Top Ehrhart coefficients of integer partition problems

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la via dell'amore

- Goal: Count the solutions of the integer restricted partition problem:

  Given $\mathfrak{a} = [\alpha_1, \alpha_2, \ldots, \alpha_N]$ positive integers and t is a non-negative integer, we consider the counting function

$$E_{\mathfrak{a}}(t) = \#\{x : \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_N x_N = t,\ x \geq 0, \quad x_i \quad \text{integer}\}.$$

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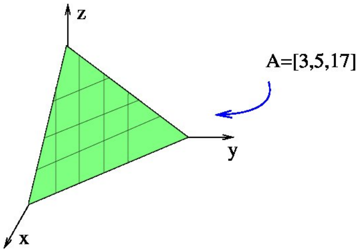
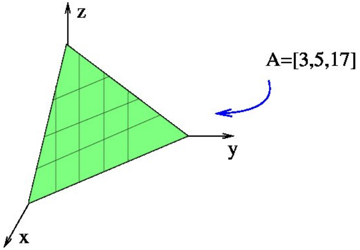
- For a given t, one wishes to decide whether $E_{\mathfrak{a}}(t) \neq 0$, but this is NP-complete and the counting problem of lattice points is $\#$P-complete.
- $E_{\mathfrak{a}}(t)$ equals number of integral points in the $(N-1)$-dimensional simplex in $\mathbf{R}^N$ $\Delta_{\mathfrak{a}} = \{\, [x_1, x_2, \dots, x_N] : x_i \geq 0, \sum_{i=1}^N \alpha_i x_i = t \,\}$ with rational vertices $\mathbf{s}_i = [0, \dots, 0, \frac{t}{\alpha_i}, 0 \dots, 0]$.



$$E_{\mathfrak{a}}(b) = \#\{(x, y, z) | 3x + 5y + 17z = b,\ x \geq 0, y \geq 0, z \geq 0\}$$

As b changes we obtain different values for $E_{\mathfrak{a}}(b)$. E.g., we see that $E_{\mathfrak{a}}(100) = 25$, $E_{\mathfrak{a}}(1110) = 2471$, etc...



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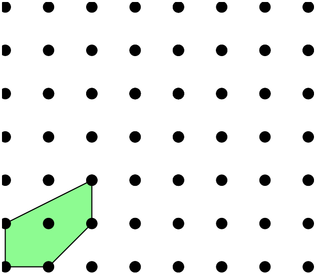
BIG QUESTION: **How does this function behave?**
Geometrically we are dilating the simplex as b grows...

For P a d-dimensional convex polytope, consider the Ehrhart function

$$E_P(n) = \#|\{a \in (nP \cap \mathbf{Z}^d)\}|$$

This is done for the lattice points in the dilation nP.



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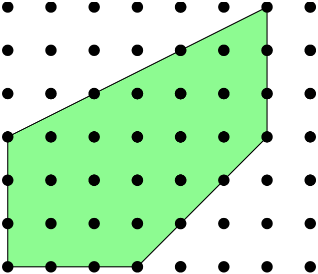

3P

Ehrhart quasipolynomials

- **Theorem** (E. Ehrhart 1962) For P a rational convex polytope on $\mathbf{R}^d$ and $n \in \mathbf{N}$ dilatation factor. Then the function
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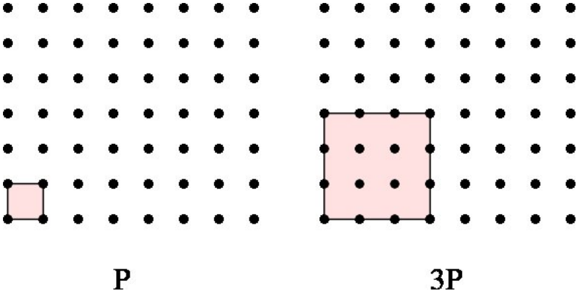
  - Its leading coefficient is the **normalized volume** of the simplex.
  - When the coordinates of the vertices of P are integers, $E_P(n)$ is a polynomial in n. It is an Ehrhart polynomial.

Example



$$i(P, n) = (n+1)^2$$

In general for a d-dimensional unit cube $i(P, n) = (n+1)^d$.

