Algebraic, Geometric, and Topological Methods in Optimization

Jesús A. De Loera

University of California, Davis

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Its goal is to maximizing or minimizing some *objective function* relative to a set of possible solutions! E.g., *Maximum profit, optimal arrangement, minimal error, etc.*

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This talk is about USING

ALGEBRA, GEOMETRY and TOPOLOGY

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Due to time I will focus onLINEAR OPTIMIZATION...

OUTLINE

1. WHAT IS LINEAR OPTIMIZATION AND WHY YOU MUST CARE!

2. TOPOLOGY INSIDE THE SIMPLEX METHOD
   - What is Simplex Method? How does it work?
   - Is there an efficient version of simplex?
   - Combinatorial Topology to the rescue!

3. ALGEBRAIC VIEW OF INTERIOR POINT METHODS
   - A quick review of interior point methods
   - The curvature of the central path
   - Tropical Algebraic Geometry to the rescue!

The classical **Linear Optimization Problem:**

$$\text{minimize } c_1x_1 + c_2x_2 + \cdots + c_dx_d$$

Subject to:

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,d}x_d \leq b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,d}x_d \leq b_2$$

$$\vdots$$

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$$\min\{\, \mathbf{c}^\intercal \mathbf{x} \;:\; A\mathbf{x} \leq \mathbf{b},\ \ \mathbf{x} \in \mathbb{R}^d \,\}.$$

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Adding/changing variables, we can rewrite it system

$$\text{Minimize} \quad \mathbf{q}^T\mathbf{y} \quad \text{subject to} \quad B\mathbf{y} = \mathbf{d} \quad \text{and} \quad \mathbf{y} \geq 0;$$

Linear Programming

$\Updownarrow$

Linear Algebra over NON-NEGATIVE reals

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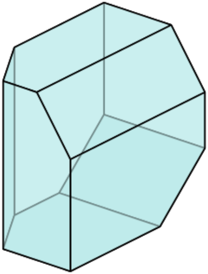
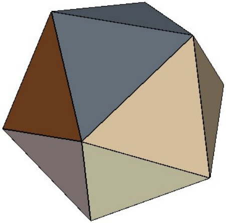
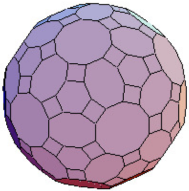
KEY POINT: Set of possible solutions is a **convex polyhedron**,

Anatomy of a Polytope P

Let P be a d-polytope. It has a nice decomposition into smaller-dimension polytopes!

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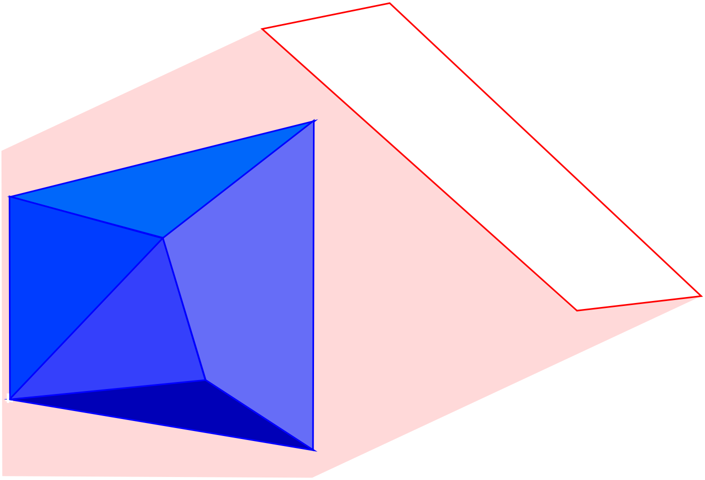
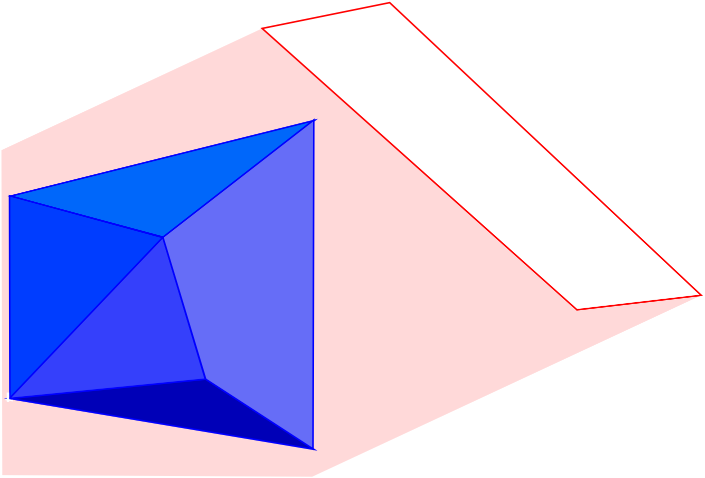
If the halfspace $H = \{x \in \mathbb{R}^c : a_1x_1 + \cdots + a_cx_c \leq a_0\} \supset P$,



the face F of P determined by H is $\partial H \cap P$.



