Easy-to-Explain but Hard-to-Solve Problems About Convex Polytopes

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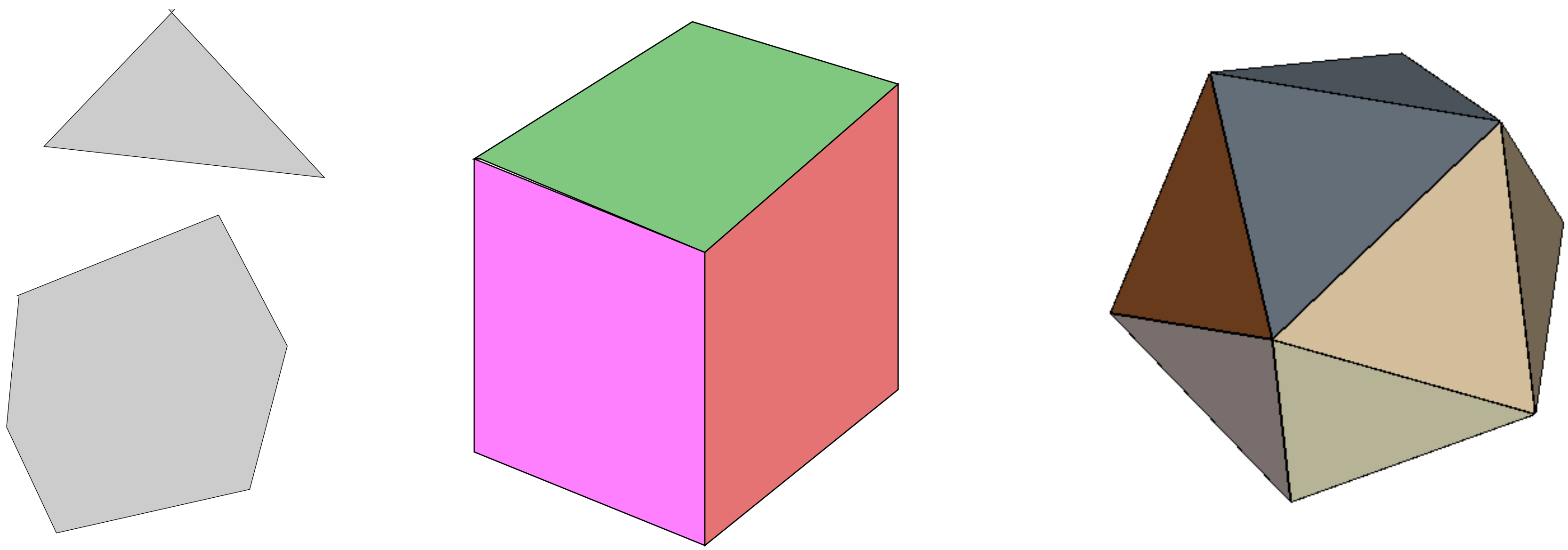

My personal crusade to show there mathematics is growing beyond calculus!

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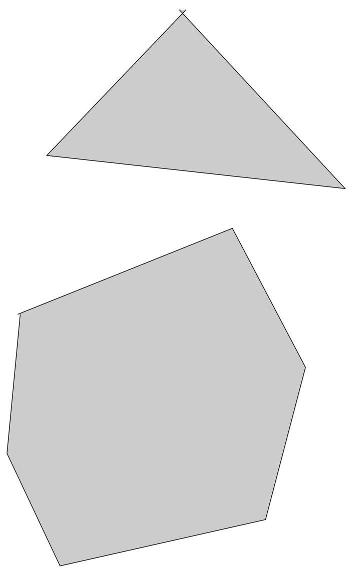
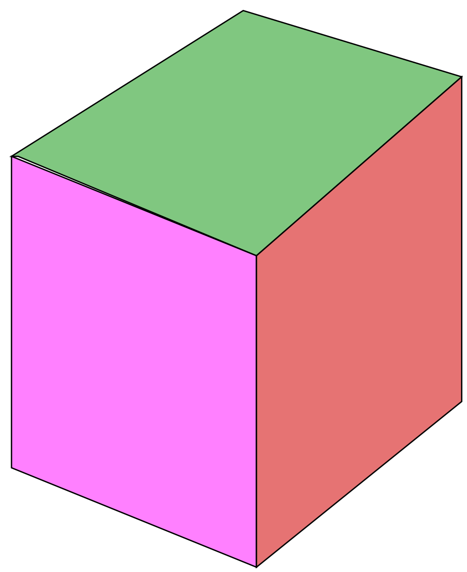
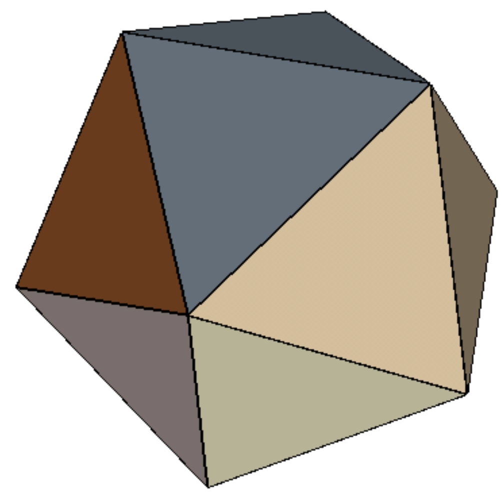
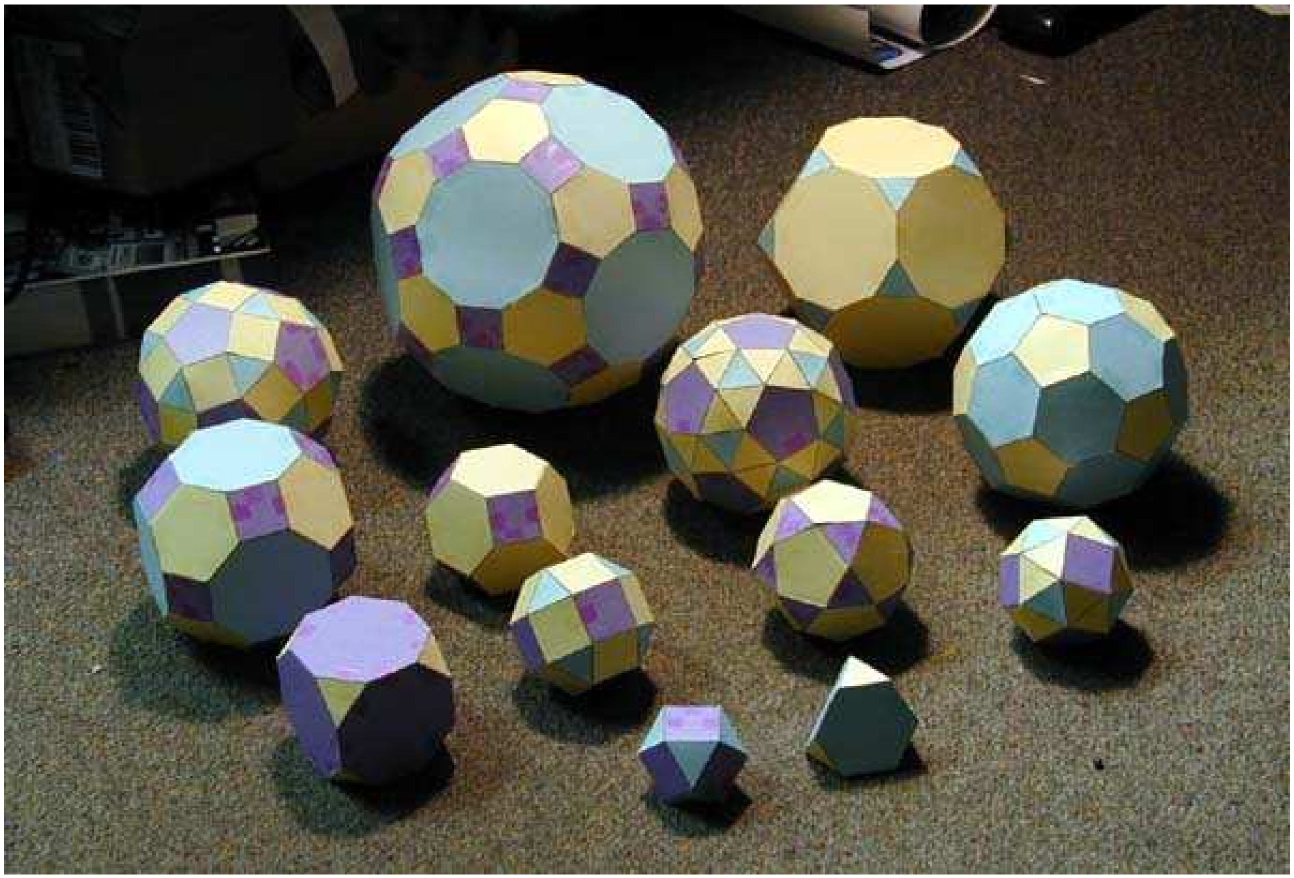




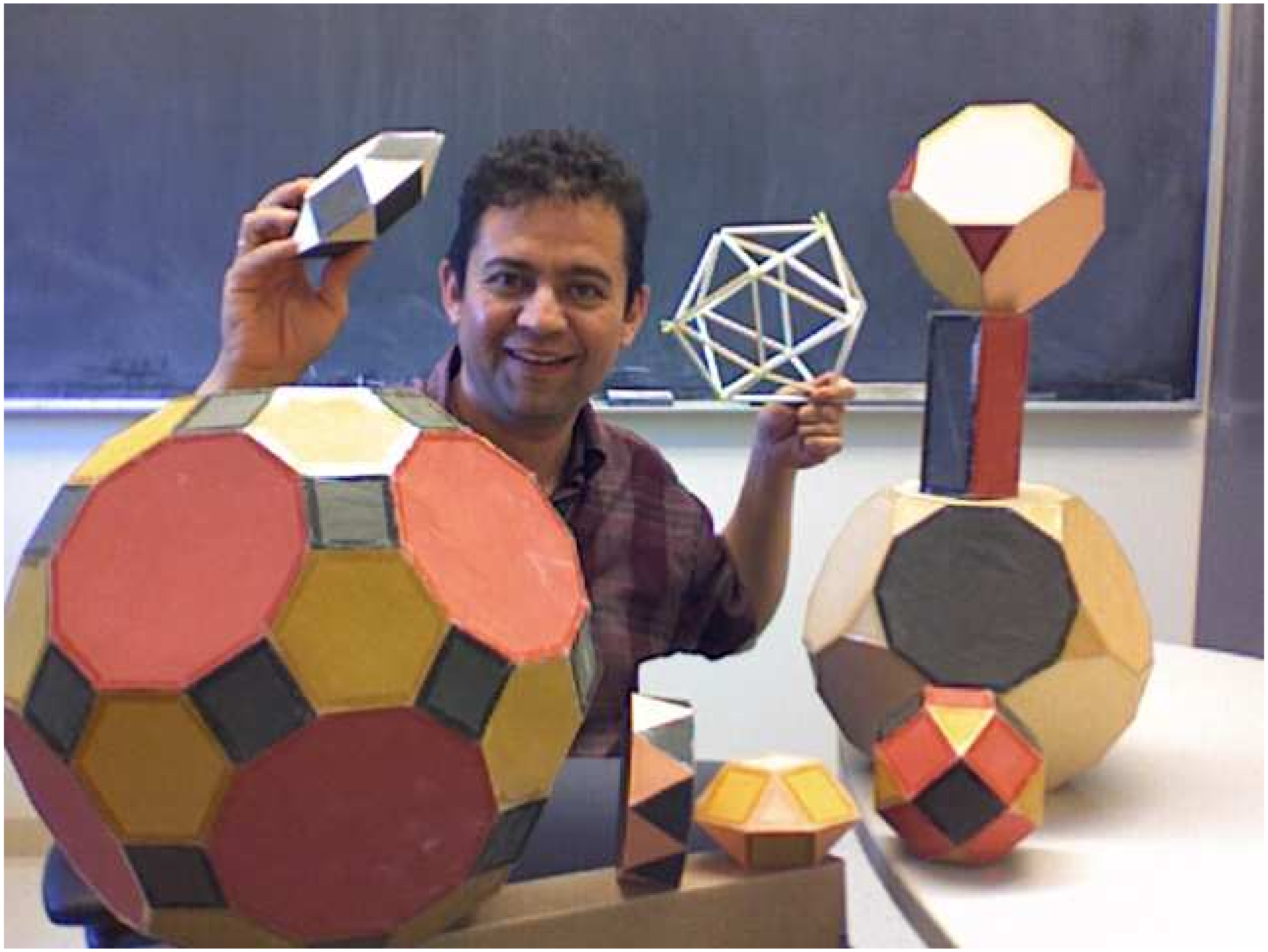


What is a Convex Polytope?

