix $A = (a_{ij})$. Let J_A denote the ideal in S generated by the binomials $x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_n^{a_{in}} - 1$ for $i = 1,2,\ldots,d$. For any positive weight vector $w \in \mathbf{R}^n$, let $in_w(J_A)$ denote the ideal generated by the w-initial forms of the binomials in J_A. If w is generic, then $in_w(J_A)$ is a monomial ideal. It was shown in [15], Corollary 8.9 that the matrix A is unimodular if and only if all initial monomial ideals $in_w(J_A)$ are square-free. Two weight vectors w and w' in $\mathbf{R}^n$ lie in the same cone of the *Gr"obner fan* if $in_w(J_A) = in_{w'}(J_A)$. By the results in [15, §8] this happens if and only if, for every linearly independent subset $\sigma = \{a_{i_1},\ldots,a_{i_r}\}$ of column vectors of A, the vector Aw lies in the cone spanned by σ if and only if the vector Aw' lies in the cone spanned by σ. This implies the following result:

Proposition 4.1 *The chamber complex of A equals the Gröbner fan of J_A.*

Algebraic algorithms for computing Gröbner fans are described in [15, §3]. The state of the art on this subject is the work of Huber and Thomas [8]. We now explain how to compute the polynomial representing ψ_A on any given chamber. Suppose that w is a positive integer vector in the interior of that chamber. Then $M = in_w(J_A)$ is a square-free monomial ideal. It was shown in [16, Corollary 7.4] that M encodes the face poset of the simple polytope

$$P_w \quad = \quad \{\, u \in \mathbf{R}^n \; : \; u \geq 0 \text{ and } Au = Aw \,\}.$$

For any $(n-d)$-element subset I of $\{1,\ldots,n\}$, the equations $u_i = 0,\ i \in I$ define a facet of P_w if and only if $\langle x_j \,:\, j \notin I \rangle$ is a minimal prime of M. Writing Σ_w for the normal fan of the simple polytope P_w, this can be restated as follows:

Proposition 4.2 *The Stanley-Reisner ideal of the fan Σ_w equals $M = in_w(J_A)$.*

In our running example, with $w = (1,1,1,1,1)$, the polytope P_w is a pentagon and the fan Σ_w has five rays in the plane. This is encoded by the ideal

$$M \;=\; \langle A,B,C\rangle \cap \langle A,B,E\rangle \cap \langle B,D,E\rangle \cap \langle C,D,E\rangle \cap \langle A,C,D\rangle. \tag{3}$$

Returning to the general case, our goal is to count the lattice points in the polytope P_w. We use known methods from toric geometry for this computation. An introduction can be found in Section

5.3 in Fulton's book [11]. For the state of the art, including computational complexity issues consult the survey article [2].

Let X_w denote the projective toric variety defined by the fan Σ_w. The variety X_w is smooth, for all w, since A is unimodular. Let L_A denote the ideal in $S = k[x_1, \ldots, x_n]$ generated by the linear forms $b_1 x_1 + \cdots + b_n x_n$ where $(b_1, \ldots, b_n)$ runs over all vectors in the kernel of the matrix A. The cohomology ring of X_w with coefficients in our field k is the artinian graded k-algebra

$$H^*(X_w; k) \quad = \quad \bigoplus_{r=0}^{n-d} H^{2r}(X_w, k) \quad = \quad S/(M + L_A). \tag{4}$$

Arithmetic operations in this algebra are performed using normal form reduction relative to any Gröbner basis of the ideal $M + L_A$. Since X_w is an irreducible complex manifold of dimension $n-d$, the top cohomology group $H^{2n-2d}(X_w, k)$ is a one-dimensional vector space. There is a canonical choice of a basis vector for that one-dimensional k-vector space, namely the (image of the) any square-free monomial $\prod_{i \in I} x_i$ which indexes a vertex of P_w, that is, $\langle x_j \, : \, j \notin I \rangle$ is a minimal prime of M. Since X_w is smooth, any two such monomials are congruent to each other modulo $M + L_A$. The resulting socle element of $H^*(X_w; k)$ represents the cohomology class which is Poincaré dual to a point on X_w.