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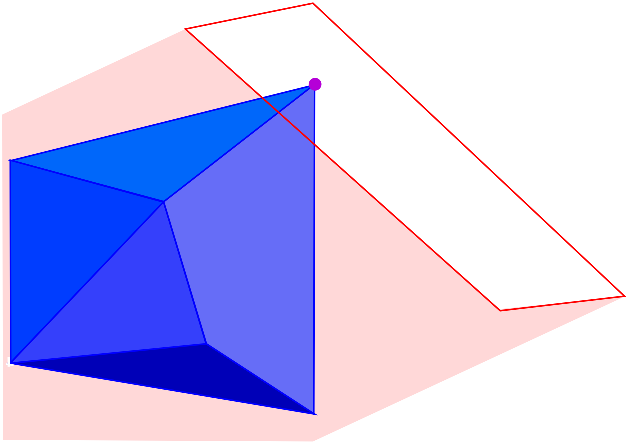
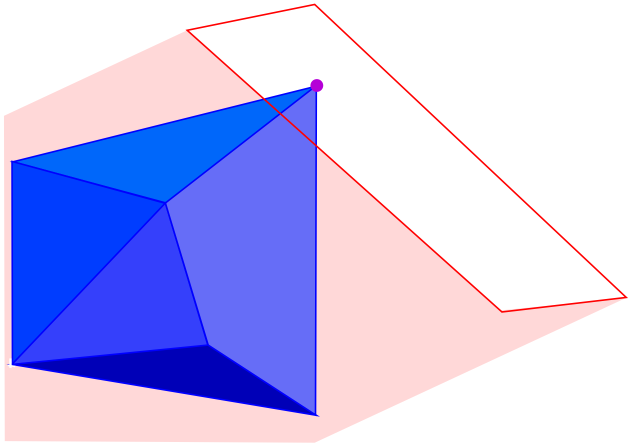
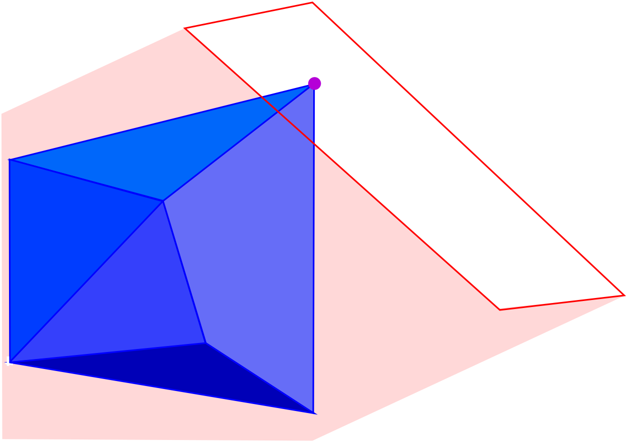


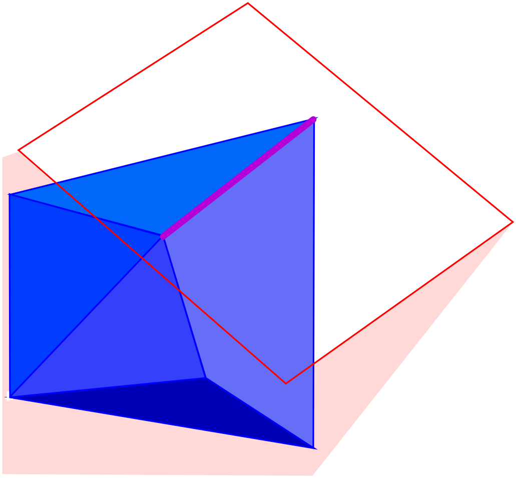
The dimension of a face is the dimension of its affine hull.



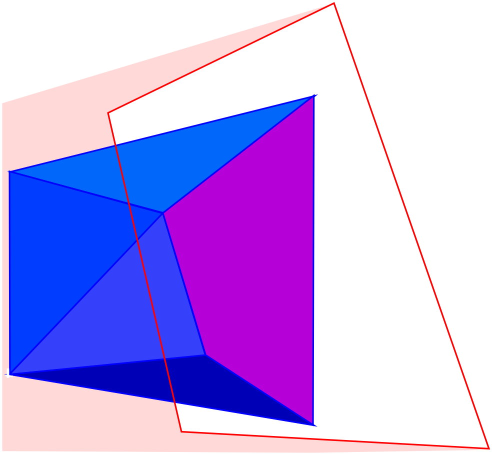
Faces of dimension 0 are called vertices.



Faces of dimension 1 are called edges. The vertices and edges of P form an abstract, finite, undirected graph called the 1-skeleton $G(P)$ of P.

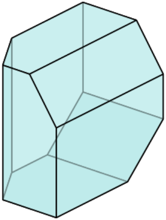


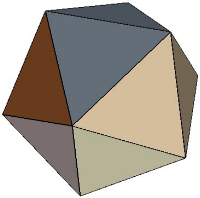
Faces of dimension $d-1$ are called facets.

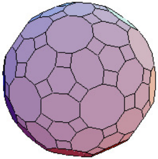


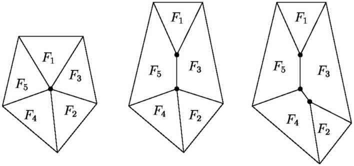
ANATOMY OF A POLYTOPE P

We assume: All polyhedra are **simple**, i.e., each **vertex** is defined by d **facets**.





