Transportation Polytopes: a Twenty year Update

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Based on various papers joint with
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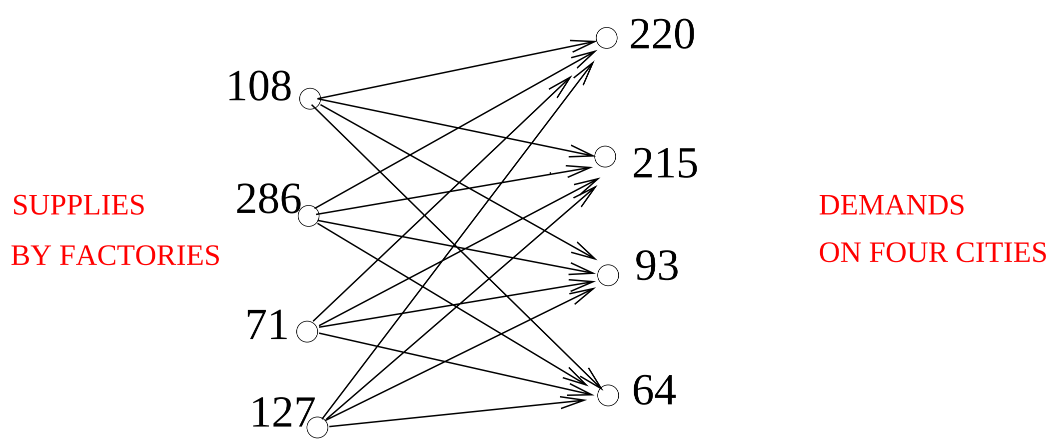


Goal and Plan of this Lecture:

Transportation polytopes arise in Optimization, Statistics and pure Mathematics! Go back to the beginning of Mathematical Programming: Kantorovich and Koopmans already studied them in the 1930's.

Two prior extended surveys and open problem collections: 1966 by Klee-Witzgall and 1984 by Kovalev-Yemelichev-Kratsov:

TODAY: a new updated survey about transportation polytopes, with many new results, open questions...

Part I: What is a transportation polytope?

Part II: Classical Two-Way Transportation Polytopes

Part III: Multiway Transportation Polytopes

Part IV: Open Problems

Part 1:

WHAT IS A TRANSPORTATION POLYTOPE ??

CLASSICAL ($n \times m$)-transportation Polytopes:

?	?	?	?	220
?	?	?	?	215
?	?	?	?	93
?	?	?	?	64
108	286	71	127	

The 2-way **transportation polytope** is the set of all possible tables whose row/column sums equal the given margins.

Multi-Way Transportation Polytopes.

A d-**way table** is an $n_1 \times n_2 \times \cdots \times n_d$ array of nonnegative real numbers $v = (v_{i_1,\ldots,i_d})$, $1 \leq i_j \leq n_j$.

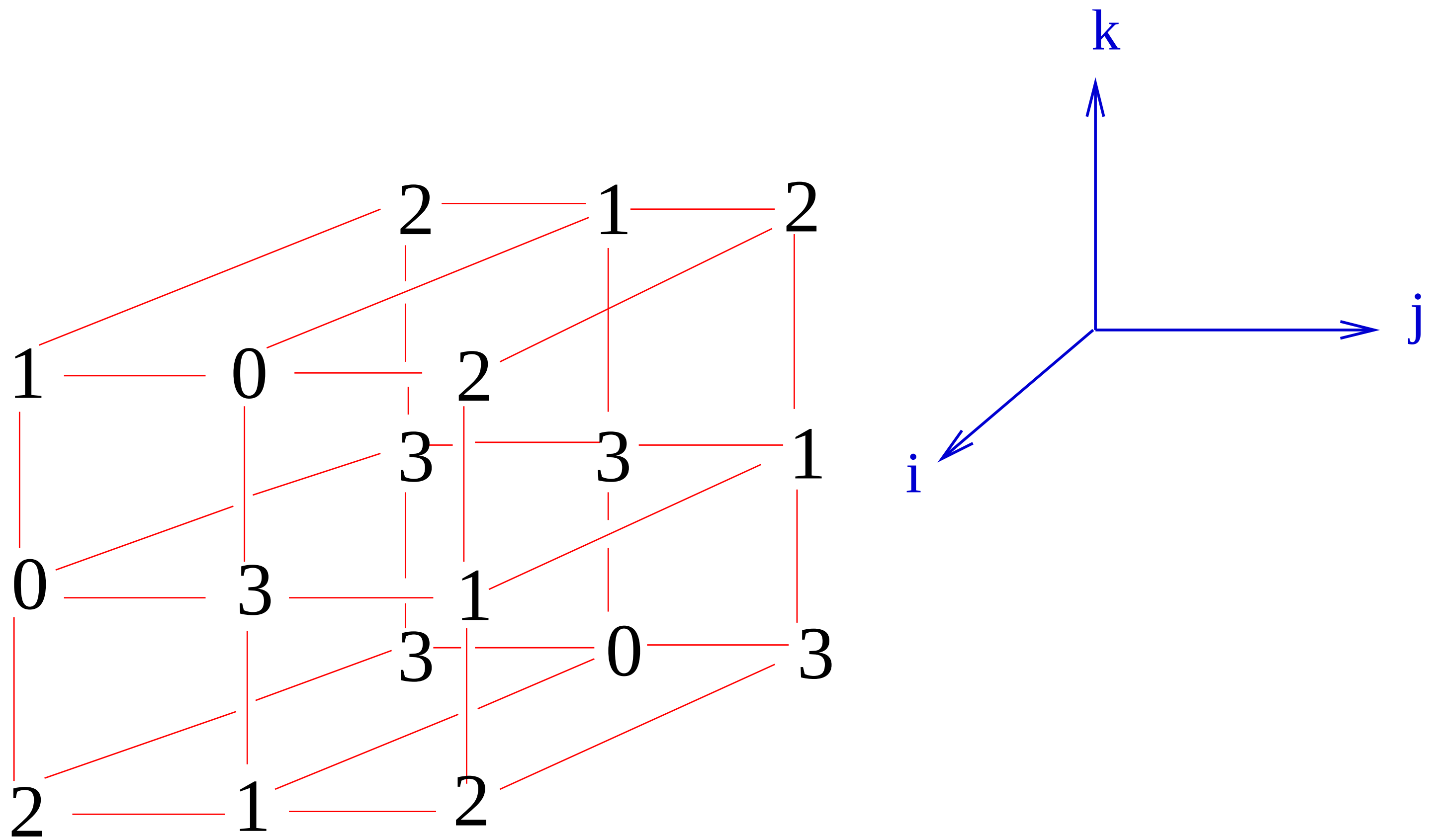
For $0 \leq m < d$, an m-**margin** of v is any of the possible m-tables obtained by summing the entries over all but m indices.