The following rule uniquely defines a k-linear functional called the *integral*:

$$H^*(X_w; k) \to k, \; p \mapsto \int_{X_w} p.$$

Writing top(p) for the degree $n - d$ component of p, we require that top$(p) - (\int p) \cdot \prod_{i \in I} x_i$ lies in $M + L_A$, where I is any index set as above.

Algorithm 1. *(Computing the integral of a cohomology class of X_w)*
Input: A polynomial $p(x_1, \ldots, x_n)$ with coefficients in a field $k \supset \mathbf{Q}$
Output: The integral $\int_{X_w} p$ of the corresponding cohomology class on X_w.

1. Compute any Gröbner basis $\mathcal{G}$ for the ideal $M + L_A$.
2. Let m denote the unique standard monomial of $n - d$.
3. Find any minimal prime $\langle x_j \, : \, j \notin I \rangle$ of M, and compute the normal form of $\prod_{i \in I} x_i$ modulo the Gröbner basis $\mathcal{G}$. It looks like $\gamma \cdot m$, where γ is a non-zero element of k.
4. Compute the normal form of p modulo the Gröbner basis $\mathcal{G}$, and let $\delta \in Q$ be the coefficient of m in that normal form.
5. Output the scalar $\delta/\gamma \in k$.

To compute number of lattice points in P_w, we note that there is a special element in the cohomology ring $H^*(X_w; k)$, denoted $td(x_1, \ldots, x_n)$ and called the *Todd class* of the toric variety X_w. The Todd class is represented (non-uniquely) by a (non-homogeneous) polynomial with rational coefficients in the variables $x_1, \ldots, x_n$. The polynomial $td(x_1, \ldots, x_n)$ does exactly what we want:

$$\phi_A(w_1, \ldots, w_n) \; = \; \#(P_w \cap \mathbf{Z}^n) \quad = \quad \int_{X_w} td(x_1, \ldots, x_n) \cdot exp\,(\sum_{i=1}^{n} w_i x_i) \tag{6}$$

Here the *exponential* of a linear form in (4) is defined by the terminating series

$$exp\,(\sum_{i=1}^{n} w_i x_i) \quad = \quad \sum_{r=0}^{n-d} \frac{1}{r\,!} \cdot (w_1 x_1 + w_2 x_2 + \cdots + w_n x_n)^r. \tag{7}$$

Pommersheim [12] gives an algorithm for computing the Todd class, which works efficiently even for non-unimodular A. For our applications, however, we prefer to use the basic formula given in the first line on page 110 in Fulton's book [11]:

$$td(x_1, \ldots, x_n) \quad = \quad \prod_{i=1}^{n} \frac{x_i}{1 - exp(-x_i)} \quad = \quad \prod_{i=1}^{n} (1 + \frac{1}{2} x_i + \frac{1}{12} x_i^2 - \frac{1}{720} x_i^4 + \cdots) \tag{8}$$

In this expansion we list only terms of degree $\leq n - d$, so that (8) becomes a polynomial in $x_1, \ldots, x_n$ with $\mathbf{Q}$-coefficients. We conclude with our main result.

Theorem 4.3 *The following algorithm computes the polynomial which represents ψ_A on a chamber containing a given non-negative vector $w \in \mathbf{R}^n$:*

1. Determine linear equalities defining the given chamber.
2. Let M be the ideal generated by the leading monomials of the Gröbner basis for J_A with respect to w and compute the ideal representing the kernel L_A of A. Use these two ideals to construct the cohomology ring.
3. Apply Algorithm 1 to the product of the polynomials in (7) and (8).

A main advantage of this algorithm over other methods is that we can do the computation parametrically, over the field $k = \mathbf{Q}(w_1, \ldots, w_n)$. Our output is the actually polynomial for ψ_A not just some numerical evaluation of it.