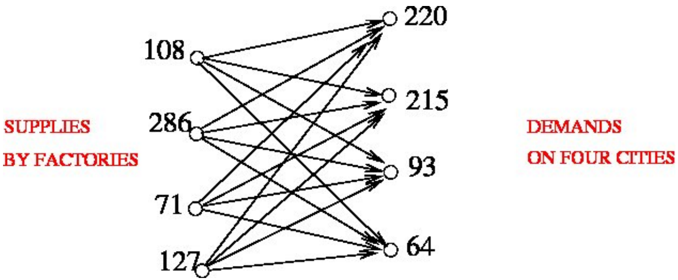




Small perturbations produce SIMPLE polyhedra!

AN EXAMPLE: TRANSPORTATION NETWORKS

$N_1 \times N_2$ Transportation problem: N_1 supply sites and N_2 demand sites. We have cost $c_{i,j}$ for transporting goods from supply site i to demand site j.



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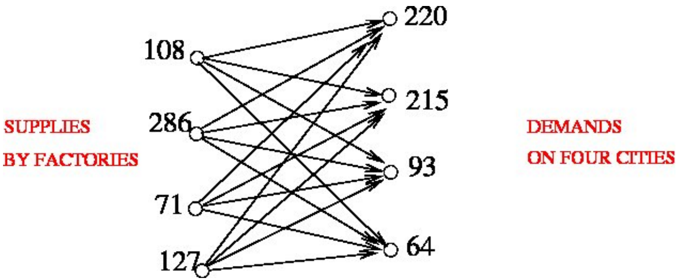

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- Oldest LPs: Kantorovich (1939) Koopmans (1941), von Neumann (1947).
- Applications: Wasserstein distance between distributions, Contingency tables.

- Linear Optimization is important for Optimization and beyond!
  - Linear Optimization is an expressive model! Besides transportation, it includes

    -*shortest paths on a network,*

    - *zero-sum two-player games*,

    -*least absolute value regression*, etc.
  - Exciting new applications keep coming:

    -*compressed sensing*,

    -*computer solution of Kepler's conjecture*,

    -*support vector machines*
  - Linear programs workhorse for the solution/approximation schemes for combinatorial and non-linear optimization.
- Many different algorithms for solution known:
  - Fourier-Motzkin elimination, Ellipsoid Method and its relatives, Projection-Relaxation methods, Others.... TODAY

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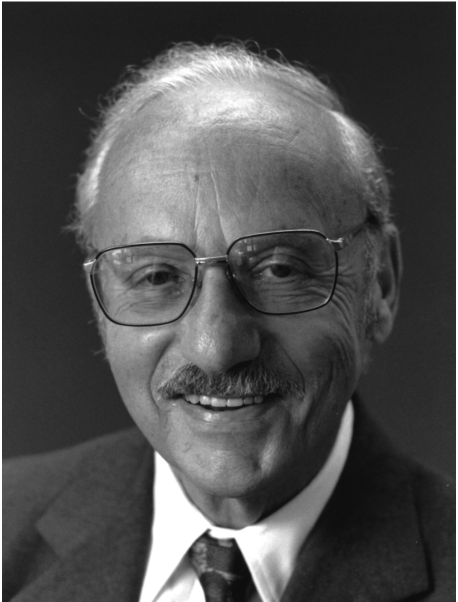
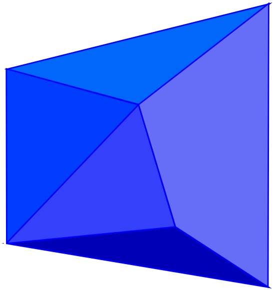
Topology inside the Simplex Method

George Dantzig, inventor of the simplex algorithm (1947)



The vertices and edges of a polyhedron form an undirected graph called the 1-skeleton $G(P)$ of P.



- The distance $d(v_1, v_2)$ between two vertices v_1 and v_2 is the length of shortest edge-path between v_1 and v_2.
- For example, $d(v_1, v_2) = 2$.
- The diameter $\delta = \max\{d(v_1, v_2) : v_1, v_2 \in V\}$ is the maximum distance among all pairs of vertices.

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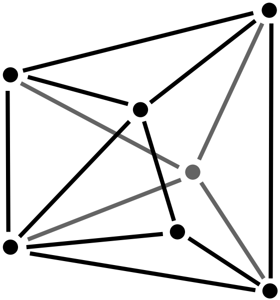

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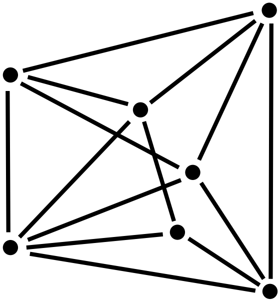

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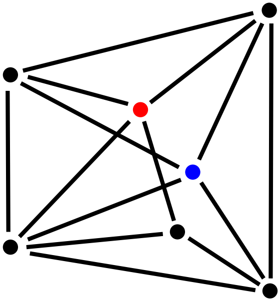

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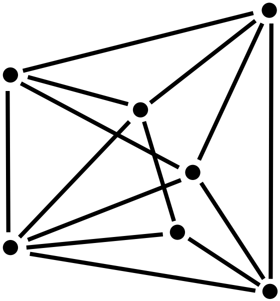

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The Simplex Method in 2 minutes

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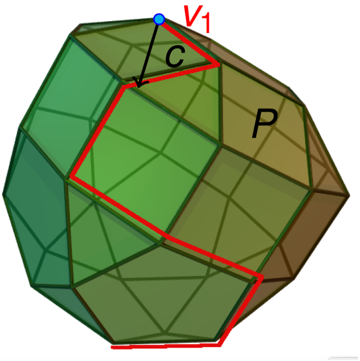