Example If $(v_{i,j,k})$ is a 3-table then its 0-margin is $v_{+,+,+} = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} v_{i,j,k}$, its 1-margins are $(v_{i,+,+}) = (\sum_{j=1}^{n_2} \sum_{k=1}^{n_3} v_{i,j,k})$ and likewise $(v_{+,j,+})$, $(v_{+,+,k})$, and its 2-margins are $(v_{i,j,+}) = (\sum_{k=1}^{n_3} v_{i,j,k})$ and likewise $(v_{i,+,k})$, $(v_{+,j,k})$.

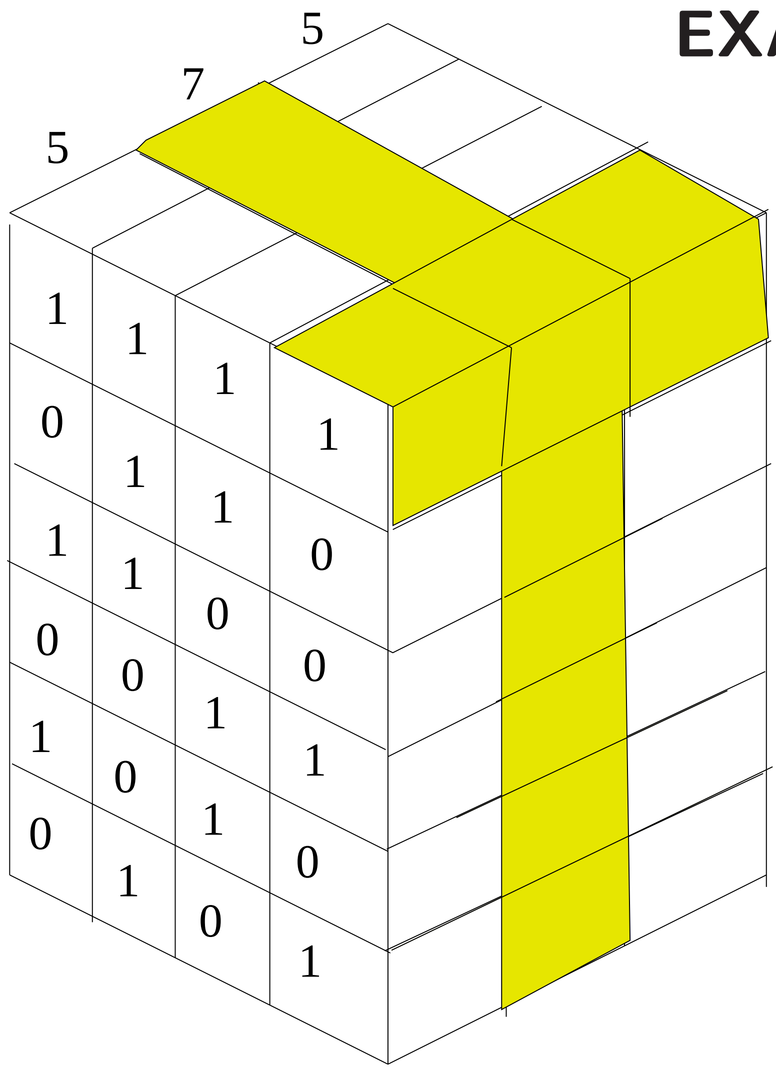
Definition: A **multi-index transportation polytope** is the set of all real d-tables that satisfy a set of given margins.