For our running example we take the polynomial ring $S = k[A, B, C, D, E]$ over the field $k = \mathbf{Q}(a, b, c, d, e)$. We fix the reverse lexicographic Gröbner basis for the ideal $M + L_A$, where $L_A = \langle A + B - D, A + C - E\rangle$ and $M = in_w(J_A)$ is the monomial ideal in (3). The Todd class (8) is computed from the formula

$$(1 + A/2 + A/12)(1 + B/2 + B/12)(1 + C/2 + C/12)(1 + D/2 + D/12)(1 + E/2 + E/12)$$

The normal form of this expression with respect to our Gröbner basis equals

$$td(A, B, C, D, E) \quad = \quad DE + C/2 + D + E/2 + 1. \tag{8$'$}$$

Likewise, the exponential of the general divisor (7) on our toric surface,

$$1 + (aA + bB + cC + dD + eE) + \frac{1}{2}(aA + bB + cC + dD + eE),$$

has the following normal form with respect to our Gröbner basis:

$$1 \,+\, (a - b + e)E + (b + c - a)C + (b + d)D \,+\, \big(ab + be + ac + cd + de - (a^2 - b^2 - c^2)/2\big)DE$$

Multiply this expression with (8'), reduce it to normal form, and extract the coefficient of the standard monomial DE. The result is the desired polynomial which represents $\psi_A(a, b, c, d, e)$ on the chamber (2). Now set $d = e = 0$.

5 Implementation in Macaulay 2

We implemented our algorithm in the computer algebra system `Macaulay 2` developed by Grayson and Stillman [7]. Our preliminary computational experience indicates that the Gröbner basis computation of the Todd class will eventually outperform the interpolation techniques proposed by Mount [10]. In our test implementation (still in progress), we did not use parallel computation but a rather straightforward and crude computer algebra code. The `Macaulay 2` program for creating the polynomial for a single chamber is very compact. Below we attach the *entire program* for one chamber in the 4 by 4 contingency table case (complex bipartite network $K_{4,4}$).

For 4x4 contingency tables, Mount reported a 3 hour calculation spent in each chamber using a divide and conquer technique where he broke the original problem into smaller problems. These subproblems were arranged to be solved only once (dynamic programming). The process used a parallel distributed system for the partial subproblems. We ran all our examples in a Pentium-III CPU with 700Mhz and 256 MB ram computer. When recomputing several of Mount's 3694 polynomials, we observed running times ranging from one hour to one day. Computing a specific numerical instance (fixed rows sums and column sums) took 11 seconds on the average. In the case of the acyclic complete network K_6 each polynomial takes only a minute of to be produced. The generation of different chambers was performed using `topcom` and `puntos` [13, 5].

```
A = {{1,0,0,0, 1,0,0,0, 1,0,0,0, 1,0,0,0},
     {0,1,0,0, 0,1,0,0, 0,1,0,0, 0,1,0,0},
     {0,0,1,0, 0,0,1,0, 0,0,1,0, 0,0,1,0},
     {0,0,0,1, 0,0,0,1, 0,0,0,1, 0,0,0,1},
     {1,1,1,1, 0,0,0,0, 0,0,0,0, 0,0,0,0},
     {0,0,0,0, 1,1,1,1, 0,0,0,0, 0,0,0,0},
     {0,0,0,0, 0,0,0,0, 1,1,1,1, 0,0,0,0},
     {0,0,0,0, 0,0,0,0, 0,0,0,0, 1,1,1,1}};

n = rank source matrix A
R = QQ[x_1..x_n, r1,r2,r3,r4,c1,c2,c3,c4];
u = (0,r1-c2-c3-c4,c2,c3,c4,r2,0,0,0,r3,0,0,0,r4,0,0,0);
component = (d,f) -> sum select(terms f, t -> d == sum degree t)
trunc = (d,f) -> sum select(terms f, t ->  sum degree t < d+1)
toBinomial = (b,R) -> (
     top := 1_R; bottom := 1_R;
     scan(#b, i -> if b_i > 0 then top = top * R_i^(b_i)
       else if b_i < 0 then bottom = bottom * R_i^(-b_i));  top - bottom);
nonfaces = ideal leadTerm ideal apply( A, a -> toBinomial(a,R));
toLinform = (b,R) -> (sss := 0_R; scan(#b, i -> sss = sss +  b_i * R_i); sss);
linearforms = ideal apply(entries transpose syz matrix A, a -> toLinform(a,R));
I = linearforms + nonfaces;
```

```
d = n - (codim nonfaces);
d, degree nonfaces
divp = 1;
divpowers = apply(1..d, i -> (divp =  sum toList apply(1..n, i ->
     u_i * (x_i * divp % I))));
todd = (d,x) -> (trunc(d,
(1+1/2*x+1/12*x^2-1/720*x^4+1/30240*x^6-1/1209600*x^8+1/47900160*x^10
      -691/1307674368000*x^12+1/74724249600*x^14)));
toddclass := 1_R;
scan(1..n, i -> toddclass = trunc(d, toddclass * todd(d,x_i)) % I);
erhart = sum toList apply(0..d-1, i ->
   (divpowers_i * (1/(i+1)!) * component(d-i-1,toddclass) % I));
makeone = ideal apply(1..n,i -> x_i - 1);
erhart = (erhart % makeone);
constt = (component(d,toddclass) % I) % makeone;
toString ((1/constt)*erhart + 1)
```