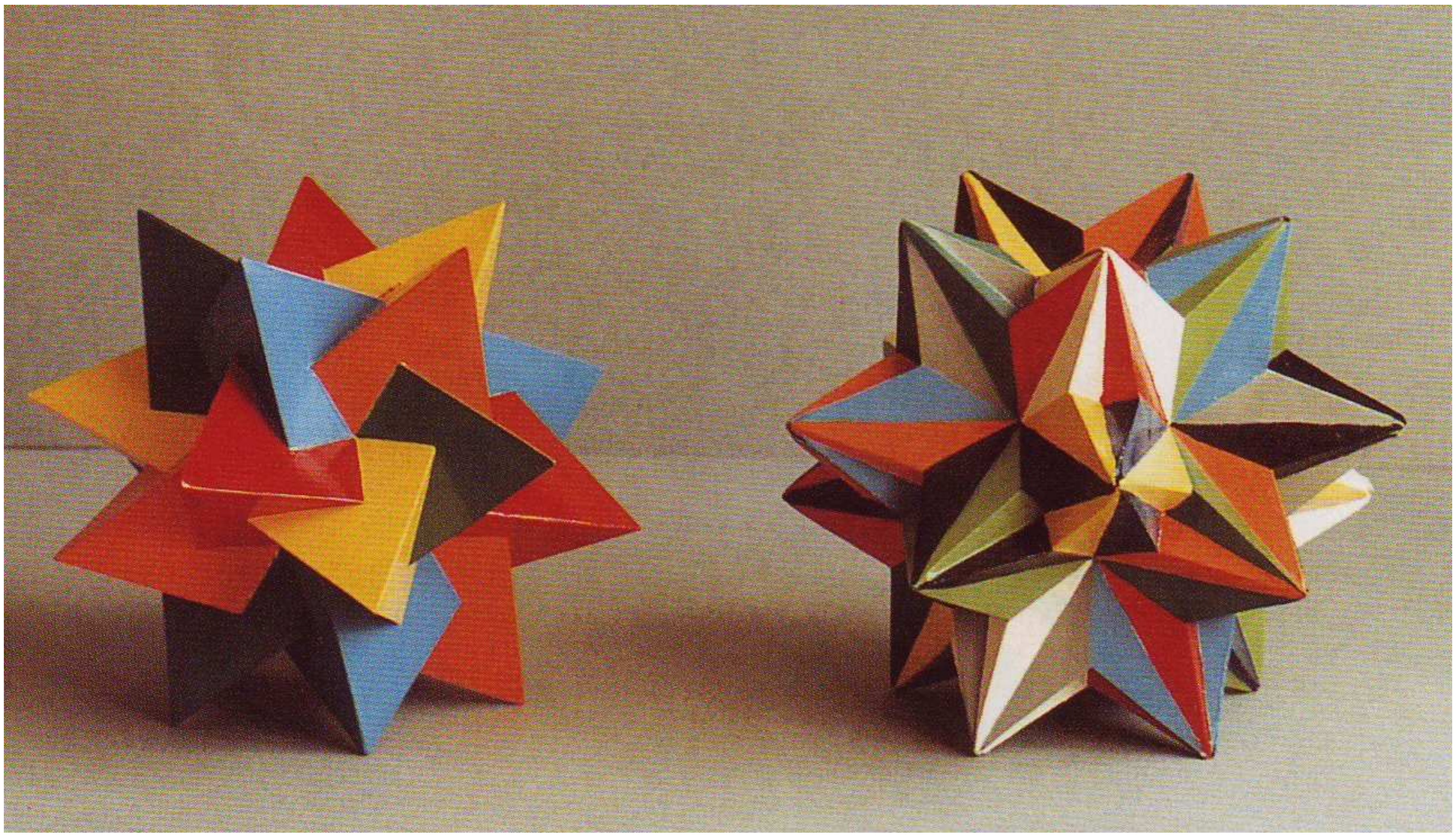
Well,something like these...



or like these

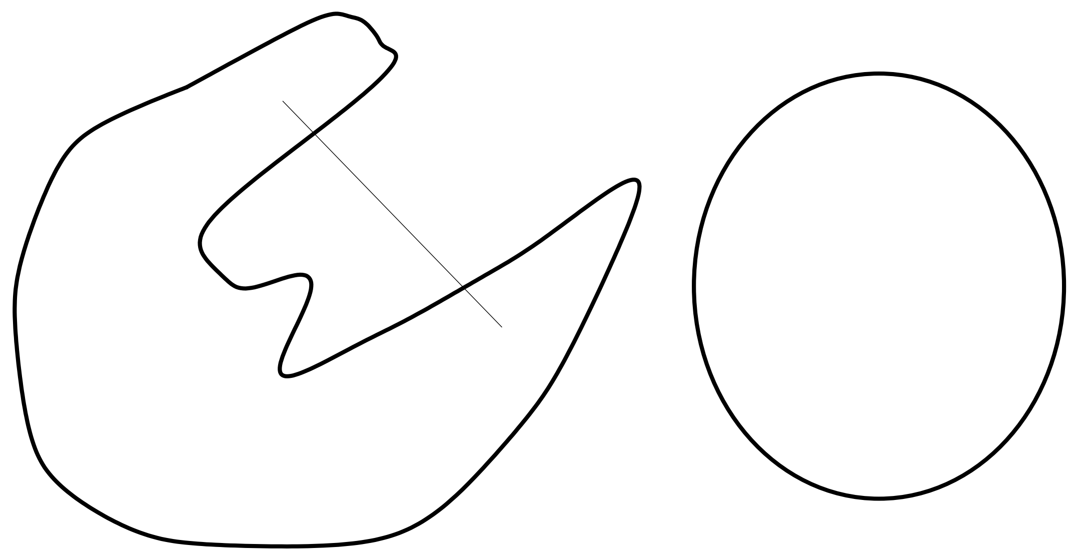


But NOT quite like these!

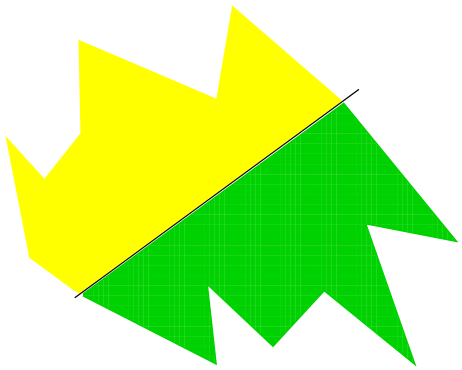


A definition PLEASE!

The word CONVEX stands for sets that contain any line segment joining two of its points:



A (hyper)plane divides spaces into two *half-spaces*. Half-spaces are convex sets! Intersection of convex sets is a convex set!



Formally a half-space is a *linear inequality*:

$$a_1x_1 + a_2x_2 + \ldots + a_dx_d \leq b$$

Definition: A polytope is a bounded subset of Euclidean space that results as the intersection of finitely many half-spaces.

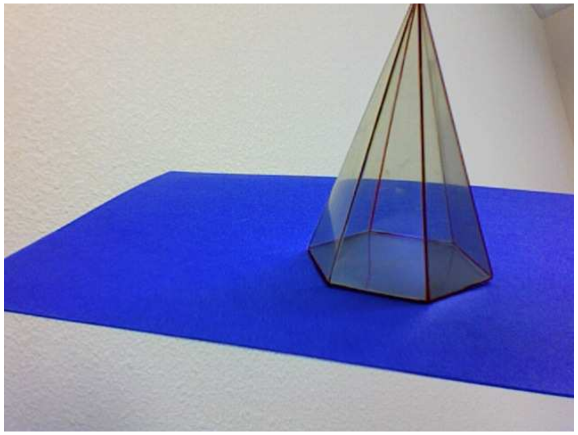
An algebraic formulation for polytopes

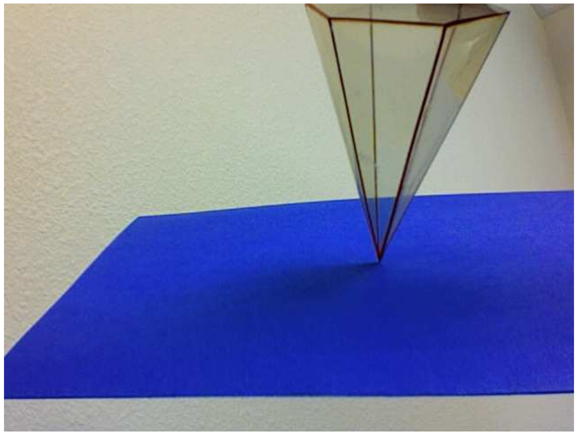
A polytope has also an algebraic representation as the set of solutions of a system of linear inequalities:

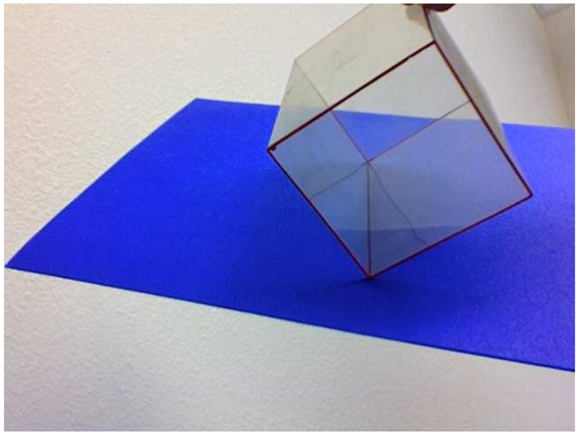
$$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,d}x_d \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,d}x_d \leq b_2$$
$$\vdots$$
$$a_{k,1}x_1 + a_{k,2}x_2 + \ldots + a_{k,d}x_d \leq b_k$$

Note: This includes the possibility of using some linear equalities as well as inequalities!! Polytopes represented by sets of the form $\{x | Ax = b, \ x \geq 0\}$, for suitable matrix A, and vector b.

Faces of Polytopes







Some Numeric Properties of Polyhedra

- **Euler's formula** $V - E + F = 2$, relates the number of vertices V, edges E, and facets F of a 3-dimensional polytope.

Given a convex 3-polytope P, if $f_i(P)$ the number of i-dimensional faces. There is one vector $(f_0(P), f_1(P), f_2(P))$. that counts faces, the f-**vector** of P.