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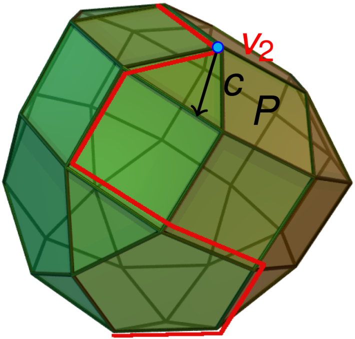

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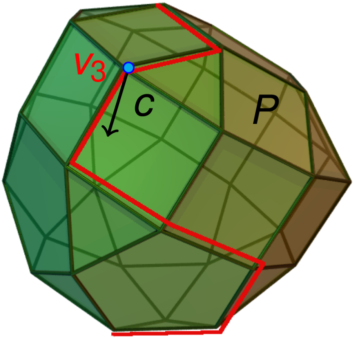

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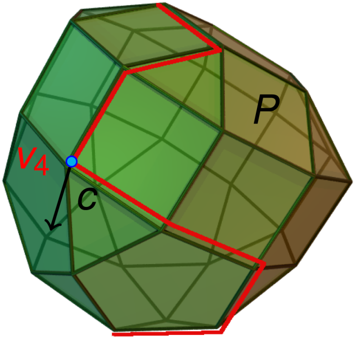

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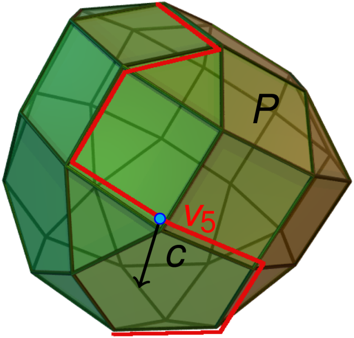

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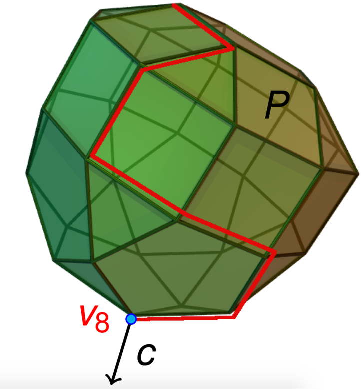

BIG QUESTION

Is there a version of the simplex method

that runs in polynomial time (in the input size)?

RELATED BIG QUESTION:

Is there a polynomial bound of the diameter??

WARNING: If diameter is exponential, then all versions of the simplex method will be exponential in the worst case.

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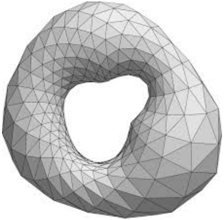
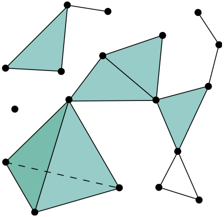
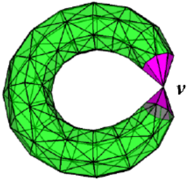
KEY IDEA: Work without coordinates!! Diameter of **simplicial complexes** instead of polyhedra Adler & Dantzig (1970's)







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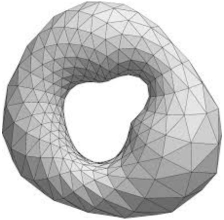

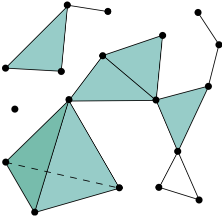

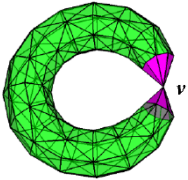

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A Topological view of the Simplex Method I

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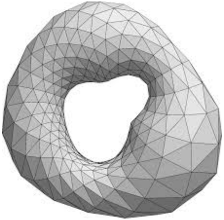
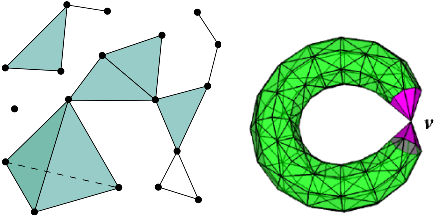




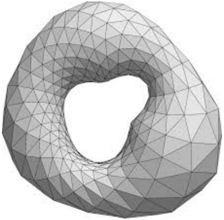
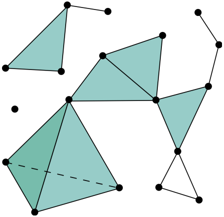
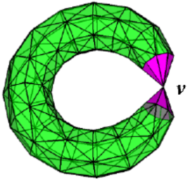
A simplicial complex K is a finite collection of simplices glued to each other in a structured manner:

a) If $\sigma \in K$ then all its faces (which are smaller simplices too!) are also in K.

b) The intersection of any two elements of K is another element of K.

A TOPOLOGICAL VIEW OF THE SIMPLEX METHOD I

KEY IDEA: Work without coordinates!! Diameter of **simplicial complexes** instead of polyhedra Adler & Dantzig (1970's)

A simplicial complex K is a finite collection of simplices glued to each other in a structured manner:

a) If $\sigma \in K$ then all its faces (which are smaller simplices too!) are also in K.

b) The intersection of any two elements of K is another element of K.

The complex K is *pure* if all of its maximal simplices are of the same dimension.

Does this abstraction make any sense??