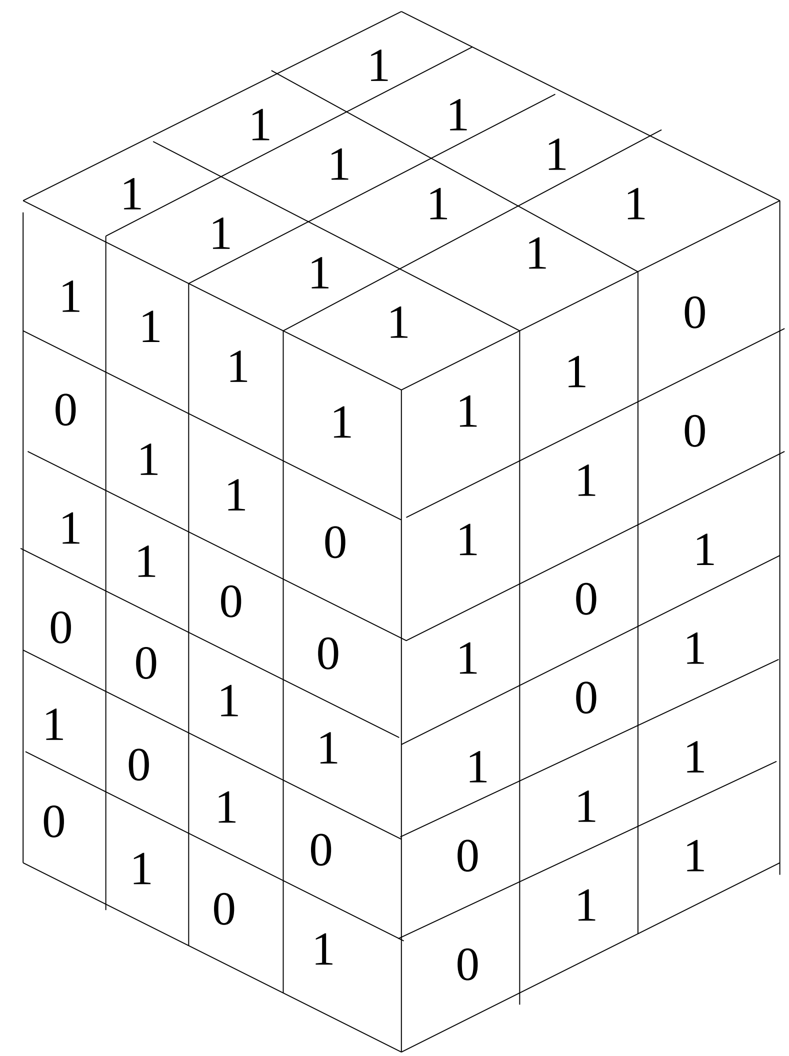
Definition An **Assignment polytope** is when all right hand-side margins have entry value one. (They also receive the names **Birkhoff-von Neumann polytopes** or **Polytope of poly-stochastic matrices**.