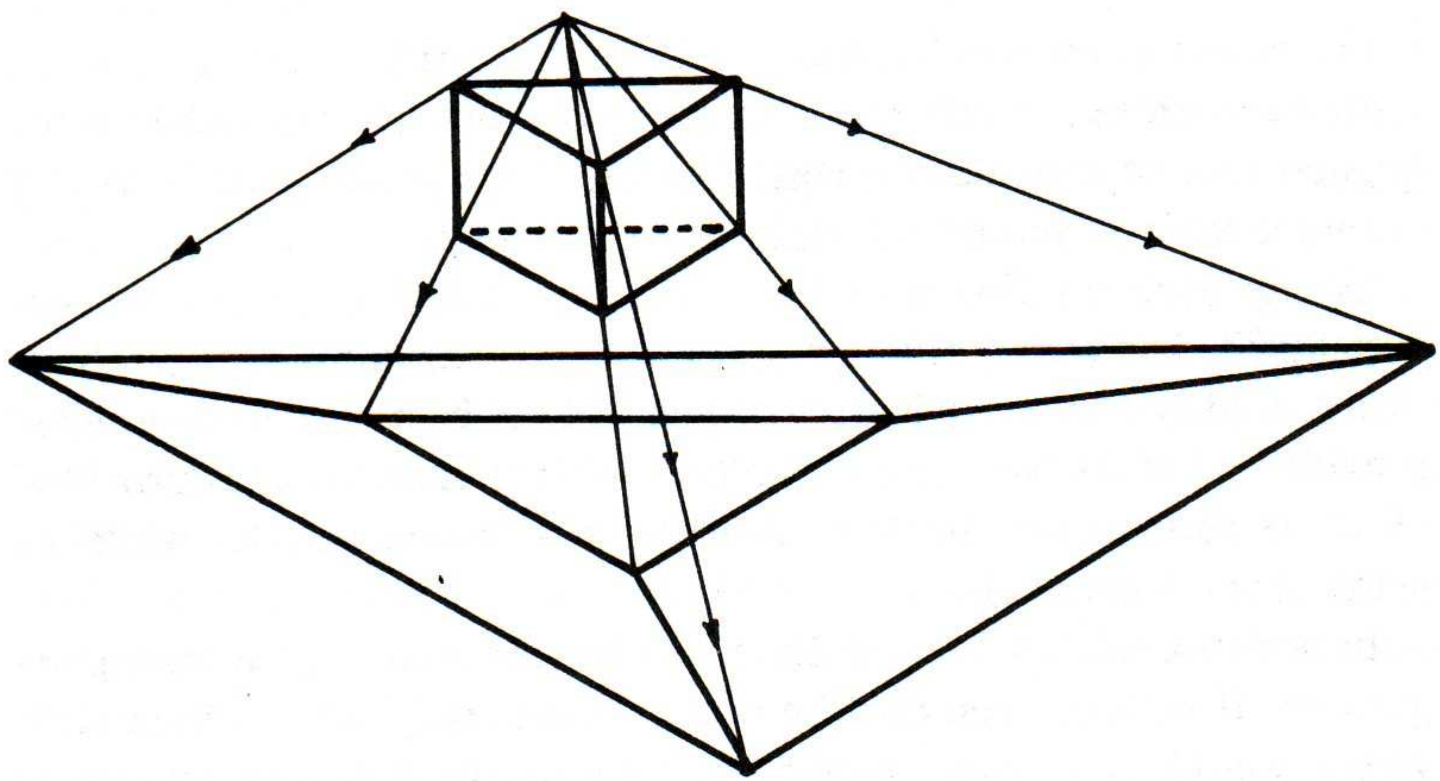
- **Theorem** (Steinitz 1906) A vector of non-negative integers $(f_0(P), f_1(P), f_2(P)) \in \mathbb{Z}^3$ is a the f-vector of a 3-dimensional polytope if and only if

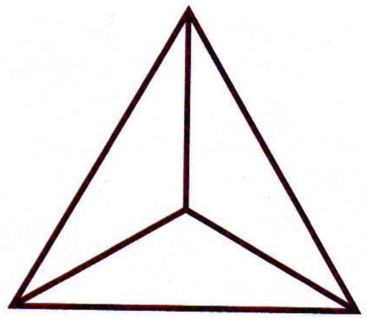
1. $f_0(P) - f_1(P) + f_2(P) = 2$
2. $2f_1(P) \geq 3f_0(P)$
3. $2f_1(P) \geq 3f_2(P)$

- OPEN PROBLEM 1: Can one find similar conditions characterizing f-vectors of 4-dimensional polytopes?

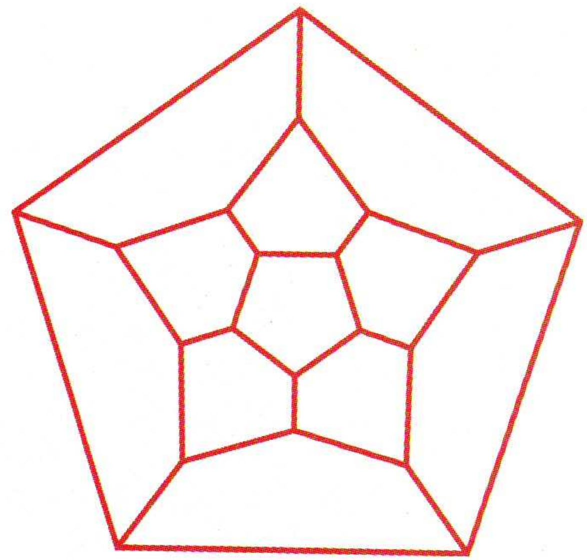
In this case the vectors have 4 components (f_0, f_1, f_2, f_3).