Example

Consider the Denumerant problem $\mathfrak{a} = [6, 2, 3]$.
On each of the cosets $q + 6\mathbf{Z}$, the function $E_{\mathfrak{a}}(t)$ coincides with a single polynomial $E^{[q]}(t)$!!
Here are the corresponding polynomials.

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\begin{aligned}
E^{[0]}(t) &= \tfrac{1}{72}t^2 + \tfrac{1}{4}t + 1, & E^{[1]}(t) &= \tfrac{1}{72}t^2 + \tfrac{1}{18}t - \tfrac{5}{72}, \\
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GOOD NEWS: **There are other (better!!!) ways to represent quasi-polynomials.**

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- **Theorem** When the number of variables is fixed, there is a polynomial-time algorithm to compute Ehrhart quasi-polynomials (shown as rational functions) (**follows from Barvinok 1993**).

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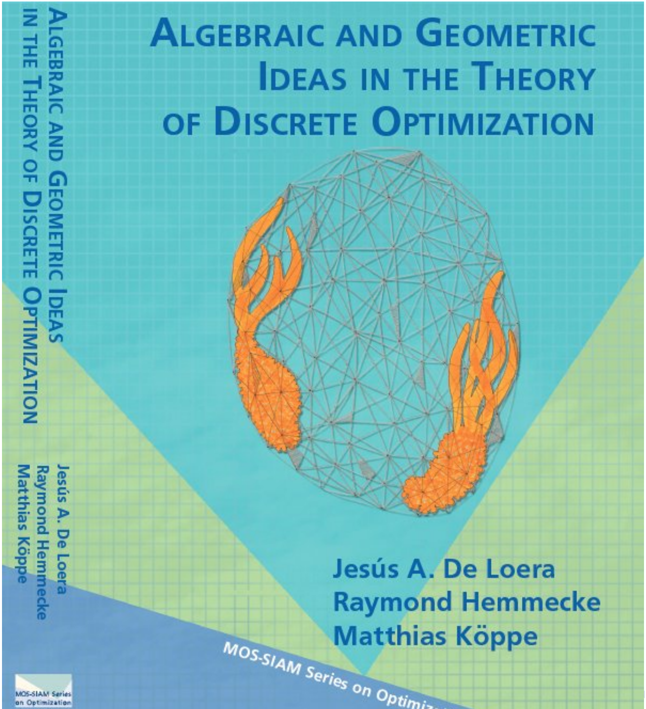
- Deep Consequences in the Theory of Optimization

And now...

THE NEW RESULTS

COMMERCIAL BREAK!!!

Are you thirsty to hear applications of Algebraic Combinatorics and Discrete Geometry?


ALGEBRAIC AND GEOMETRIC IDEAS IN THE THEORY OF DISCRETE OPTIMIZATION
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MOS-SIAM Series on Optimization

Main Theorem (2012) (Pisa Team)

There is a polynomial time algorithm for the following problem.
Fix k_0 positive integer.
Input: $\mathfrak{a} = [\alpha_1, \alpha_2, \ldots, \alpha_N]$ be a sequence of positive integers.

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Output: The $k_0 + 1$ top degree Ehrhart coefficients of the quasi-polynomial function

$$E_{\mathfrak{a}}(t) = \#\{x : \mathfrak{a}^T x = t,\, x \geq 0, \quad x \quad \text{integral}\}.$$

This will be presented as a Step polynomials.

The dimension not fixed!!!!!.

What are step polynomials?

(i) Let $\{s\} = \lceil s \rceil - s \in [0, 1)$ for $s \in \mathbf{R}$, where $\lceil s \rceil$ denotes the smallest integer larger or equal to s. The function $\{s + 1\} = \{s\}$ is a periodic function of s modulo 1.

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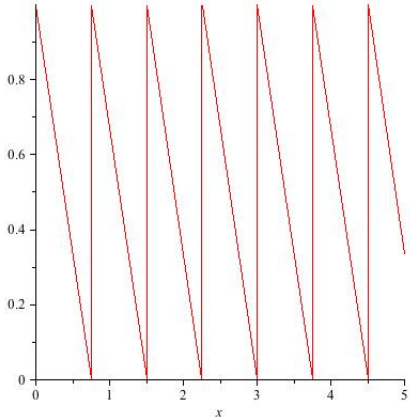

1. **Definition** A periodic function $\phi(T)$ of the form

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3. ϕ is of *period* q if all the rational numbers r_j have common denominator q.