EXAMPLE 1

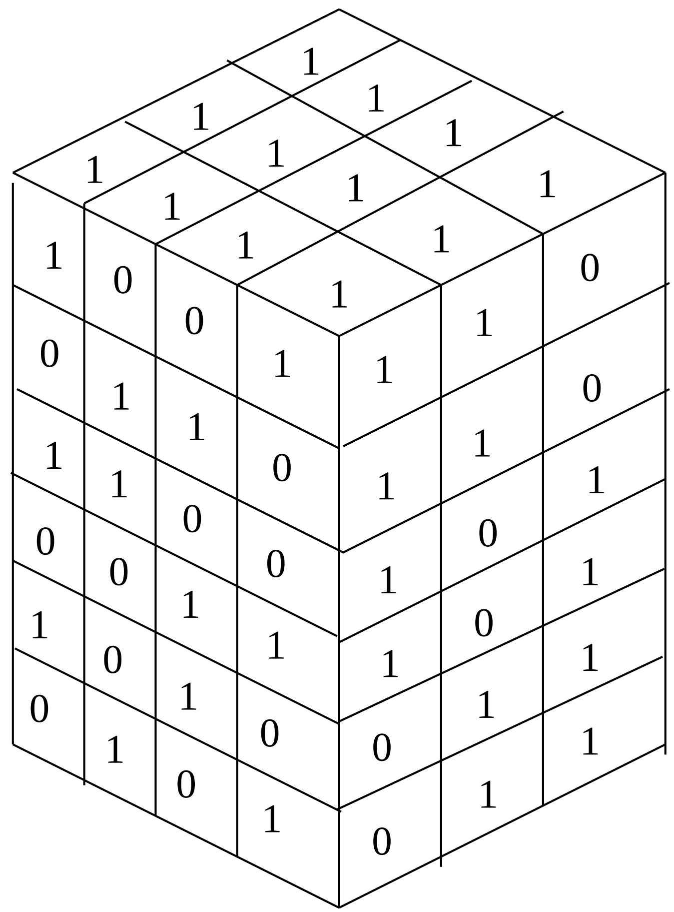


EXAMPLE 2





EXAMPLE 3


1 1 1 1
1 1 1 1
1 1 1 1
1 0 0 1
0 1 1 0
1 1 0 0
0 0 1 1
1 0 1 0
0 1 0 1
1 1 0
1 1 0
1 0 1
1 0 1
0 1 1
0 1 1

Part 2:

TWO-WAY TRANSPORTATION POLYTOPES

Constraint Matrix has SPECIAL shape

We can easily see that the system $Ax = b,\ x \geq 0$ has as constraint matrix A the vertex-edge incidence matrix of the complete bipartite graph $K_{n,m}$.

Example: For the transportation problem

?	?	?	?	220
?	?	?	?	215
?	?	?	?	93
?	?	?	?	64
108	286	71	127	

we have

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Background facts: We know a lot!

Theorem: In the classical $n \times m$ transportation polytope we are given 1-marginals $a = (a_1, \dots, a_{n_1})$ and $b = (b_1, \dots, b_m)$,

(1) (dimension and feasibility) If $\sum a_i = \sum b_i$, then polytope is non-empty and it has dimension $(n-1)(m-1)$. The *northwest-corner Rule* can find a solution!

(2) (vertices and lattice points) Given integral consistent marginals a, b, all vertices of the corresponding transportation polytope are integral. Constraint matrix is totally unimodular!

Proposition 2-way transportation polytopes have the *"integral interval property"* for all entries: If $l < x_{ij} < u$, then there is a lattice point with $x_{ij} = s$ for all integers in between l, u. No gaps!

(3) (optimization) All Integer linear programs can be solved efficiently.

Graphs of two-way transportation polytopes

Theorem: Let $X = (x_{ij})$ be a matrix that belongs to the $n \times m$ transportation polytope $T(a,b)$. Create the auxiliary graph F_X with $n \times m$ nodes and edge $\{(i,j) : x_{i,j} > 0\}$. Then the matrix X is a vertex of the transportation polytope $T(a,b)$ if and only if F_X forms a spanning forest in the complete bipartite graph $K_{n,m}$.

Lemma:(Adjacency conditions of vertices) Let X and Y be distinct vertices of the transportation polytope $T(a,b)$. The vertices are adjacent if and only if the union of the associated graphs F_X and F_Y contains a unique cycle.

Theorem:(2004 Brightwell-van den Heuvel-Stougie) The diameter of the transportation polytope of size $n \times m$ does not exceed $8(m+n-2)$.

breaking news: (2008 Edward Kim) The diameter is less than or equal to $3(m+n-2)$.

Part 3:

MULTI-WAY TRANSPORTATION POLYTOPES

Several Open Questions from the 1980's answered

(0) **Real feasibility and Dimension**:

(Vlach Problems 1986): Do the conditions on the Several conditions on the margins proposed by Schell, Haley, Moravek and Vlack during the 1980's. Do they suffice to guarantee that the 2-margin 3-way Transportation polytope is non-empty?

NO! These conditions are necessary but not sufficient

But, is there some easy way to find a feasible solution in terms of the 2-margins of those 3-way transportation polytopes which are empty? Something like the *northwest-corner rule*???

Unlikely, as hard a solving an arbitrary LP!!!!

What is the dimension of the polytope?

Dimension can be any number from 1 up to $(n-1)(m-1)(k-1)$

(1) **Integer Feasibility Problems:** Given integral 2-margins, that describe a 3-transportation polytope, is there an integer table with these margins? can one such integral d-table be efficiently determined ?

NO, problem is NP-complete already for $3 \times n \times m$ transportation polytopes

Do k-way tables have the “integral interval property” for entry values?

No, any set of non-negative integers can be the values of a given entry! There are Gaps!

(3) **The graphs of 2-margin k-way transportation polytopes** Is there a nice easy characterization of vertices and edges? Do we have a good bound for the diameter?

This problem is as hard as proving a bound for ALL polytopes!!!