Ways to Visualize Polytopes

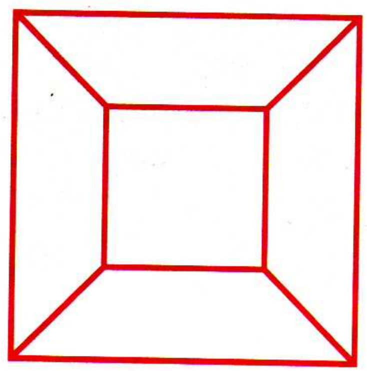




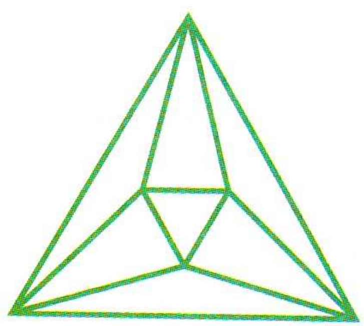
Tetrahedron



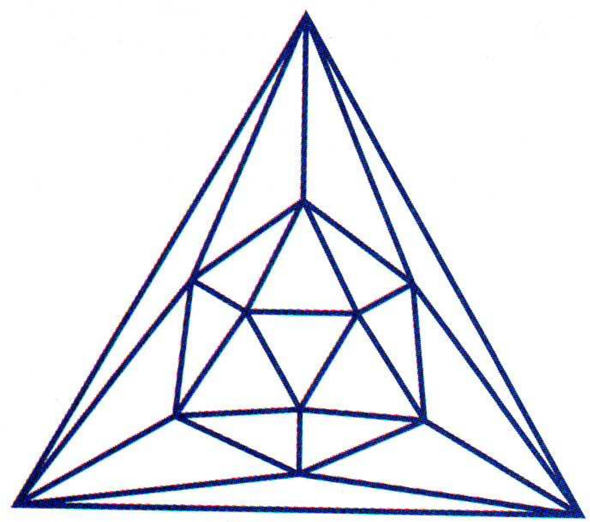
Dodecahedron



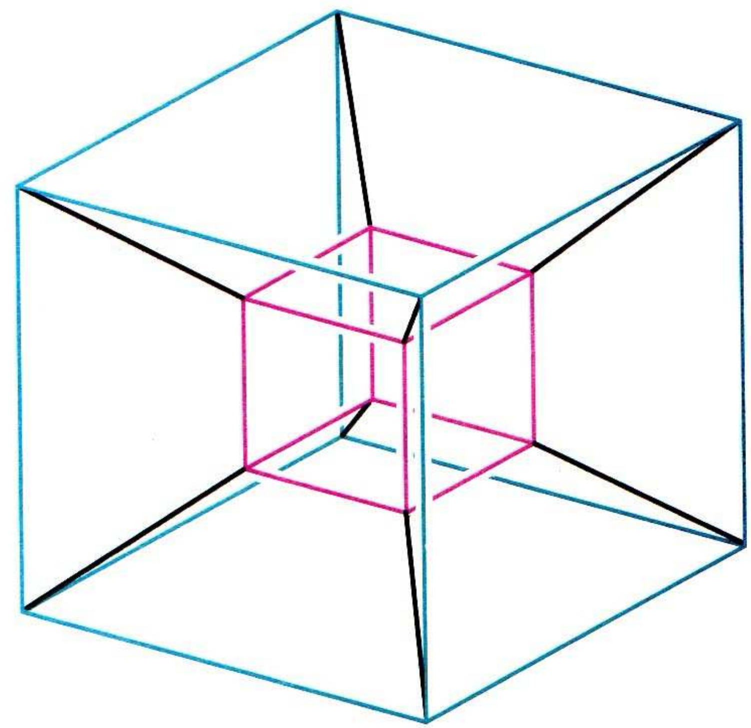
Cube



Octahedron



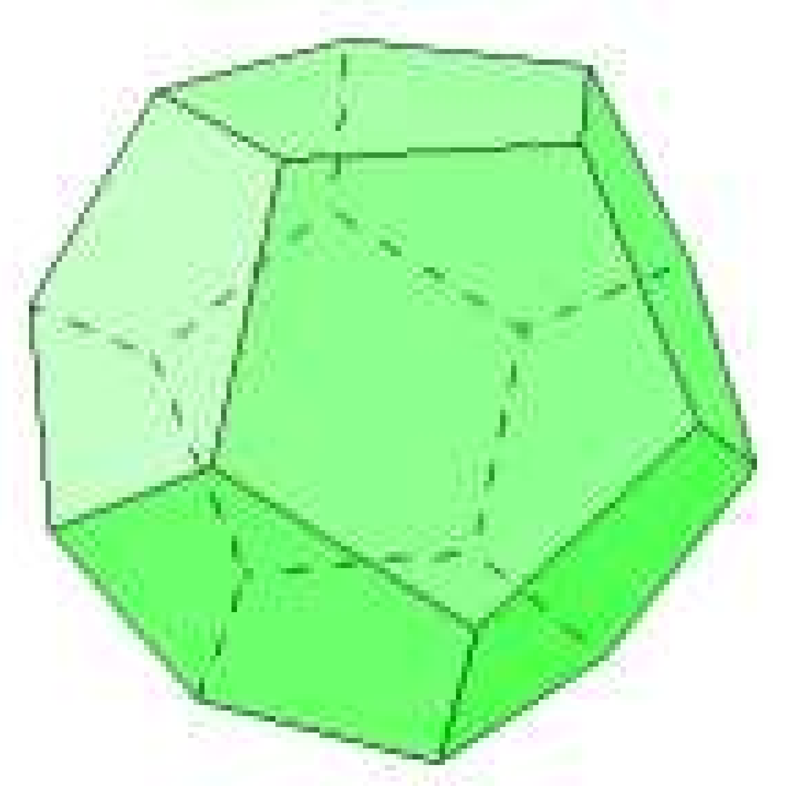
Icosahedron

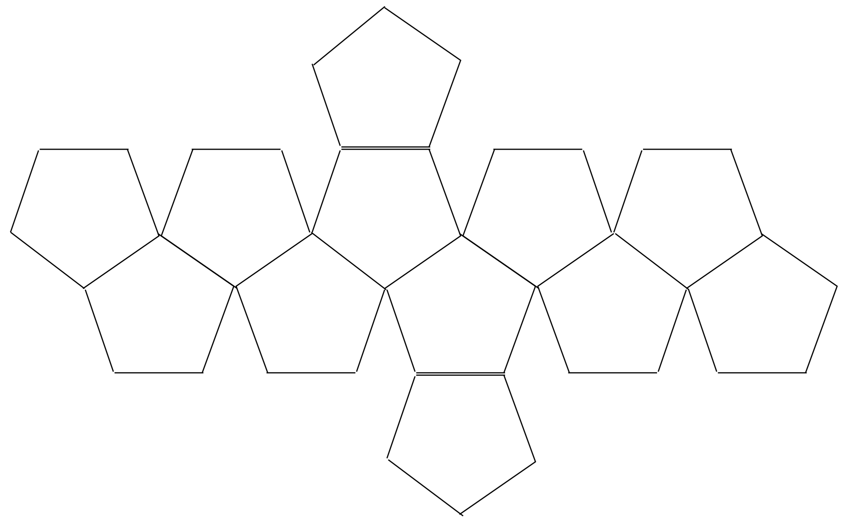


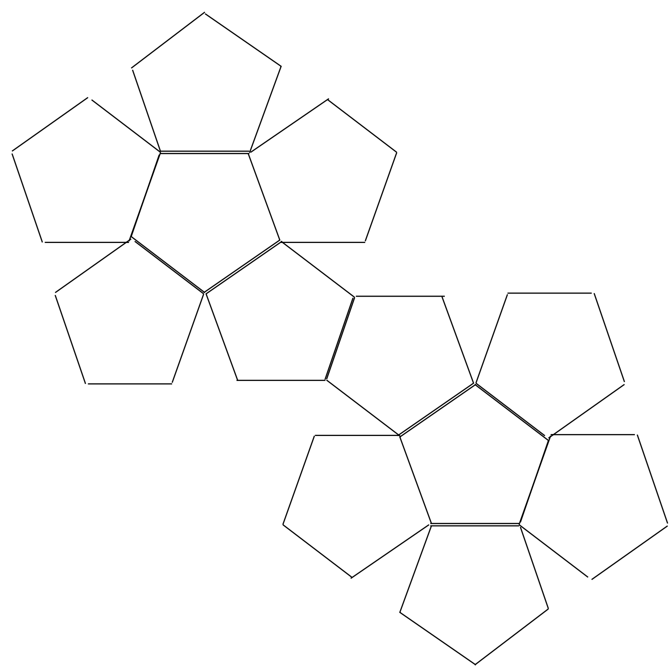
The central projection of a hypercube from four-space to three-space appears as a cube within a cube.

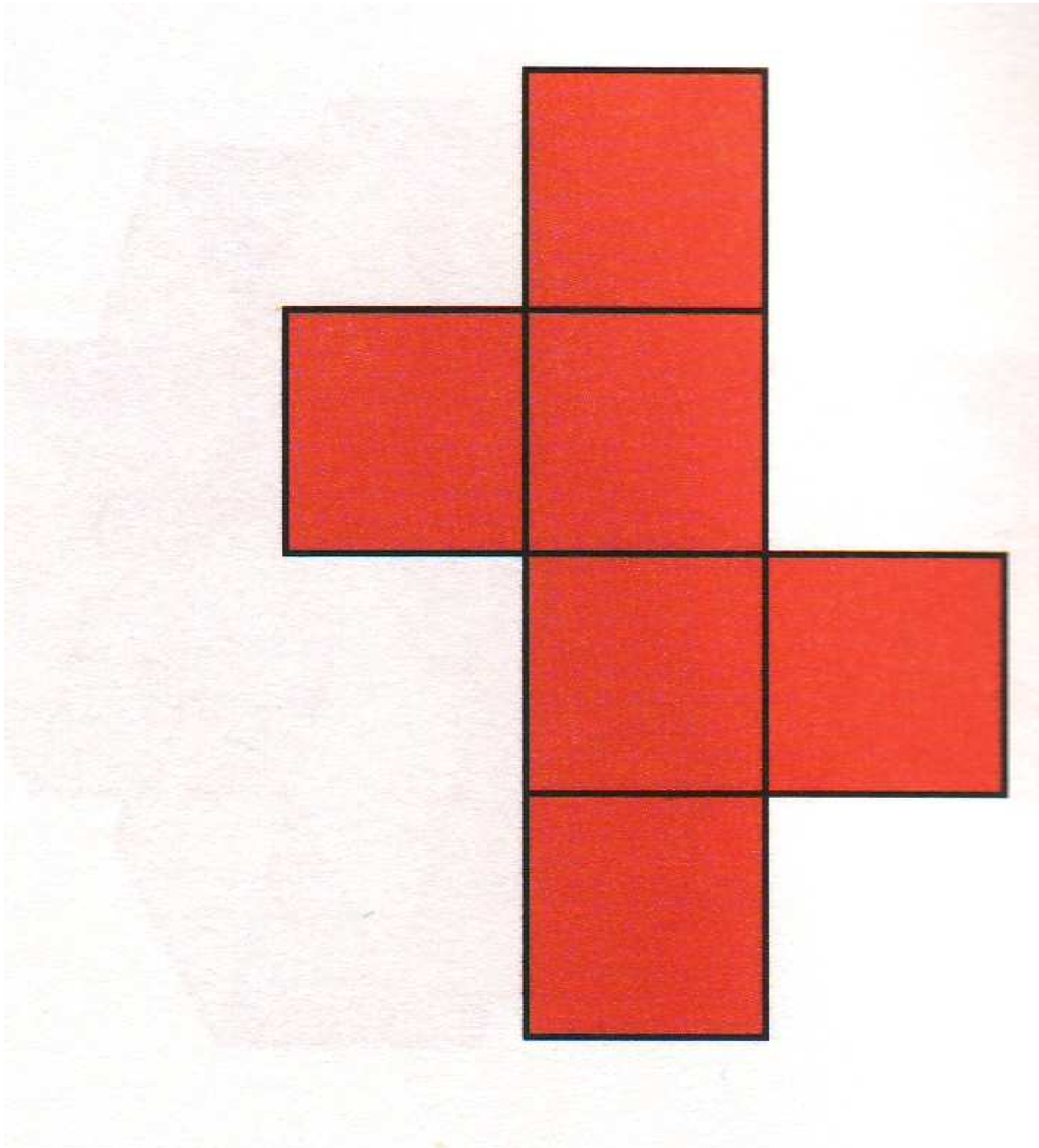
Unfolding Polyhedra

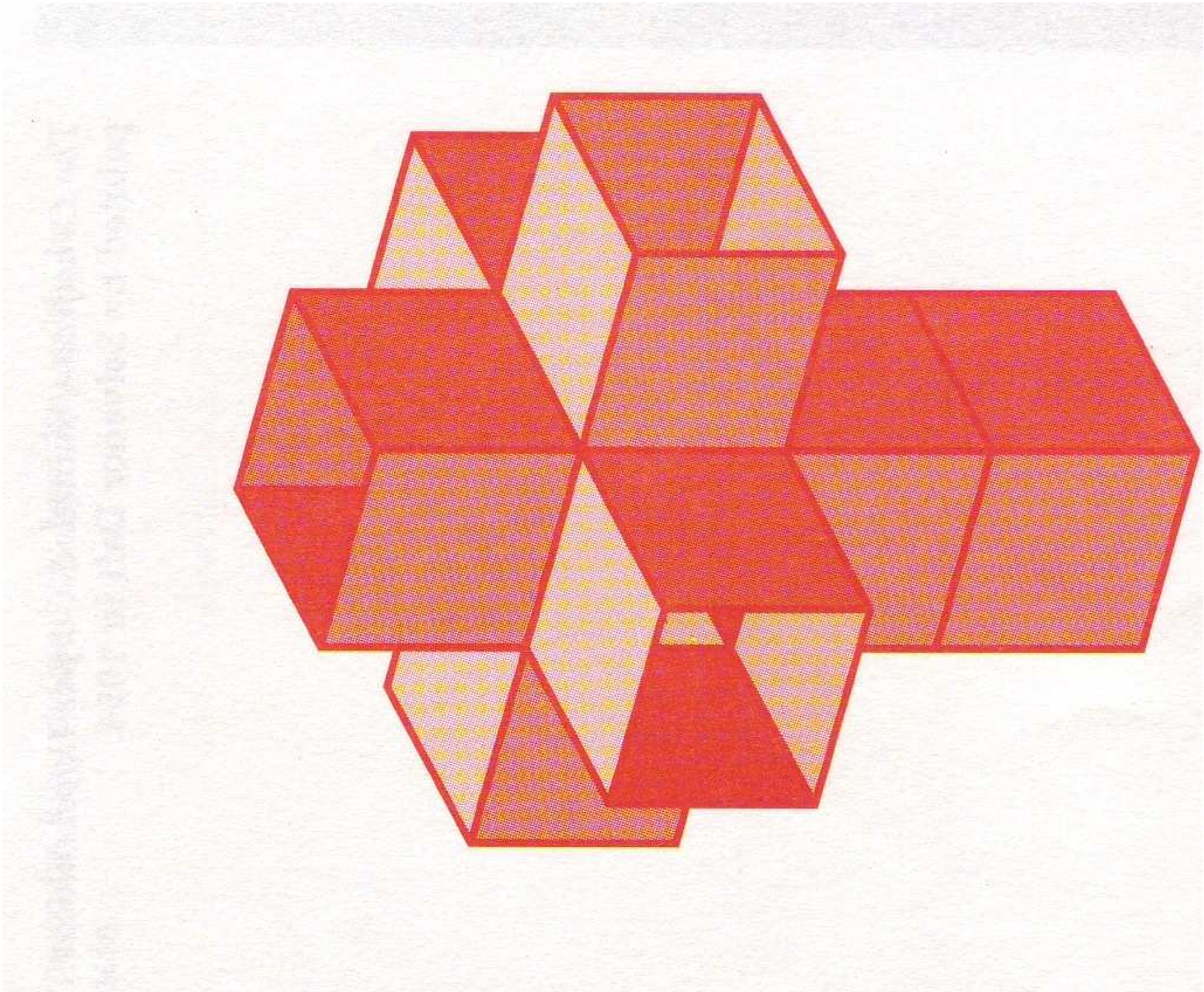
What happens if we use scissors and cut along the edges of a polyhedron? What happens to a dodecahedron?