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KEY IDEAS + METHODS

Counting through generating functions

- Given $\mathfrak{a} = [\alpha_1, \alpha_2, \ldots, \alpha_{N+1}]$. We can construct a generating function

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- Basic complex analysis: Compute the values of $E_{\mathfrak{a}}(n)$, through the poles of the complex function $F_{\mathfrak{a}}(z)$.

- NOTE: The poles of $F_{\mathfrak{a}}(z)$ are roots of unity $\mathcal{P} = \bigcup_{i=1}^{N+1} \{ \zeta \in \mathbf{C} : \zeta^{\alpha_i} = 1 \}$
- **Lemma:** Let $\mathfrak{a} = [\alpha_1, \alpha_2, \ldots, \alpha_N]$ be a list of integers with greatest common divisor equal to 1, and let

$$F(\mathfrak{a})(z) := \frac{1}{\prod_{i=1}^{N}(1 - z^{\alpha_i})}.$$

If t is a non-negative integer, then

$$E(\mathfrak{a})(t) = -\sum_{\zeta \in \mathcal{P}} \mathrm{Res}_{z=\zeta}\, z^{-t-1} F_{\mathfrak{a}}(z)\, \mathfrak{d}z \qquad (1)$$

and the ζ-term of this sum is a quasi-polynomial function of t with degree less than or equal to $p(\zeta) - 1$.

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- Given an integer $0 \leq k \leq N$, we partition the set of poles $\mathcal{P}$ in two disjoint sets:

$$\mathcal{P}_{>N-k} = \{\, \zeta : p(\zeta) > N-k \,\}, \qquad \mathcal{P}_{\leq N-k} = \{\, \zeta : p(\zeta) \leq N-k \,\}.$$

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- For computing what we need it is sufficient to compute the function $E_{\mathcal{P}_{>N-k}}(t)$.

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- For computing what we need it is sufficient to compute the function $E_{\mathcal{P}_{>N-k}}(t)$.
  The function $E_{\mathcal{P}_{\leq N-k}}(t)$ is a quasi-polynomial function of t of degree in t strictly less than $N-k$.

IDEA 2: Posets+Groups on the higher-order poles

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- the set $\mathcal{G}_{>N-k}(\mathfrak{a})$ is a set of integers stable by the operation of taking greatest common divisors. Thus, $\mathcal{G}_{>N-k}(\mathfrak{a})$ is a partially ordered set, where $f \preceq f'$ if f divides f'.

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- the set $\mathcal{G}_{>N-k}(\mathfrak{a})$ is a set of integers stable by the operation of taking greatest common divisors. Thus, $\mathcal{G}_{>N-k}(\mathfrak{a})$ is a partially ordered set, where $f \preceq f'$ if f divides f'.
- Using the group $G(f) \subset \mathbf{C}^\times$ of f-th roots of unity,

$$G(f) = \{ \zeta \in \mathbf{C} : \zeta^f = 1 \},$$

  we have thus $\mathcal{P}_{>N-k} = \bigcup_{f \in \mathcal{G}_{>N-k}(\mathfrak{a})} G(f)$. Not a disjoint union!!!

- **Lemma** Inclusion–exclusion principle says: We can write the characteristic function of $\mathcal{P}_{>N-k}$ as a linear combination of characteristic functions of the groups $G(f)$:

$$[\mathcal{P}_{>N-k}] = \sum_{f \in \mathcal{G}_{>N-k}(\mathfrak{a})} \mu(f)[G(f)],$$

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- **Lemma** If f is a positive integer define

$$E(\mathfrak{a}, f)(t) = -\sum_{\zeta^f=1} \mathrm{Res}_{z=\zeta}\, z^{-t-1} F(\mathfrak{a})(z)\, \mathfrak{d}z.$$

  Let k be a fixed integer. Then

$$E_{\mathcal{P}_{>N-k}}(t) = -\sum_{f \in \mathcal{G}_{>N-k}(\mathfrak{a})} \mu(f) E(\mathfrak{a}, f)(t).$$

IDEA 3: Use polyhedral cones and special lattices!

Use convex geometry to compute

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NEWS Very Nice paper with experiments coming soon !!!