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6 Appendix: The case of A_4

Now we go to the smallest interesting case A_4^+ (others are presented on-line). The f-vector of the chamber complex is $(1, 19, 77, 107, 48)$. There are 48 chambers, only two of these 48 chambers are not simplices, they are bipyramids over a triangle. Remarkably, a virtual chamber appears. This is related to the existence of non-regular triangulations for the Gale diagram of A_4^4+, which is a 10-dimensional vector configuration We present all 48 polynomials and chambers in the web page. Here we only present those polynomials that can be factored over Q.

1. If $\min\{b_1, b_2, b_3, b_4\} \geq 0$ then
$\frac{1}{360}(b_1+3)(b_1+2)(b_1+1)\left(b_1{}^2+5\,b_1\,b_2+9\,b_1+20+10\,b_2{}^2+30\,b_2\right)(b_2+3+b_1+3\,b_3)$
2. If $\min\{b_4, b_1+b_2+b_3, -b_3, -b_2\} \geq 0$ then
$\frac{1}{360}(b_2+5+b_1+b_3)(b_2+4+b_1+b_3)(b_2+3+b_1+b_3)(b_2+2+u1+b_3)$
$(b_2+1+b_1+b_3)(b_2+3+b_1-2\,b_3)$
3. If $\min\{-b_3, -b_2-b_4, -b_1-b_4, b_1+b_2+b_3+b_4\} \geq 0$ then
$\frac{1}{360}(b_4+3+b_1+b_3+b_2)(b_4+2+b_1+b_3+b_2)(b_4+1+b_1+b_3+b_2)$
$(60+56\,b_1+6\,b_2-14\,b_3-54\,b_4+9\,b_3\,b_4\,b_1+6\,b_2\,b_3-3\,b_2\,b_4\,b_1-9\,b_3{}^2b_1+$
$3\,b_1\,b_4{}^2+3\,b_2\,b_4{}^2-6\,b_3\,b_4{}^2+27\,b_1\,b_2-9\,b_1\,b_3-9\,b_1{}^2b_4+6\,b_2{}^2b_4+9\,b_1{}^2b_2-6\,b_1\,b_2{}^2+3\,b_3{}^2b_4+6\,b_3{}^2b_2+$
$24\,b_3\,b_4-45\,b_1\,b_4-6\,b_2\,b_3\,b_4-b_3{}^3+6\,b_1{}^2-9\,b_2{}^2-15\,b_3{}^2-2\,b_1{}^3+3\,b_2{}^3+9\,b_4{}^2)$