A UNIVERSALITY THEOREM

Theorem: (JD & S. Onn 2004) Any rational polytope $P = \{y \in \mathbb{R}^n : Ay = b \ y \geq 0 \ \}$ is polynomial-time representable as a 3-way $(r \times c \times 3)$ transportation polytope

$$\{ \, x \in \mathbb{R}^{r \times c \times 3}_{\geq 0} \ : \ \sum_i x_{i,j,k} = w_{j,k} \, , \ \sum_j x_{i,j,k} = v_{i,k} \, , \ \sum_k x_{i,j,k} = u_{i,j} \, \} \ .$$

The polytope $P \subset \mathbb{R}^p$ is *representable* as a polytope $T \subset \mathbb{R}^q$ if there is an injection $\sigma : \{1, \ldots, p\} \longrightarrow \{1, \ldots, q\}$ such that the projection

$$\pi : \mathbb{R}^q \longrightarrow \mathbb{R}^p \ : \ x = (x_1, \ldots, x_q) \mapsto \pi(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(p)})$$

is a bijection between T and P and between the sets of integer points $T \cap \mathbb{Z}^q$ and $P \cap \mathbb{Z}^p$.

Theorem (JD & S. Onn)(now with complexity estimate).

Any polytope $P=\{y \in \mathbb{R}^n_{\geq 0} : Ay = b\}$ with integer $m \times n$ matrix $A = (a_{i,j})$ and integer b is polynomial-time representable as a slim transportation polytope

$$T \quad = \quad \{\, x \in \mathbb{R}^{r\times c\times 3}_{\geq 0} \ : \ \sum_i x_{i,j,k} = w_{j,k} \,,\ \sum_j x_{i,j,k} = v_{i,k} \,,\ \sum_k x_{i,j,k} = u_{i,j} \,\} \,,$$

with $r = O(m^2(n+L)^2)$ rows and $c = O(m(n+L))$ columns,

where $L := \sum_{j=1}^{n} \max_{i=1}^{m} \lfloor \log_2 |a_{i,j}| \rfloor$.

Three Independent Steps

Step 1 Reduce Size of Coefficients:

Step 2 Encode polytope INTO transportation polytope with 1-margins with some entries bounded

Step 3 Encode any transportation polytope with 1-margins and bounded entries INTO transportation polytope with 2-margins

Step 1

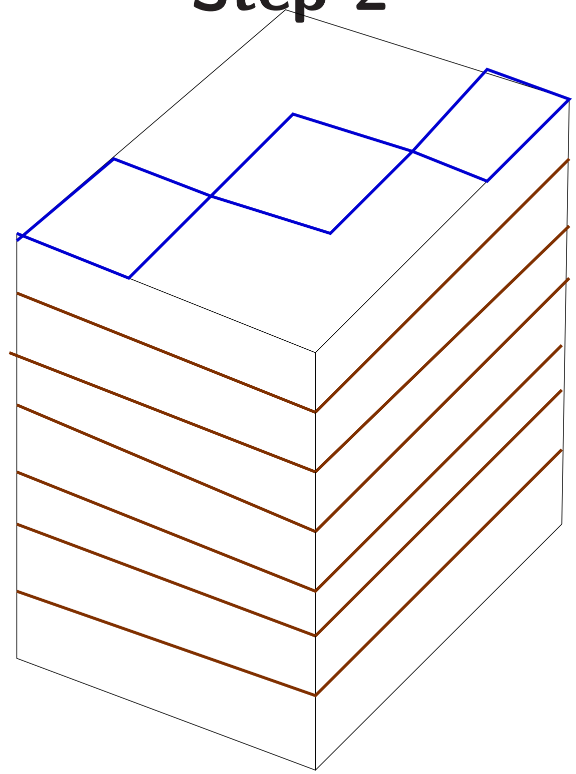
Given $P = \{y \geq 0 : Ay = b\}$ where $A = (a_{i,j})$ is an integer matrix and b is an integer vector. We represent it as a polytope $Q = \{x \geq 0 : Cx = d\}$, in polynomial-time, with a $\{-1, 0, 1, 2\}$-valued matrix $C = (c_{i,j})$ of coefficients.

Use the binary expansion $|a_{i,j}| = \sum_{s=0}^{k_j} t_s 2^s$ with all $t_s \in \{0, 1\}$, we rewrite this term as $\pm \sum_{s=0}^{k_j} t_s x_{j,s}$.

EXAMPLE: The equation $3y_1 - 5y_2 + 2y_3 = 7$ becomes

$$\begin{array}{cccccccccc}
2x_{1,0} & -x_{1,1} & & & & & & = & 0 \\
 & & 2x_{2,0} & -x_{2,1} & & & & = & 0 \\
 & & & 2x_{2,1} & -x_{2,2} & & & = & 0 \\
 & & & & & 2x_{3,0} & -x_{3,1} & = & 0 \\
x_{1,0} & +x_{1,1} & -x_{2,0} & & -x_{2,2} & & +x_{3,1} & = & 7
\end{array} .$$

Step 2



Each equation $k = 1, \ldots, m$ will be encoded in a "horizontal table" plus an extra layer of "slacks". Each variable y_j, $j = 1, \ldots, n$ will be encoded in a "vertical box" Other entries are zero.