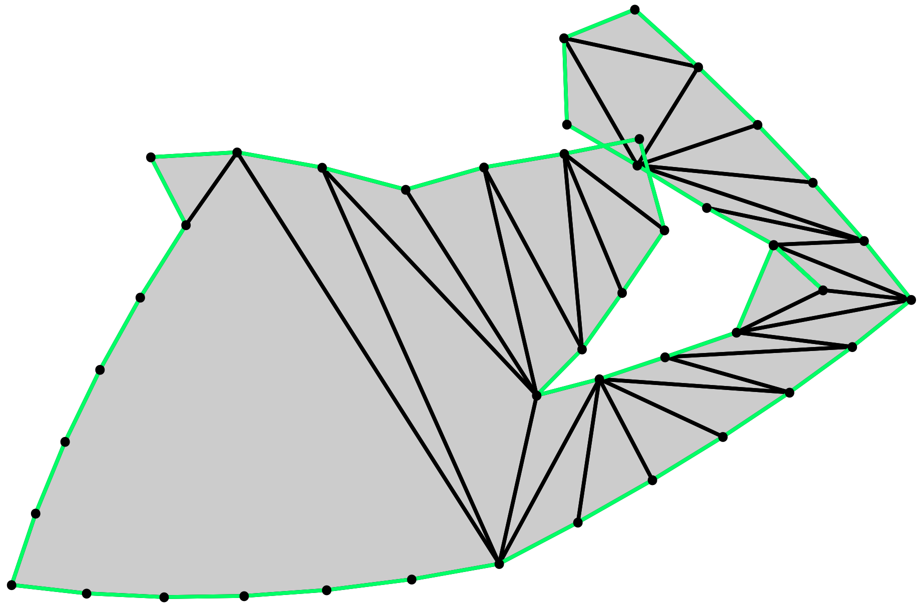


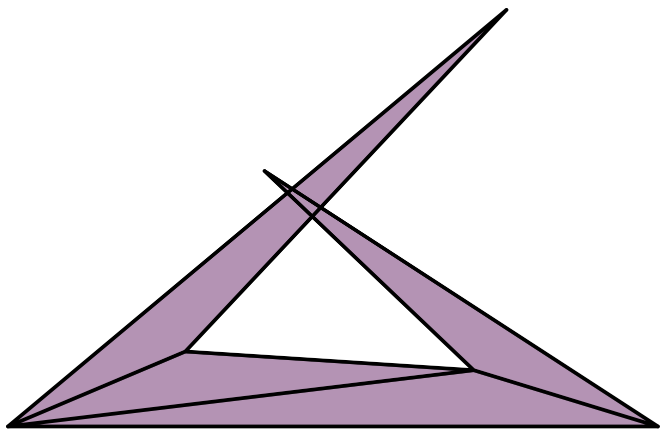








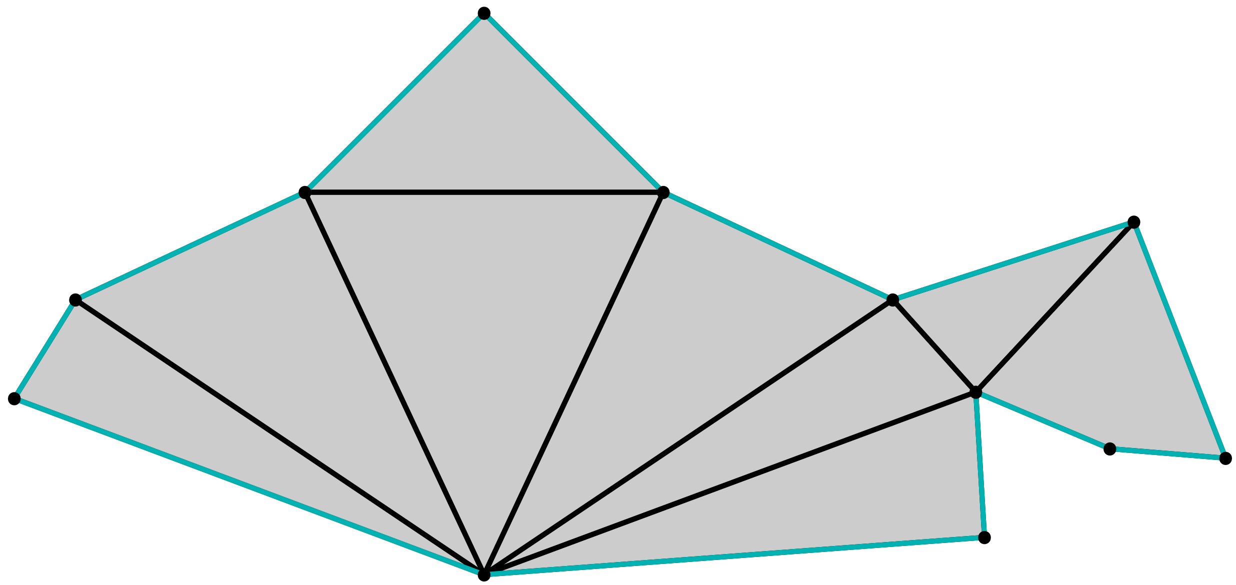


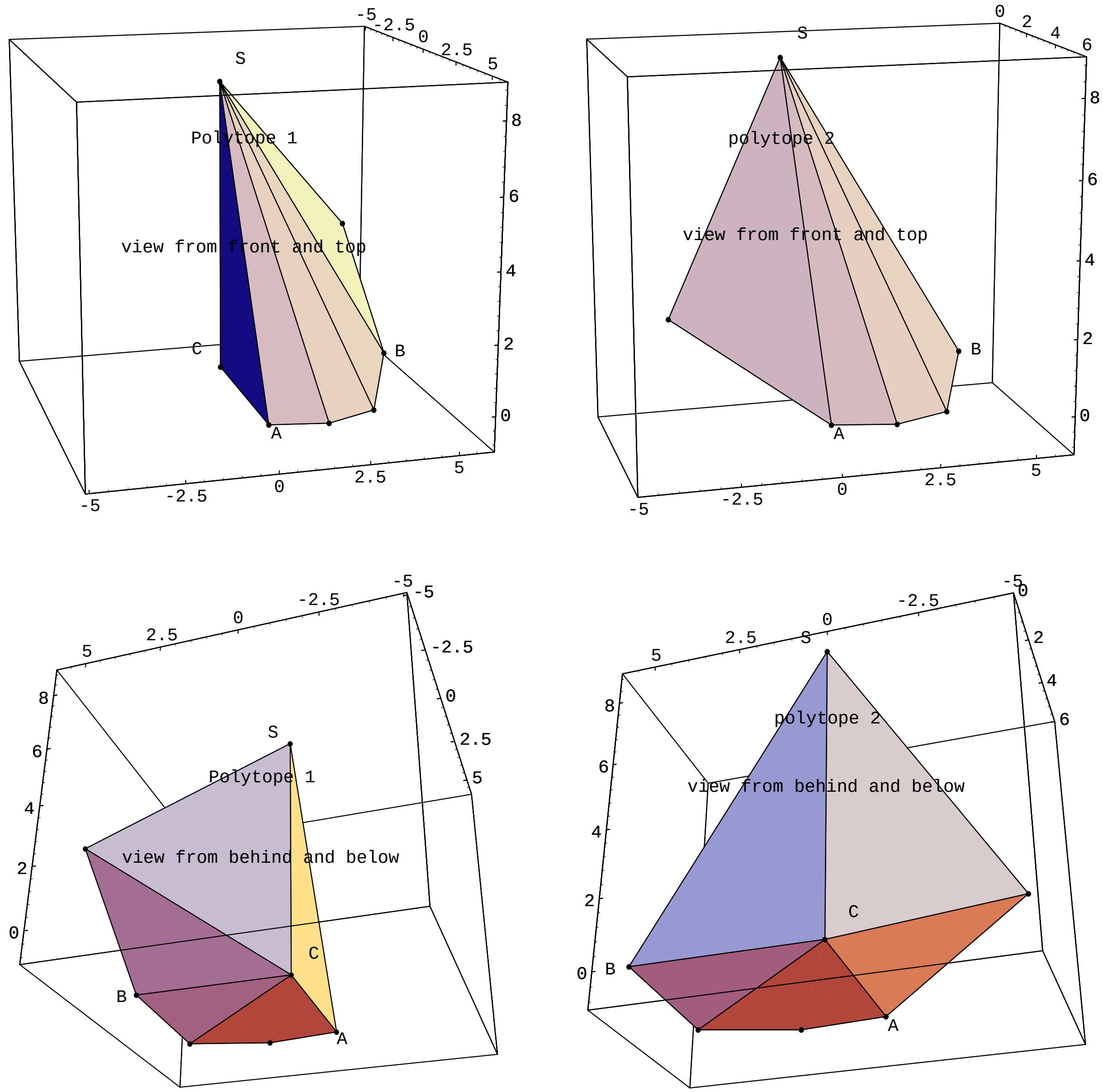


Open Problem 2: Can one always find an unfolding that has no self-overlappings?

A Challenge to intuition

Question: Is there always a single way to glue together an unfolding to reconstruct a polyhedron?




Polytope 1
view from front and top
S
C
B
A
polytope 2
view from front and top
S
B
A
Polytope 1
view from behind and below
S
C
B
A
polytope 2
view from behind and below
S
C
B
A

Linear Programming: Polytopes are useful!!

You may not know it but, We all need to solve the **Linear Programming Problems:**

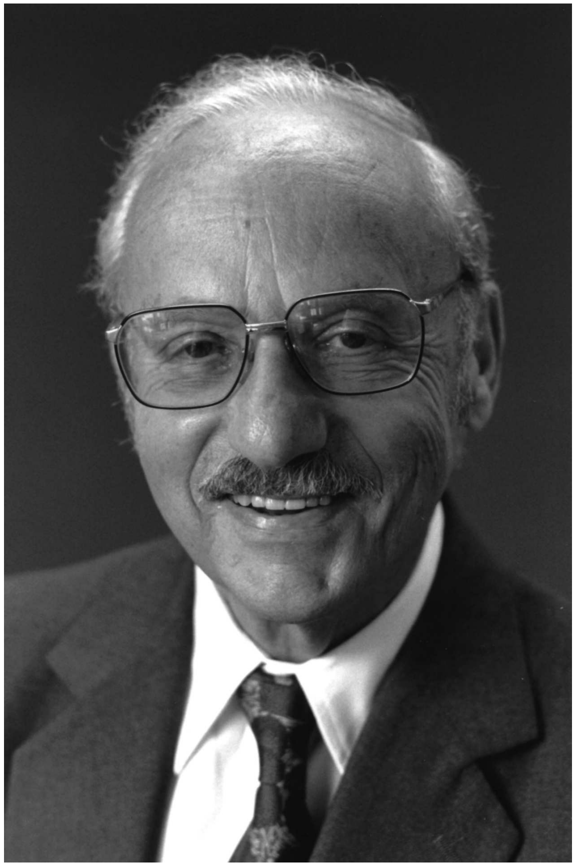
maximize $C_1x_1 + C_2x_2 + \ldots + C_dx_d$

among all $x_1, x_2, \ldots, x_d,$ satisfying:

$$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,d}x_d \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,d}x_d \leq b_2$$
$$\vdots$$
$$a_{k,1}x_1 + a_{k,2}x_2 + \ldots + a_{k,d}x_d \leq b_k$$

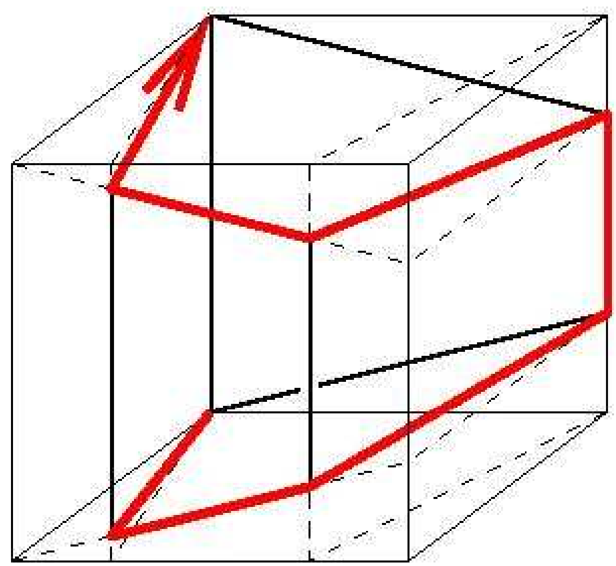
The Simplex Method

George Dantzig, inventor of the simplex algorithm



The simplex method

- **Lemma:** A vertex of the polytope is always an optimal solution for a linear program. We need to find a special vertex of the polytope!
- The simplex method **walks** along the graph of the polytope, each time moving to a better and better cost!



Hirsch Conjecture

- Performance of the simplex method depends on the diameter of the graph of the polytope: largest distance between any pair of nodes.