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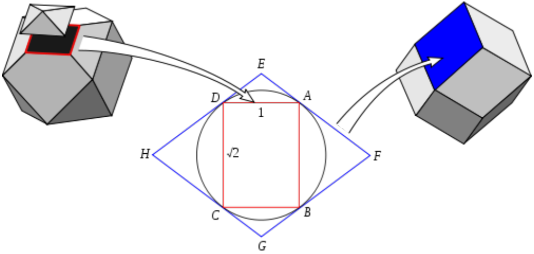
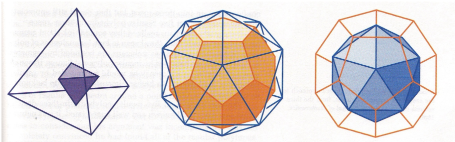
YES! Every polyhedron can be naturally associated a pure simplicial complex via POLARITY

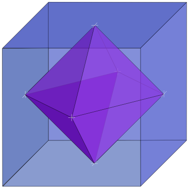




The self-dual tetrahedron.

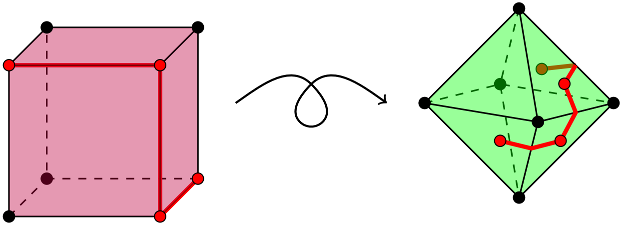
The dodecahedron is dual to the icosahedron.

The icosahedron is dual to the dodecahedron.



We reduced the **vertex** diameter of LPs to the **facet** diameter of Simplicial Polytopes!

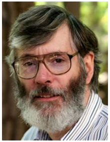
- The **distance** between two facets, F_1, F_2, is the length k of the shortest simplicial path $F_1 = f_0, f_1, \ldots, f_k = F_2$.
- The **diameter** of a simplicial complex is the maximum over all distances between all pairs of vertices.



Best Upper Bounds on the Diameter

- Kalai-Kleitman: $(facets(P))^{\log(dim(P))+1}$.



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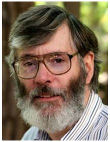


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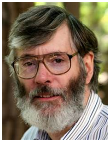


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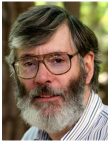


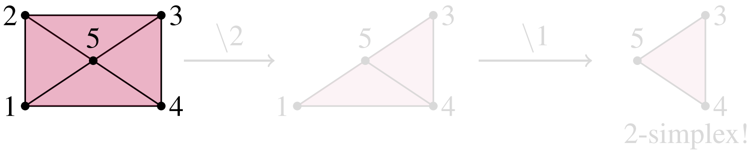

Theorem: These bounds work for diameter of simplicial complexes!
Eisenbrand-Hähnle-Razborov-Rothvoss (2010)

Nice simplicial complexes have low diameter: Weak Vertex-Decomposability

Definition

A d-dimensional complex Δ is **weakly vertex-decomposable** if

1. All the maximal-dimension simplices are of dimension d, and
2. either Δ is a d-simplex, or there exists a vertex τ of Δ (called a **shedding vertex**) such that $\Delta \setminus \tau$ is d-dimensional and weakly vertex-decomposable.



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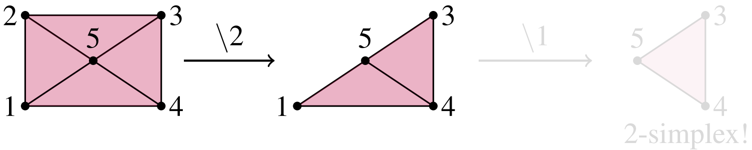

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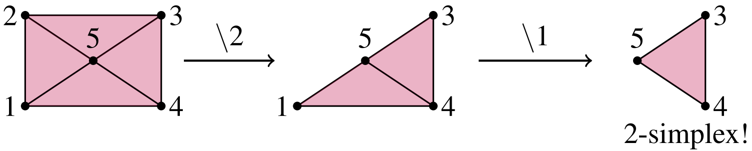

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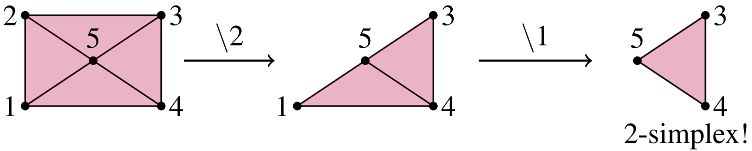

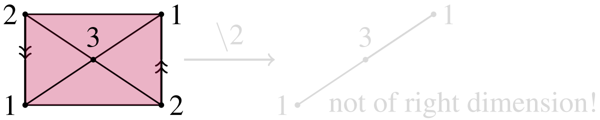

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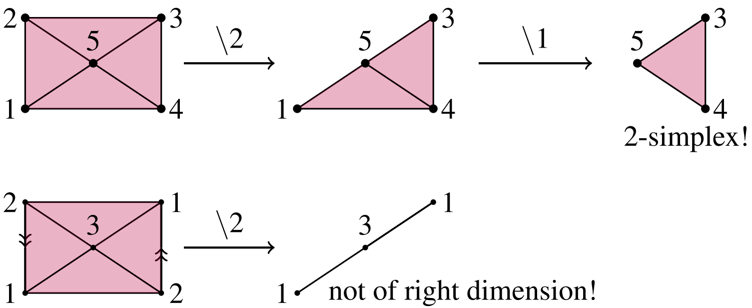

Why should we care??