EXAMPLE

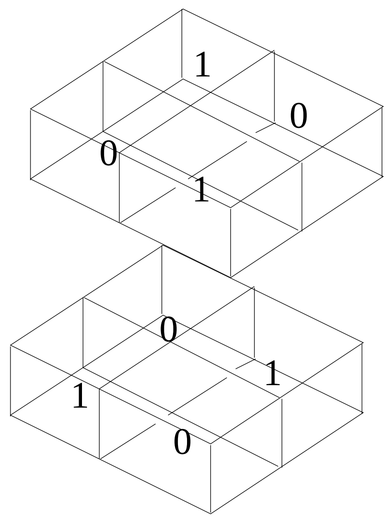
Starting one dimensional polytope $P = \{y | 2y = 1,\ y \geq 0\}$ we obtain

INPUT

table size (2,2,2)

1−marginals = 1

entry upper bounds (see below)



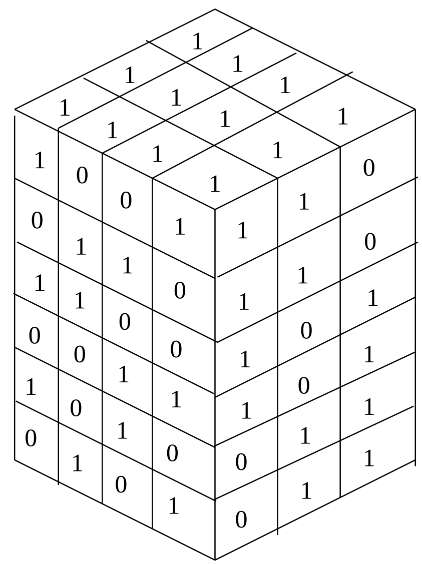
Unique real feasible array
All entries = 1/2 or 0

No integer table

OUTPUT

table size (3,4,6)

2−marginals (see below)



Unique real feasible array
All entries 1/2 and 0

No integer table.

MORE CONSEQUENCES

(3) **All linear and integer programming problems are Slim 3-way Transportation Problems**. Any linear or integer programming problem is equivalent to one that has only $\{0, 1\}$-valued coefficients, with exactly three 1's per column, and depends only on the right-hand side data.

(4) **Problem reduction** If there is an polynomial-time algorithm for 3-way transportation problems would imply the existence of a polynomial-time algorithm for linear programs in general. For example, if a polynomial diameter is true for 3-way transportation polytopes given by 2-marginals, then it is true for all rational polytopes!

(5) **Hardness of approximation** The family of 3-way transportation problems of $(r, c, 3)$ and specified by 2-margins contains subproblems that admit fully polynomial approximation schemes as well as subproblems that do not have arbitrarily close approximation (unless $NP = P$).

k-way 1-margin transportation problems

Theorem: (JD & S. Onn 2004) Any rational polytope $P = \{y \in \mathbb{R}^n : Ay = b,\ y \geq 0\ \}$ is polynomial-time representable as **a face** of a 3-way $(r \times c \times 3)$ transportation polytope with 1-margins

$T = \{\, x \in \mathbb{R}^{r \times c \times 3}_{\geq 0} \ : \ \sum_{i,j} x_{i,j,k} = w_k \,,\ \sum_{j,k} x_{i,j,k} = v_i \,,\ \sum_{i,k} x_{i,j,k} = u_j \,\}\ .$

Theorem: [JD, E. Kim, F. Santos, S. Onn 2007] Any 3-way transportation polytope of size $l \times m \times n$ with fixed 1-margins has diameter no more than $8(l + m + n)^2$.

Question: Can one bound the diameter of a face of the 1-margin transportation polytope??

k-way 1-margin Assignment polytopes are Nasty too!!

Theorem: (Karp 1972) The optimization problems

maximize/minimize $\sum p_{i,j,k} x_{i,j,k}$ subject to

$$\{ x \in \mathbb{Z}_{\geq 0}^{n\times n\times n} \ : \ \sum_{i,j} x_{i,j,k} = 1 \, , \ \sum_{j,k} x_{i,j,k} = 1 \, , \ \sum_{i,k} x_{i,j,k} = 1 \} \ .$$

are NP-hard.

Theorem: (Crama-Spieksma 1992) For the minimization case, no polynomial time algorithm can even achieve a constant performance ratio unless NP=P.

Theorem (Gromova 1992) Any positive decreasing vector v with components less than one is a permutation of part of a vertex of some 3-way assignment polytope.

New Positive Results

Theorem 1 (JDL, Hemmecke, Onn, Weismantel) Fix any r, s. Then there is a polynomial time algorithm that, given l, integer objective vector c, and integer line-sums $(u_{i,j})$, $(v_{i,k})$ and $(w_{j,k})$, solves the integer transportation problem

$$\min\{\, cx \;:\; x \in \mathbb{N}^{r\times s\times l}\,,\ \sum_i x_{i,j,k} = w_{j,k}\,,\ \sum_j x_{i,j,k} = v_{i,k}\,,\ \sum_k x_{i,j,k} = u_{i,j}\,\}\ .$$