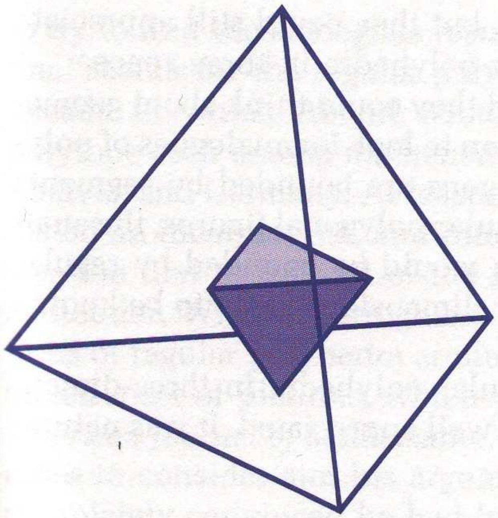
- Open Problem 3: (the Hirsch conjecture) The diameter of a polytope P is at most $\#$ of facets$(P) - dim(P)$.

- It has been open for 40 years now! It is known to be true in many instances, e.g. for polytopes with $0/1$ vertices.

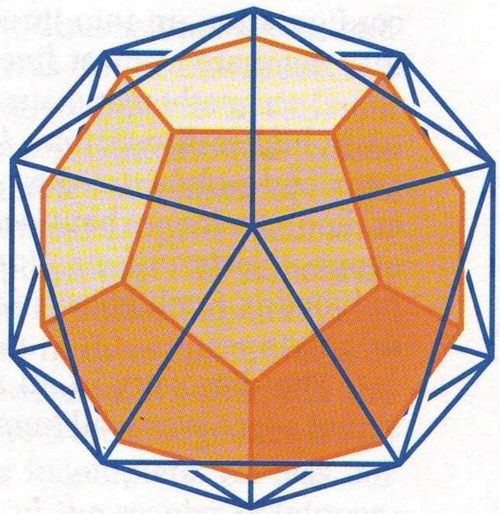
- It is best possible tight bound for general polytopes. Best known general bound is

$$\frac{2^{dim(P)-2}}{3}(\#\text{ facets of } P - dim(P) + 5/2).$$

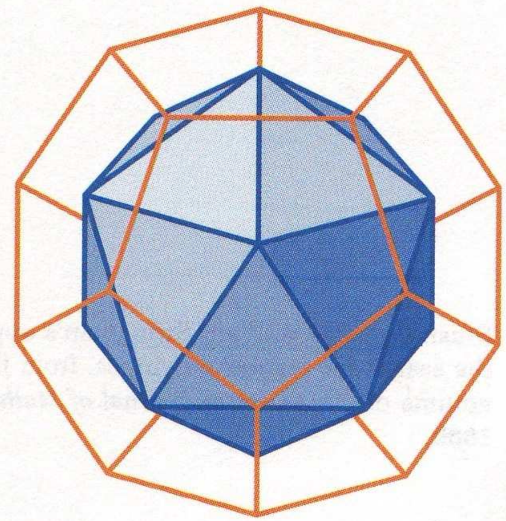
Duality



The self-dual tetrahedron.



The dodecahedron is dual to the icosahedron.

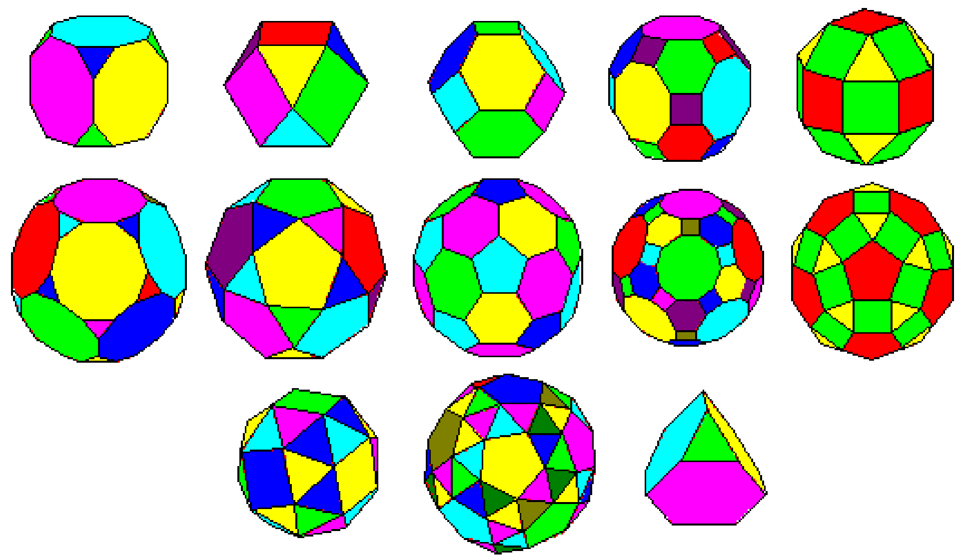


The icosahedron is dual to the dodecahedron.

Problems about faces can also be rephrased as problems about vertices!

Coloring Faces/Vertices

Given a 3-dimensional polyhedron we want to color its faces or vertices, with the minimum number of colors possible, in such a way that two adjacent elements have different colors.

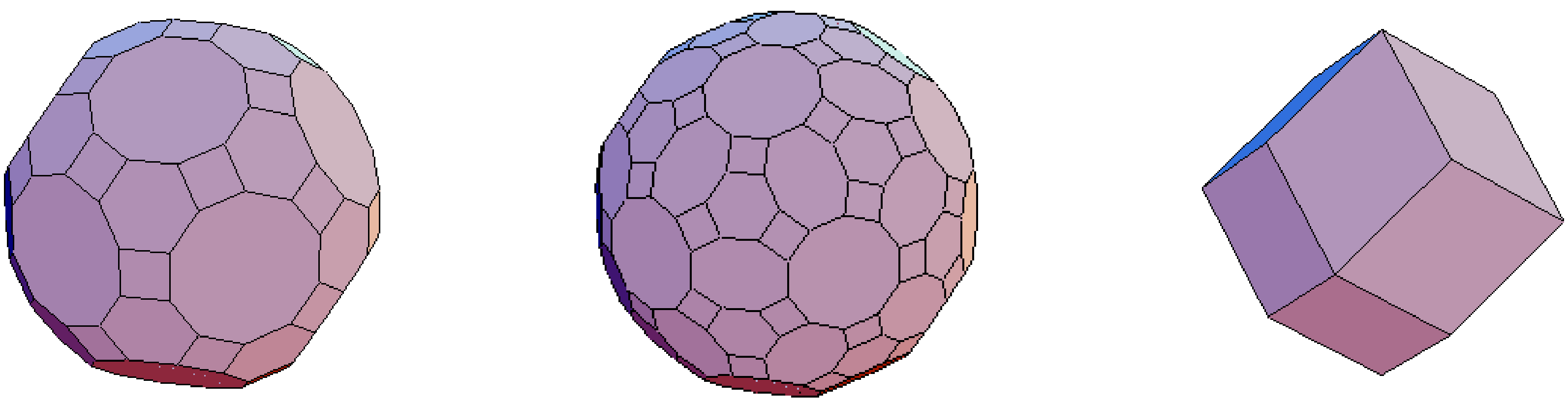


Theorem[The four-color theorem] Four colors always suffice!

Zonotopes

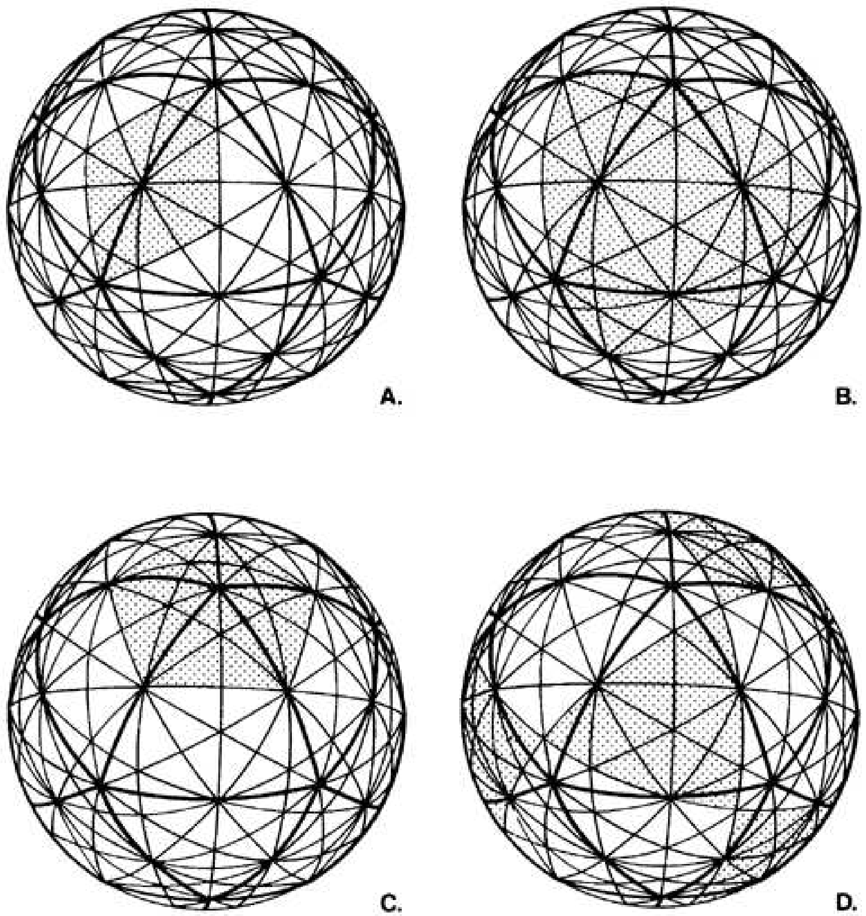
Question: Are there special families of 3-colorable 3-polytopes?

A zonotope is the linear projection of a k-dimensional cube.

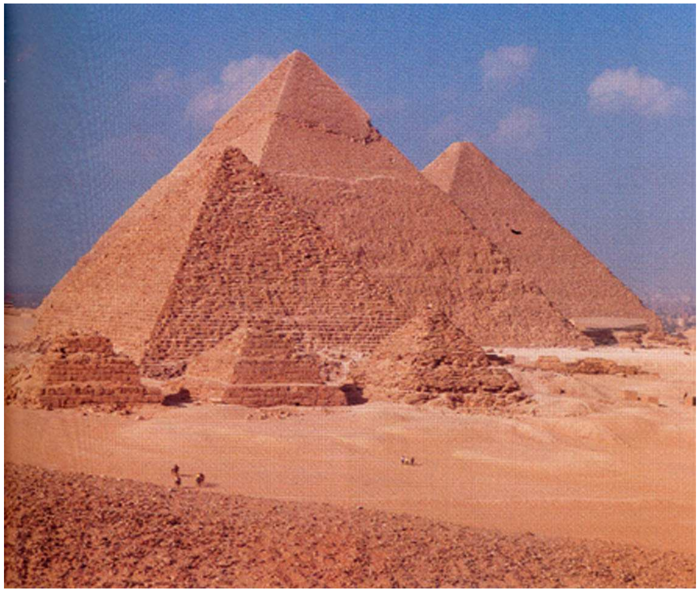


Open Problem 4 Are the vertices of the graph of 3-zonotopes always 3-colorable.

Equivalent to: The regions of any great-circle arrangement can be colored with 3-colors!