4. If $\min\{-b_3, -b_2, -b_4, b_1 + b_2 + b_3 + b_4\} \geq 0$ then
$\frac{1}{360}(b_2 + 3 + b_1 - 2\,b_3)(b_4 + 3 + b_1 + b_3 + b_2)(b_4 + 2 + b_1 + b_3 + b_2)$
$(b_4 + 1 + b_1 + b_3 + b_2)(b_2{}^2 + 2\,b_2\,b_3 + 9\,b_2 + 2\,b_1\,b_2 - 3\,b_2\,b_4 + 9\,b_3 + b_1{}^2 + 20 + 9\,b_1 + 2\,b_1\,b_3 + b_3{}^2 - 3\,b_3\,b_4 - 21\,b_4 - 3\,b_1\,b_4 + 6\,b_4{}^2)$

5. If $\min\{b_3, b_4, -b_2, b_1 + b_2\} \geq 0$ then
$\frac{1}{360}(b_2 + b_1 + 3)(b_2 + 2 + b_1)(b_2 + 1 + b_1)(b_2 + 5 + b_1)(b_2 + 4 + b_1)(b_2 + 3 + b_1 + 3\,b_3)$

6. If $\min\{b_1, b_3, b_2 + b_4, -b_1 - b_3 - b_4\} \geq 0$ then
$\frac{1}{360}(b_1 + 3)(b_1 + 2)(b_1 + 1)$
$(60 + 56\,b_1 + 110\,b_2 + 70\,b_3 + 50\,b_4 - 30\,b_3\,b_4\,b_1 + 90\,b_2\,b_3 + 30\,b_2{}^2 b_3 - 15\,b_2\,b_4\,b_1$
$-6\,b_1{}^2 b_3 - 15\,b_3{}^2 b_1 - 30\,b_1\,b_4{}^2 - 30\,b_2\,b_4{}^2 - 30\,b_3\,b_4{}^2 + 57\,b_1\,b_2 + 21\,b_1\,b_3 - 15\,b_1{}^2 b_4 + 3\,b_1{}^2 b_2 + 15\,b_1\,b_2{}^2 - 30\,b_3{}^2 b_4 + 30\,b_2\,b_4 - 15\,b_1\,b_4 + 15\,b_2\,b_3\,b_1 - 10\,b_3{}^3 - 20\,b_4{}^3 + 6\,b_1{}^2 + 60\,b_2{}^2 - 2\,b_1{}^3 + 10\,b_2{}^3 - 30\,b_4{}^2)$

7. If $\min\{b_1, b_3 + b_4, b_2 + b_4, -b_1 - b_4\} \geq 0$ then
$\frac{1}{360}(b_1 + 3)(b_1 + 2)(b_1 + 1)$
$(3\,b_1{}^2 b_3 - 6\,b_1{}^2 b_4 + 9\,b_1{}^2 + 3\,b_1{}^2 b_2 + 51\,b_1 + 57\,b_1\,b_2 + 15\,b_1\,b_2{}^2 - 9\,b_1\,b_4$
$-15\,b_1\,b_4{}^2 + 27\,b_1\,b_3 - 15\,b_2\,b_4\,b_1 + 15\,b_2\,b_3\,b_1 + 30\,b_2\,b_4 + 60 + 60\,b_3 + 60\,b_2{}^2 + 40\,b_4 + 90\,b_2\,b_3 - 30\,b_4{}^2 + 110\,b_2 - 10\,b_4{}^3 + 30\,b_2{}^2 b_3 + 10\,b_2{}^3 - 30\,b_2\,b_4{}^2)$