Theorem 2 (JDL, Hemmecke, Onn, Rothblum, Weismantel) Let d, r, s be fixed positive integers. Given integer line-sums margins tables $(u_{i,j})$, $(v_{i,k})$ and $(w_{j,k})$, arrays $w_1, \ldots, w_d \in \mathbb{N}^{r\times s\times l}$, and a convex function $c : \mathbb{R}^d \longrightarrow \mathbb{R}$, presented by comparison oracle, then there is a polynomial oracle-time algorithm that, solves the convex **maximization** integer 3-way transportation problem

$$\max\ c(w_1x,\ldots,w_dx) \text{ subject to}$$

$$\{\ x\in\mathbb{N}^{r\times s\times l}\ ,\ \sum_i x_{i,j,k}=w_{j,k}\ ,\ \sum_j x_{i,j,k}=v_{i,k}\ ,\ \sum_k x_{i,j,k}=u_{i,j}\ \}\ .$$

Hemmecke-Onn-Weismantel (2007) proved now (using algebra again!) that **convex minimization** can be also done efficiently

What is behind these theorems?

Algebra + Graver Test Sets!! Closely related to Commutative Algebra!

The lattice $L(A) = \{x \in \mathbb{Z}^n : Ax = 0\}$ has a natural partial order. For $u, v \in \mathbb{Z}^n$ we say that u is *conformal* to v, denoted $u \sqsubseteq v$, if $|u_i| \leq |v_i|$ and $u_i v_i \geq 0$ for $i = 1, \ldots, n$, that is, u and v lie in the same orthant of $\mathbb{R}^n$ and each component of u is bounded by the corresponding component of v in absolute value.

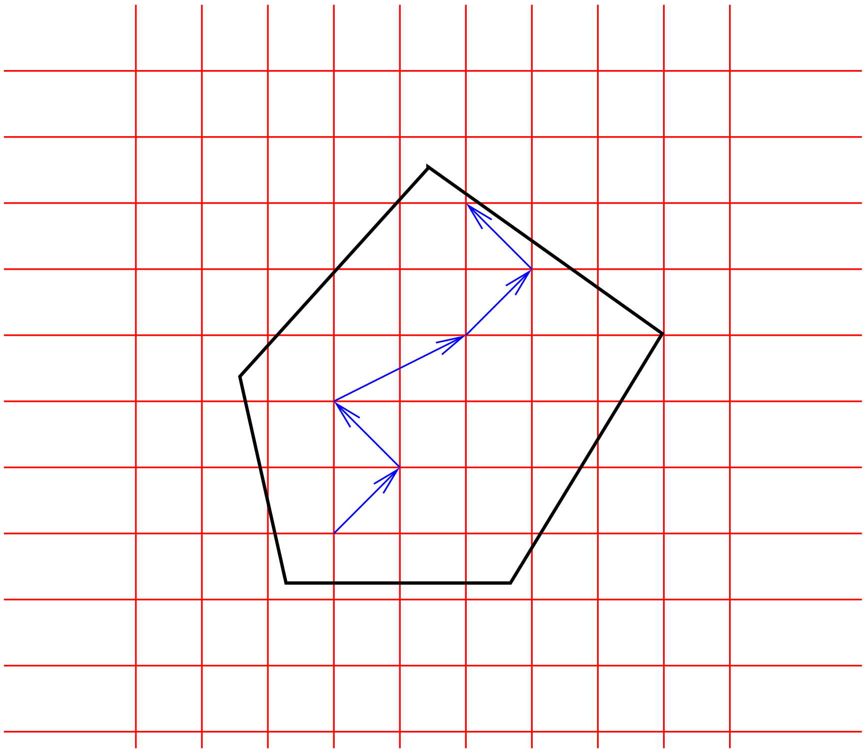
The Graver basis of an integer matrix A is the set of conformal-minimal nonzero integer dependencies on A.