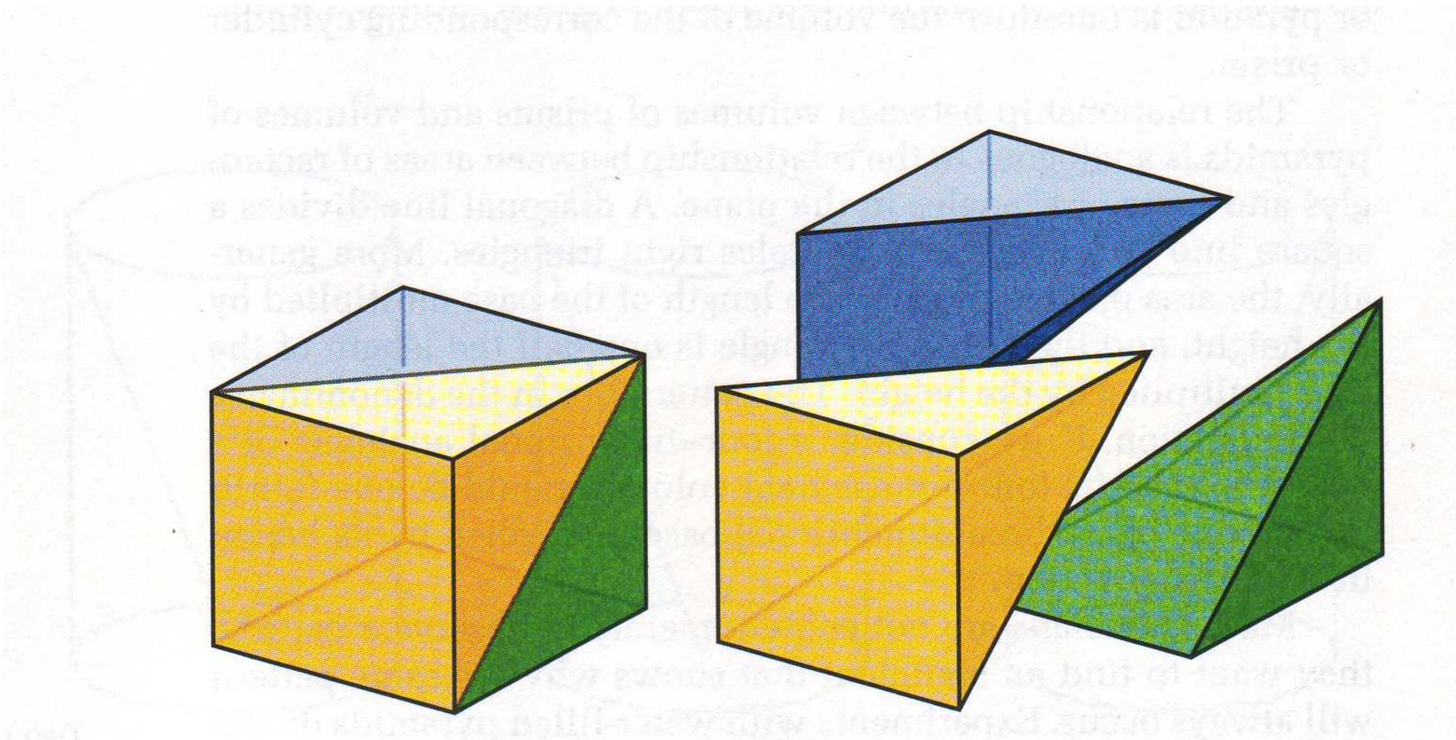


What is the volume of a Polytope?

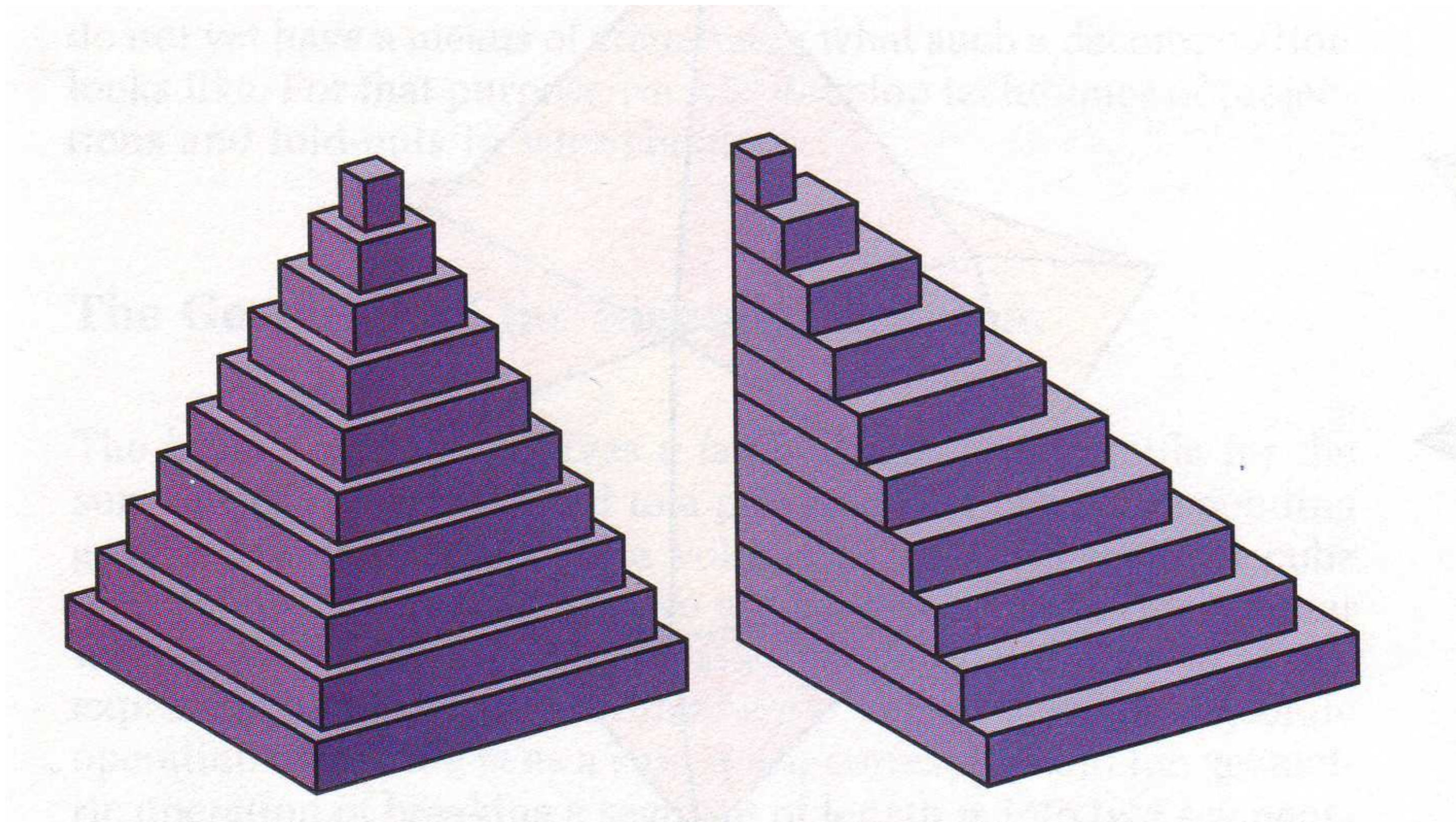


$$\text{volume of egyptian pyramid} = \frac{1}{3}(\text{area of base}) \times \text{height}$$

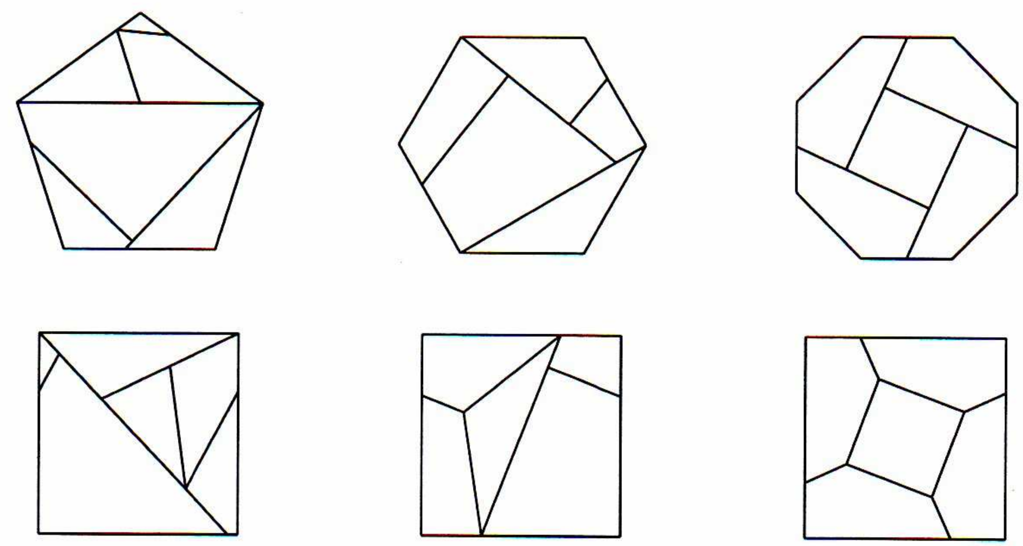
Easy and pretty in some cases...



But general proofs seem to rely on an infinite process!



But not in dimension two!



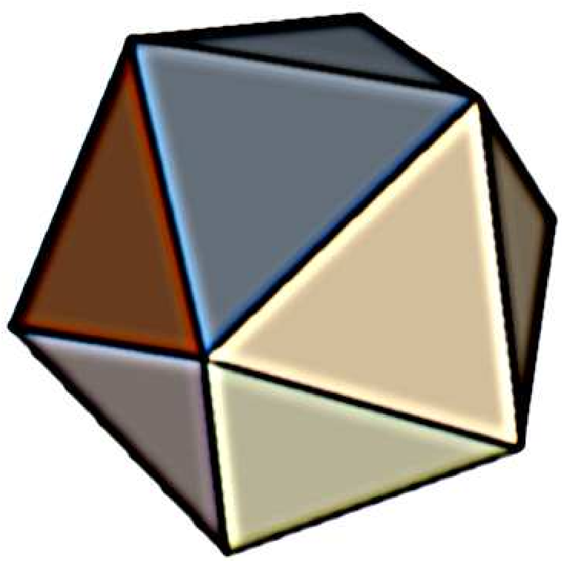
Polygons of the same area are equidecomposable, i.e., one can be partitioned into pieces that can be reassembled into the other.

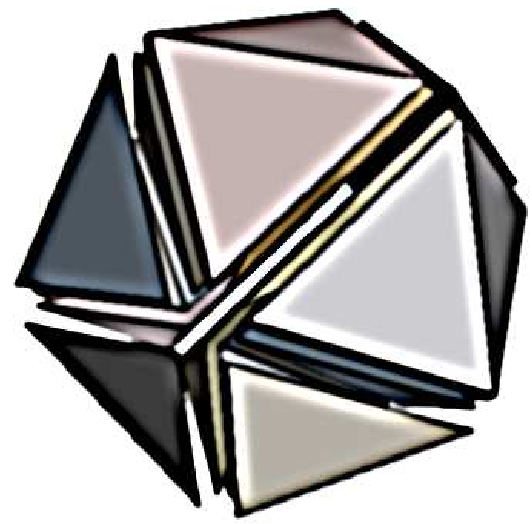
Hilbert's Third Problem

Are any two convex 3-dimensional polytopes of the same volume equidecomposable?

Enough to know how to do it for tetrahedra!