Theorem (Billera, Provan, 1980)

If Δ is a weakly vertex-decomposable complex, then we have a linear bound ($n = f_0(\Delta)$):

$$\operatorname{diam}(\Delta) \leq 2f_0(\Delta) = 2n.$$

Which simplicial polytopes are weakly vertex-decomposable?

Theorem (JDL + S. Klee, 2012) Not all simplicial polytopes are weakly vertex decomposable!

We constructed explicit transportation problems whose polars are not weakly vertex-decomposable (from dimension 4 onward).

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  But today, the best we know is still LINEAR!!

- **Theorem** Criado and Santos (2015) Pure simplicial complexes, even pseudomanifolds, can have exponential diameter!!!
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SIMPLICIAL COMPLEXES OF LARGE DIAMETER

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Narendra Karmarkar, inventor of interior point methods

The Central Path of a Linear Program: Optimizers view

Linear Program: $\text{Maximize}_{\mathbf{x}\in\mathbb{R}^n} \quad \mathbf{c}^T \cdot \mathbf{x}$ s.t. $A \cdot \mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$.

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Replace by : $\quad \text{Maximize}_{\mathbf{x}\in\mathbb{R}^n} \; f_\mu(\mathbf{x}) \quad$ s.t. $\quad A \cdot \mathbf{x} = \mathbf{b}$,

where $\mu \in \mathbb{R}_+$ and $f_\mu(\mathbf{x}) \; := \mathbf{c}^T \cdot \mathbf{x} + \mu \sum_{i=1}^n \log |x_i|$.

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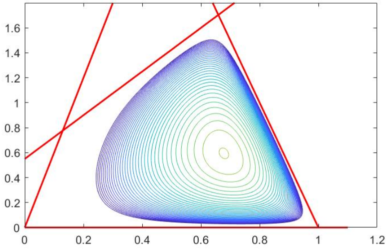

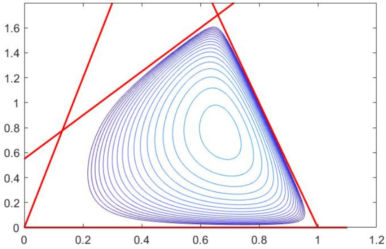

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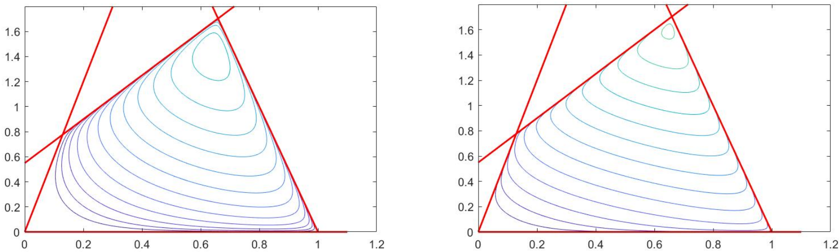

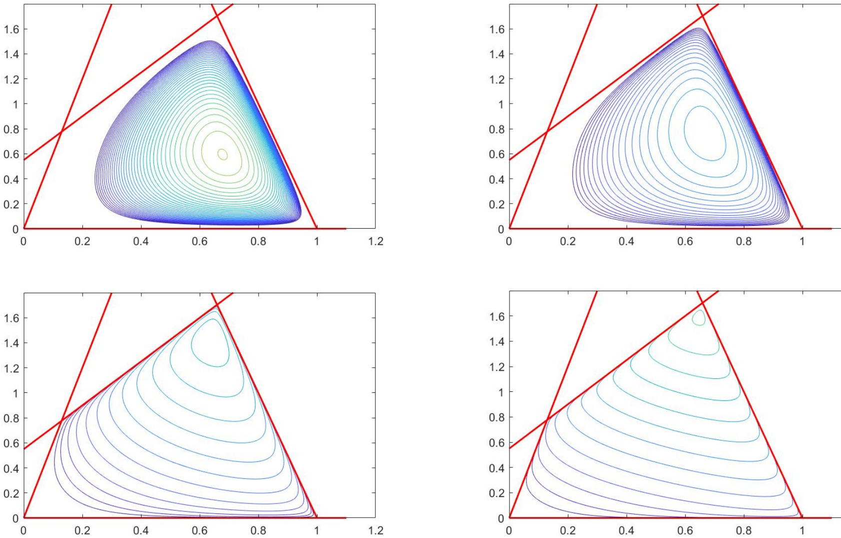
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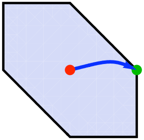
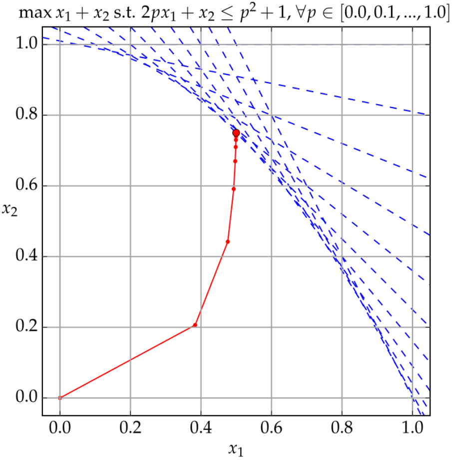
The central path is $\{\mathbf{x}^*(\mu)\ :\ \mu > 0\}$.
As $\mu \to 0$, the path leads from the analytic center of the polytope, $\mathbf{x}^*(\infty)$, to the optimal vertex, $\mathbf{x}^*(0)$.