Example: If $A = [1\ 2\ 1]$ then its Graver basis is

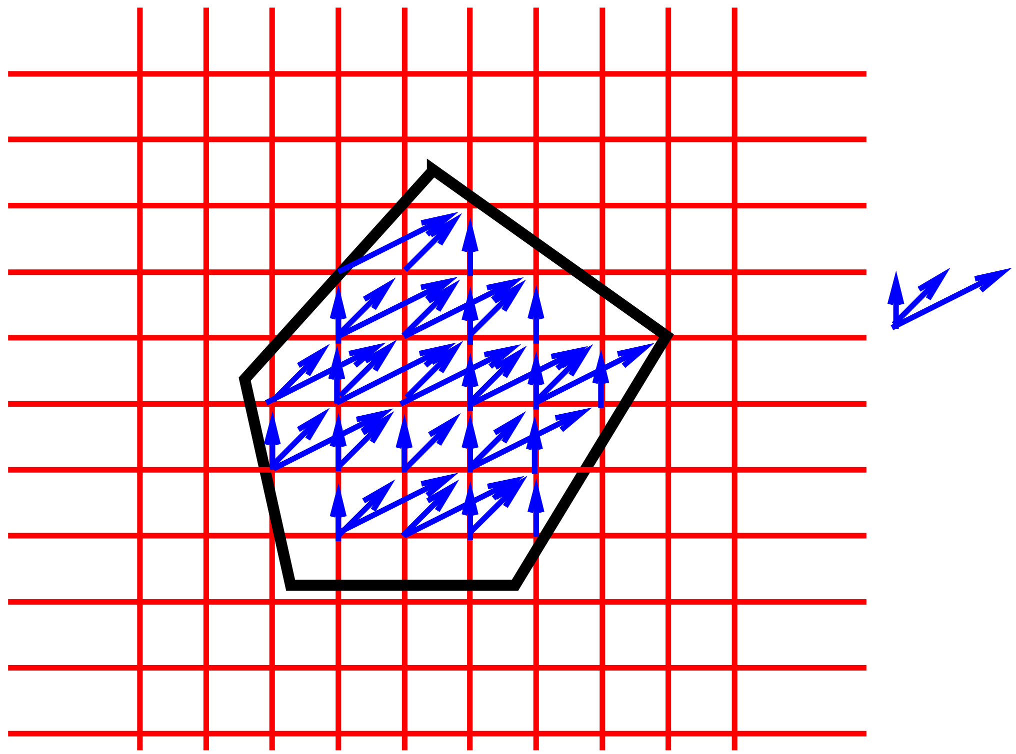
$$\pm\{[2, -1, 0], [0, -1, 2], [1, 0, -1], [1, -1, 1]\}$$

.

Theorem [J. Graver 1975] Graver bases for A can be used to solve the augmentation problem. Given $A \in \mathbb{Z}^{m\times n}$, $x \in \mathbb{N}^n$ and $c \in \mathbb{Z}^n$, either find an improving direction $g \in \mathbb{Z}^n$, namely one with $x - g \in \{y \in \mathbb{N}^n \,:\, Ay = Ax\}$ and $cg > 0$, or assert that no such g exists.



For a fixed cost vector c, visualize a Graver basis of of an integer program by creating a graph!!



Key ingredient [Schulz-Weismantel] The number of *directed* augmentation steps needed to reach optimality is polynomially bounded.

Part 4:

SOME OPEN PROBLEMS

- Open Question: Determine the exact value for the volume of the $n \times n$ Birkhoff polytope B_n. Can this volume be computed efficiently?

  For general polytopes volume computation is a $\#P$-hard problem!

  Recently, R. Canfield and B. Mckay gave a rather good asymptotic value for the volume of the Birkhoff polytope.

  Also recently, we (joint with F. Liu and R. Yoshida) provided an exact formula for the volume of B_n.

- Open Question: What are the possible number of vertices of simple $m \times n$ transportation polytopes?

sizes	Distribution of number of vertices in transportation polytopes
2×3	3 4 5 6
2×4	4 6 8 10 12
2×5	5 8 11 12 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30
3×3	9 12 15 18
3×4	16 21 24 26 27 29 31 32 34 36 37 39 40 41 42 44 45 46 48 49 50 52 53 54 56 57 58 60 61 62 63 64 66 67 68 70 71 72 74 75 76 78 80 84 90 96
4×4	108 116 124 128 136 140 144 148 152 156 160 164 168 172 176 180 184 188 192 196 200 204 208 212 216 220 224 228 232 236 240 244 248 252 256 260 264 268 272 276 280 284 288 296 300 304 312 320 340 360

We have a complete classification of **all** combinatorial types of simple $n \times m$ transportation polytopes. 5×5 is the smallest value for which we do not know all types yet! We used the theory of regular triangulations (Gelfand-Kapranov-Zelevinsky).

- Open Question: What is the distribution of number of vertices for $m \times n \times k$ non-degenerate 3-way transportation polytopes, specified by 2-margin matrices A, B, C?

- Conjecture: All integer numbers between 1 and $m + n - 1$, and only these, are realized as the diameters of $m \times n$ transportation polytopes.

- Conjecture: Are all graphs of $n \times m$ transportation polytopes hamiltonian.

  Theorem:(Bolker 1972) When n, m are relatively prime, the maximum possible number of vertices on an $m \times n$ transportation polytopes is achieved by the central transportation polytope whose marginals are the m-vector $a^* = (n, n, \ldots, n)$ and n-vector $b^* = (m, m, \ldots, m)$.

- Open Question: What is the largest possible number of vertices in a 3-way $n \times m \times k$ transportation polytope?

The YKK conjecture The $m \times n \times k$ *generalized central transportation polytope* is the 3-way transportation polytope of line sums whose 2-margins are given by the $n \times k$ matrix $U(j,k) = m$, the $m \times k$ matrix $V(i,k) = n$, and the $m \times n$ matrix $W(i,j) = k$.

Conjecture: (Yemelichev-Kovalev-Kratsov) The generalized central transportation polytope has the largest number of vertices among 3-way transportation polytopes.

FALSE!! (JD, E. Kim, F. Santos, S. Onn)