8. If $\min\{-b_2 - b_4, -b_2 - b_3, -b_1 - b_3 - b_4, b_1 + b_2 + b_3 + b_4\} \geq 0$ then
$\frac{1}{360}(b_4 + 3 + b_1 + b_3 + b_2)(b_4 + 2 + b_1 + b_3 + b_2)(b_4 + 1 + b_1 + b_3 + b_2)$
$(60 + 51\,b_1 + 11\,b_2 - 9\,b_3 - 59\,b_4 + 3\,b_3\,b_4\,b_1 - 6\,b_2{}^2 b_3 - 9\,b_2\,b_4\,b_1 - 3\,b_1{}^2 b_3$
$-6\,b_3{}^2 b_1 + 9\,b_1\,b_4{}^2 - 9\,b_3\,b_4{}^2 + 27\,b_1\,b_2 - 9\,b_1\,b_3 - 3\,b_1{}^2 b_4 + 9\,b_2{}^2 b_4 + 6\,b_1{}^2 b_2 - 3\,b_1\,b_2{}^2 + 6\,b_3{}^2 b_4 + 24\,b_3\,b_4 - 39\,b_1\,b_4 + 6\,b_2\,b_3\,b_1 - 3\,b_3{}^3 + 2\,b_4{}^3 + 9\,b_1{}^2 - 12\,b_2{}^2 - 18\,b_3{}^2 + b_2{}^3 + 12\,b_4{}^2)$

9. If $\min\{-b_2, -b_3 - b_4, b_1 + b_2 + b_3 + b_4, -b_1 - b_2 - b_4\} \geq 0$ then
$\frac{1}{360}(b_4 + 3 + b_1 + b_3 + b_2)(b_4 + 2 + b_1 + b_3 + b_2)(b_4 + 1 + b_1 + b_3 + b_2)(b_2 + 4 - b_4 - b_3 + b_1)(2\,b_2{}^2 + 13\,b_2 + 4\,b_1\,b_2 - b_2\,b_4 - b_2\,b_3 + 15 + 13\,b_1 - b_1\,b_3 + 2\,b_1{}^2 - 7\,b_3 - 7\,b_4 - b_1\,b_4 + 2\,b_4{}^2 + 4\,b_3\,b_4 + 2\,b_3{}^2)$

10. If $\min\{b_4, b_1 + b_2 + b_3, -b_2 - b_3, -b_1 - b_3\} \geq 0$ then
$-\frac{1}{360}(b_2 - 3 + b_3 - 2\,b_1)(b_2 + 5 + b_1 + b_3)(b_2 + 4 + b_1 + b_3)(b_2 + 3 + b_1 + b_3)(b_2 + 2 + b_1 + b_3)(b_2 + 1 + b_1$

11. If $\min\{b_1, b_2 + b_3 + b_4, -b_1 - b_4, -b_1 - b_3\} \geq 0$ then
$\frac{1}{360}(b_1 + 3)(b_1 + 2)(b_1 + 1)(b_3 - 2\,b_4 + b_2 + 3)(10\,b_2{}^2 + 30\,b_2 + 15\,b_1\,b_2 + 20\,b_2\,b_3 + 20\,b_2\,b_4 + 20 + 15\,b_1\,b_4 + 30\,b_3 + 30\,b_4 + 15\,b_1\,b_3 + 6\,b_1{}^2 + 10\,b_4{}^2 + 10\,b_3{}^2 + 24\,b_1 + 20\,b_3\,b_4)$

12. If $\min\{b_1, b_4, b_2 + b_3, -b_1 - b_3\} \geq 0$ then
$\frac{1}{360}(b_1 + 3)(b_1 + 2)(b_1 + 1)(b_1 + 4 + 2\,b_2 + 2\,b_3)$
$\left(5\,b_2{}^2 + 5\,b_1\,b_2 + 10\,b_2\,b_3 + 20\,b_2 + 13\,b_1 + 5\,b_1\,b_3 + 20\,b_3 + 2\,b_1{}^2 + 5\,b_3{}^2 + 15\right)$