The Central Path of a Linear Program

In practice = piecewise-linear approx. of the entire central path



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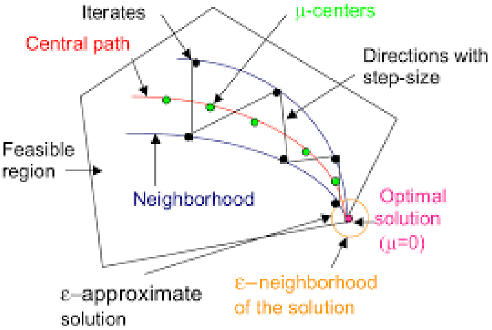
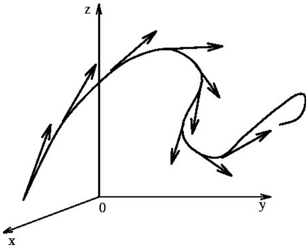
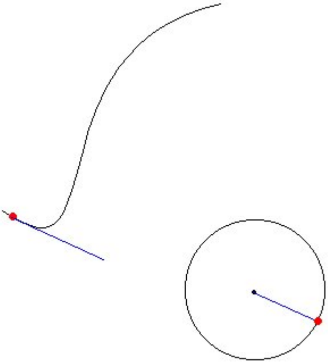


How "curvy" or "twisted" is the central path?

Intuition: The number of steps will depend on how "curvy" how "twisted" is the central path! What is the curvature?





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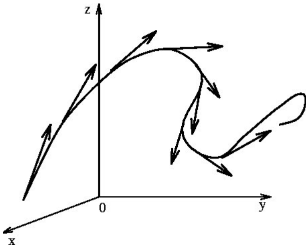

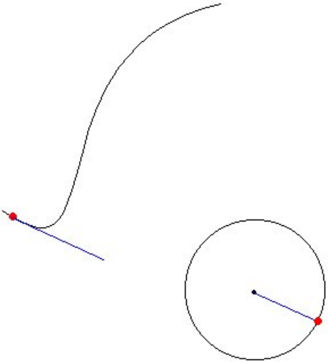

The total curvature of a curve $\mathcal{C}$ in $\mathbb{R}^m$ is the arc length of its image under the Gauss map. $\gamma : \mathcal{C} \to \mathbb{S}^{m-1}$.

Bounds on curvature $\to$ bounds on $\#$ steps.

Megiddo-Shub (1989), Sonnevend-Stoer-Zhao (1991),Todd-Ye (1996), Vavasis-Ye (1996), Dedieu-Malajovich-Shub (2005), Deza-Terlaky-Zinchenko (2008),Monteiro-Tsuchiya (2008)....

Question: What is the **exact** total curvature of the central path?

Conjecture:(Deza, Terlaky, Zinchenko) **The total curvature of the central path in a polyhedron is** $\leq 2\pi(\#$**number of facets**$)$**.**

There is a recent solution of this conjecture!

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High curvature LPs exist!

Theorem (Allamigeon, Benchimol, Gaubert, Joswig, 2017)

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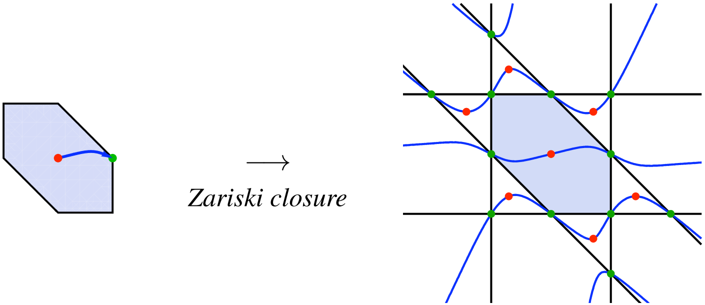
Corollary

The number of iterations of any primal-dual path-following interior point algorithm with a log-barrier function which iterates in the wide neighborhood of the central path is exponential in r (for sufficiently large parameter value).

IDEA 1: CENTRAL PATH IS PART OF AN ALGEBRAIC CURVE!!!!

The central curve $\mathcal{C}$ is the Zariski closure of the central path.

Theorem: JDL, B. Sturmfels, C. Vinzant (2014): computed equations, curvature, degree.

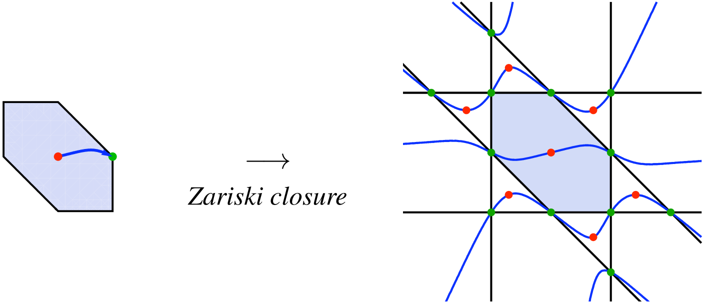


From an algebraic variety V, one can obtain a combinatorial model, its tropical variety $T(V)$. We do a tropicalization!

IDEA 1: CENTRAL PATH IS PART OF AN ALGEBRAIC CURVE!!!!