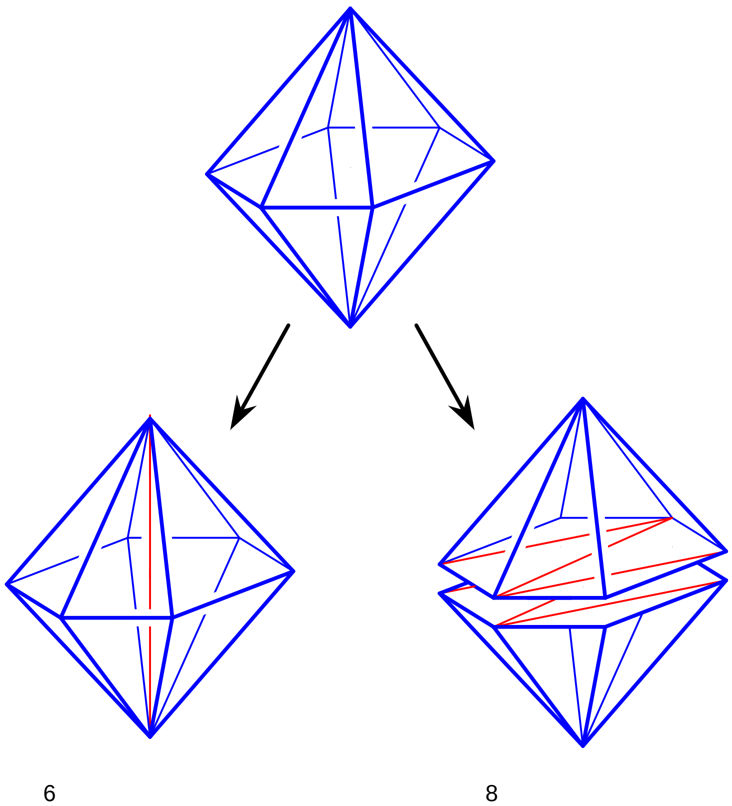
To compute the volume of a polyhedron divide it as a disjoint union of tetrahedra. Calculate volume for each tetrahedron (an easy determinant) and then add them up!





The size of a triangulation

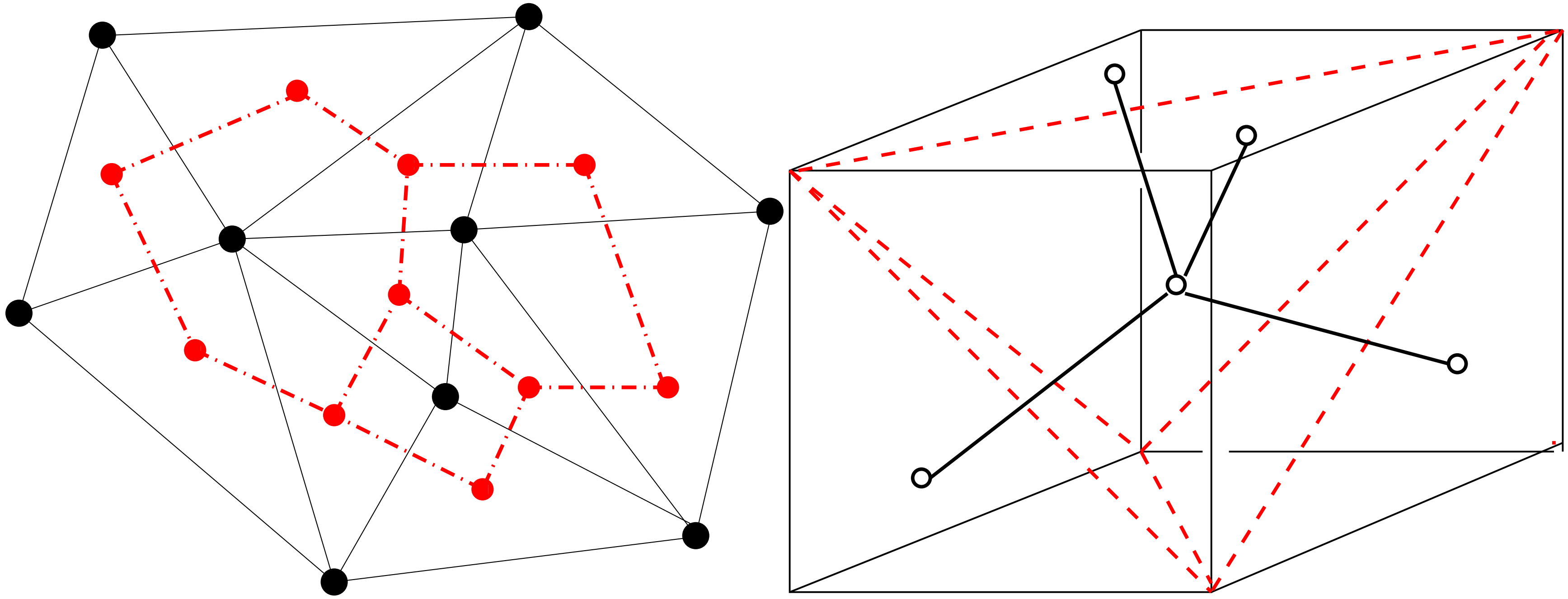
Triangulations of a convex polyhedron come in different sizes! i.e. the number of tetrahedra changes.



Open Problem 5: If for a 3-dimensional polyhedron P we know that there is triangulation of size k_1 and triangulations of size k_2, with $k_2 > k_1$ is there a triangulation of every size k, with $k_1 < k < k_2$?

The Hamiltonicity of a triangulation

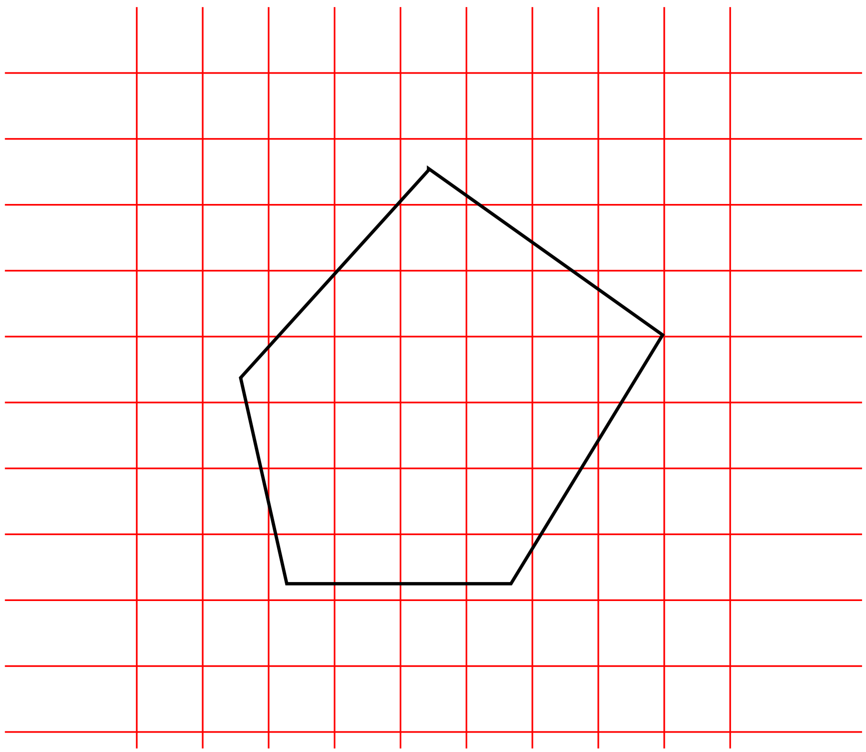
The dual graph of a triangulation: it has one vertex for each tetrahedron and an edge joining two such vertices if the two tetrahedra share a triangle:



Open Problem 6 Is it true that every 3-dimensional polyhedron has a triangulation whose dual graph is Hamiltonian?

Counting lattice points

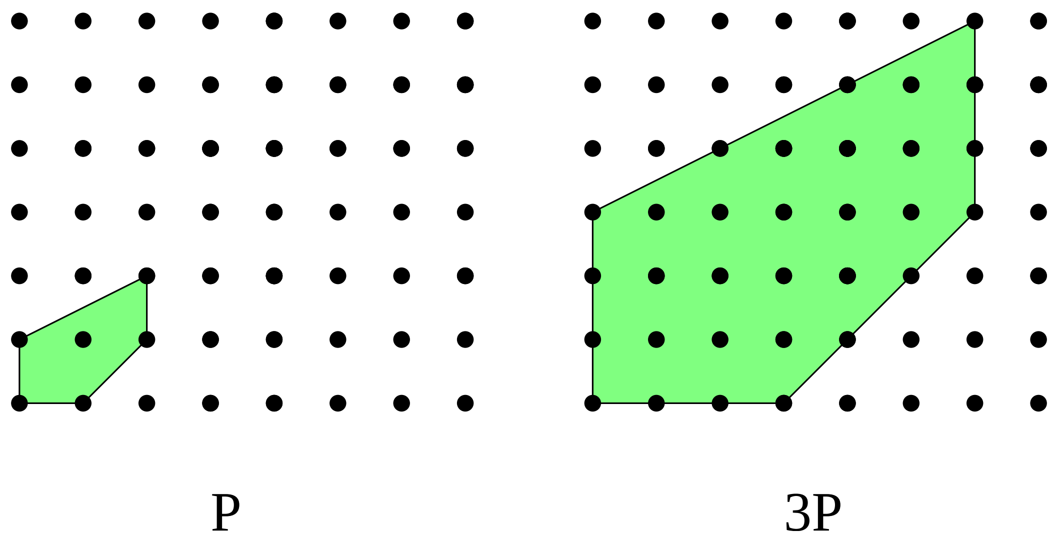
Lattice points are those points with integer coordinates: $\mathbb{Z}^n = \{(x_1, x_2, \ldots, x_n) | x_i \text{ integer}\}$ We wish to count how many lie inside a given polytope!



We can approximate the volume!

Let P be a convex polytope in $\mathbb{R}^d$. For each integer $n \geq 1$, let

$$nP = \{nq | q \in P\}$$



Counting function approximates volume

For P a d-polytope, let

$$i(P,n) = \#(nP \cap \mathbb{Z}^d) = \#\{q \in P \ |nq \in \mathbb{Z}^d\}$$

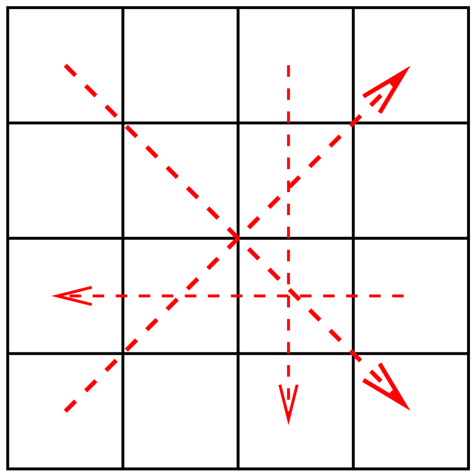
This is the number of lattice points in the dilation nP.

$$\text{Volume of } P = limit_{n\to\infty} \frac{i(P,n)}{n^d}$$

At each dilation we can approximate the volume by placing a small unit cube centered at each lattice point:

Combinatorics via Lattice points

Many objects can be counted as the lattice points in some polytope: E.g., Sudoku configurations, matchings on graphs, and **MAGIC squares:**



12	0	5	7
0	12	7	5
7	5	0	12
5	7	12	0

5

CHALLENGE: **HOW MANY 4×4 magic squares with sum n are there?** Same as counting the points with integer coordinates inside the n-th dilation of a "magic square" polytope!

Indeed, we can describe it by linear constraints!

The possible magic squares are non-negative integer solutions of a system of equations and inequalities: Ten equations, one for each row sum, column sum, and diagonal sum. For example,

$x_{11} + x_{12} + x_{13} + x_{14} = 220$, **first row**
$x_{13} + x_{23} + x_{33} + x_{43} = 71$, **third column**, and of course $x_{ij} \geq 0$

Open Problem 7: Find a formula for the volume of $n \times n$ magic squares polytope or, more strongly, find a formula for the number of lattice points of each dilation.

And many more open problems!

Thank you! Muchas Gracias!