13. If $\min\{-b_2 - b_3, -b_4, b_1 + b_2 + b_3 + b_4, -b_1 - b_3\} \geq 0$ then
$-\frac{1}{360}(b_2 - 3 + b_3 - 2\,b_1)(b_4 + 3 + b_1 + b_3 + b_2)(b_4 + 2 + b_1 + b_3 + b_2)(b_4 + 1 + b_1 + b_3 + b_2)$
$(b_2{}^2 + 2\,b_2\,b_3 + 9\,b_2 + 2\,b_1\,b_2 - 3\,b_2\,b_4 + 9\,b_3 + b_1{}^2 + 20 + 9\,b_1 + 2\,b_1\,b_3 + b_3{}^2 - 3\,b_3\,b_4 - 21\,b_4 - 3\,b_1\,b_4 + 6\,b_4{}^2)$

14. If $\min\{-b_2, b_3 + b_4, b_1 + b_2, -b_1 - b_2 - b_4\} \geq 0$ then
$\frac{1}{360}(b_2 + 1 + b_1)(b_2 + 5 + b_1)(b_2 + 4 + b_1)(b_2 + b_1 + 3)(b_2 + 2 + b_1)(2\,b_2 + 2\,b_1 + 3\,b_4 + 3 + 3\,b_3)$

15. If $\min\{-b_1 - b_4, -b_2 - b_3 - b_4, b_1 + b_2 + b_3 + b_4, -b_1 - b_3\} \geq 0$ then
$\frac{1}{360}(b_3 - 2\,b_4 + b_2 + 3)(b_4 + 1 + b_1 + b_3 + b_2)(b_4 + 3 + b_1 + b_3 + b_2)(b_4 + 2 + b_1 + b_3 + b_2)(b_2{}^2 + 2\,b_2\,b_4 + 2\,b_2\,b_3 - 6\,b_2 - 3\,b_1\,b_2 + 20 + b_4{}^2 + 2\,b_3\,b_4 + b_3{}^2 - 6\,b_4 - 3\,b_1\,b_4 - 6\,b_3 - 3\,b_1\,b_3 + 24\,b_1 + 6\,b_1{}^2)$

16. If $\min\{b_1, b_2, b_3 + b_4, -b_1 - b_2 - b_4\} \geq 0$ then
$\frac{1}{360}\,(b_1+3)\,(b_1+2)\,(b_1+1)\left(b_1{}^2+5\,b_1\,b_2+9\,b_1+20+10\,b_2{}^2+30\,b_2\right)(2\,b_2+2\,b_1+3\,b_4+3+3\,b_3)$

17. If $\min\{b_1, b_2 + b_3 + b_4, -b_1 - b_3 - b_4, -b_1 - b_2 - b_4\} \geq 0$ then
$\frac{1}{360}\,(b_1+3)\,(b_1+2)\,(b_1+1)\,(-b_4+3-b_3+2\,b_2)\,(10\,b_2{}^2+30\,b_2+15\,b_1\,b_2+20\,b_2\,b_3+20\,b_2\,b_4+20+15\,b_1\,b_4+30\,b_3+30\,b_4+15\,b_1\,b_3+6\,b_1{}^2+10\,b_4{}^2+10\,b_3{}^2+24\,b_1+20\,b_3\,b_4)$

18. If $\min\{-b_2 - b_3 - b_4, -b_1 - b_3 - b_4, b_1 + b_2 + b_3 + b_4, -b_1 - b_2 - b_4\} \geq 0$ then
$\frac{1}{360}\,(-b_4+3-b_3+2\,b_2)\,(b_4+2+b_1+b_3+b_2)\,(b_4+1+b_1+b_3+b_2)\,(b_4+3+b_1+b_3+b_2)\,(b_2{}^2+2\,b_2\,b_4+2\,b_2\,b_3-6\,b_2-3\,b_1\,b_2+20+b_4{}^2+2\,b_3\,b_4+b_3{}^2-6\,b_4-3\,b_1\,b_4-6\,b_3-3\,b_1\,b_3+24\,b_1+6\,b_1{}^2)$