The central curve $\mathcal{C}$ is the Zariski closure of the central path.

Theorem: JDL, B. Sturmfels, C. Vinzant (2014): computed equations, curvature, degree.



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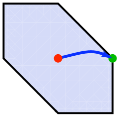
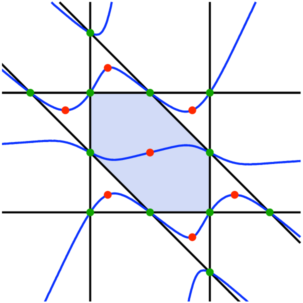


$\longrightarrow$

Zariski closure



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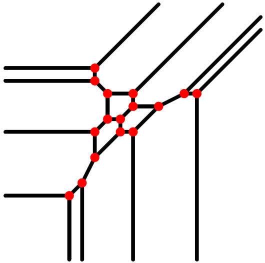
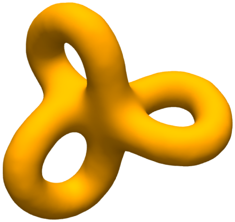
IDEA 2: TROPICAL GEOMETRY

- The tropicalization of V is a polyhedral complex in $\mathbb{R}^d$. We turn algebra into combinatorics!





- **KEY POINT:** Features of the algebraic variety V are easier to see/compute in its tropicalization $T(V)$.

  E.g., the central curve is piecewise linear, the curvature is now a sum of piece-wise linear angular turns.

Tropical Arithmetic

- Tropical Geometry is geometry over the Tropical semiring:
- $T(\mathbb{R})$, this is the reals union $-\infty$ with two binary operations $\oplus = \max$ and $\otimes = +$.

$$1 \oplus 3 = 3 \qquad 5 \otimes 0 = 5$$

- Tropical Arithmetic is associative distributive, the additive identity is $-\infty$ and the multiplicative identity is 0. We do not have subtraction!
- Tropical arithmetic extends to matrices and polynomials equations.
- The polynomials we use have parametric coefficients.

Tropicalization of parametric polynomials

- The **tropicalization** replaces a **parametric** polynomial (coefficients depend on t) into a tropical polynomial.

$$f(x,t) := x^3 - (t^3 + 2t + 1)x^2 - 2t^4$$

goes to a **tropical polynomial.**

$$F(x) = x^{\otimes 3} \oplus 3 \otimes x^{\otimes 2} \oplus 4$$

- The coefficients in t replaced by the leading power dominating term. Signs do not matter!!
- Now, in the usual arithmetic $F(x)$ is a **piece-wise linear function!**

$$\max(3x, 3 + 2x, 4)$$

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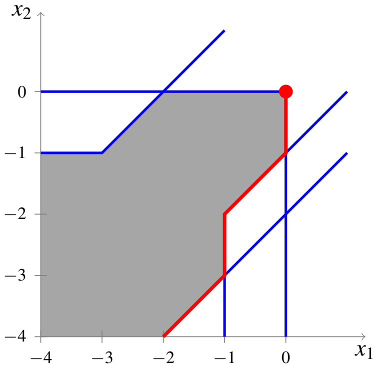
- A parametric polyhedron $\mathcal{P} \subset \mathbb{K}^2$ is defined by inequalities.

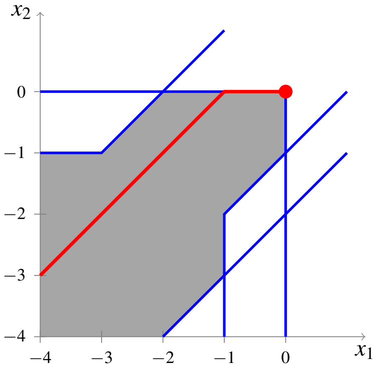
$$\begin{aligned} \boldsymbol{x}_1 + \boldsymbol{x}_2 &\leq 2 \\ t\boldsymbol{x}_1 &\leq 1 + t^2\boldsymbol{x}_2 \\ t\boldsymbol{x}_2 &\leq 1 + t^3\boldsymbol{x}_1 \\ \boldsymbol{x}_1 &\leq t^2\boldsymbol{x}_2 \\ \boldsymbol{x}_1, \boldsymbol{x}_2 &\geq 0 \,. \end{aligned} \tag{1}$$

- The tropicalization of $\mathcal{P}$ is described by five *tropical* linear inequalities:

$$\begin{aligned} \max(x_1, x_2) &\leq 0 \\ 1 + x_1 &\leq \max(0, 2 + x_2) \\ 1 + x_2 &\leq \max(0, 3 + x_1) \\ x_1 &\leq 2 + x_2 \,. \end{aligned} \tag{2}$$



$\min \boldsymbol{x}_1$



$\min t\boldsymbol{x}_1 + \boldsymbol{x}_2$

Figure shows the tropicalization of the polyhedron, but it also shows, in red, two different tropical central paths (for two different objective functions).

Note: The tropical central path may degenerate to a vertex-edge path, like the simplex method.

Conclusions

- Similar methods can be applied in other areas of optimization!
  - Global and Conic optimization and Real algebraic geometry
  - Tropical Geometry and Game Theory
  - Lattices and Geometry of Numbers and Integer Optimization
- Diversity of methodology is a powerful way to approach problems!!

  Just imagine what we could do if we have a larger and more diverse group of mathematicians working together?

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Gracias!
Thank you!