When combinatorial computing meets algebraic computing: Hilbert's Nullstellensatz and Combinatorial Infeasibility

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based on joint work with J. Lee, S. Margulies, P. Malkin, and S. Onn

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  We transfer the Combinatorial feasibility problem to the solvability of a system of polynomials
  We then solve a Polynomial Feasibility Problem by a finite sequence of linear algebra problems!.

Le Menu

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- The NulLA ALGORITHM and BEYOND.

Part I

Combinatorics and Polynomials

A Typical Combinatorial Feasibility Problem

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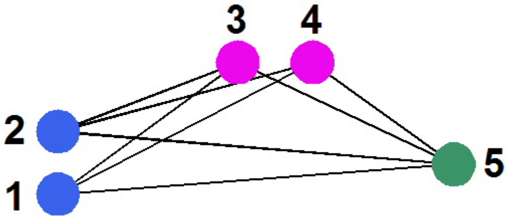
- Recall, the *stability* number of a graph is the size of the largest stable set in the graph, and is denoted by $\alpha(G)$.
- **Turán Graph $T(5,3)$:** no stable set of size bigger than 2.



Stable Set as a System of Polynomial Equations (L. Lovász 1989)

Given a graph G and an integer k:

- one **variable** per **vertex**
- For every vertex $i = 1, \ldots, n$, let $x_i^2 - x_i = 0$
- For every edge $(i, j) \in E(G)$, let $x_i x_j = 0$
- Finally, let

$$\left(-k + \sum_{i=1}^{n} x_i \right) = 0$$

Turán Graph $T(5,3)$: $\Longrightarrow$ System of Polynomial Equations

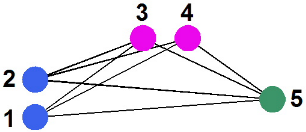


Figure: Does $T(5,3)$ have a stable set of size 3?

$$x_1x_3 = 0,\ x_1x_4 = 0,\ x_1x_5 = 0,\ x_2x_3 = 0, \qquad x_1^2 - x_1 = 0,\ x_2^2 - x_2 = 0$$

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$$x_1 + x_3 + x_5 + x_2 + x_4 - 3 = 0, \qquad x_5^2 - x_5 = 0$$

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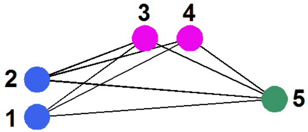

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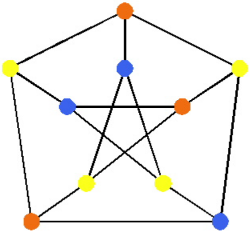
$$x_1 + x_3 + x_5 + x_2 + x_4 - 3 = 0, \qquad x_5^2 - x_5 = 0$$

Proposition: Let G be a graph, k an integer, encoded as the above $(n+m+1)$ system of equations. Then this system has a solution over $\mathbb{C}$ if and only if G has a stable set of size k. Bijection between stable sets of size k and solutions of the equations.

- **Graph coloring:** Given a graph G, and an integer k, can the vertices be colored with k colors in such a way that no two adjacent vertices are the same color?
- **Is the Petersen Graph 3-colorable?**



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- **Proposition:**(1988 D. Bayer) Let G be a graph, k an integer, then the system of equations has a solution over $\mathbb{C}$ if and only if G is k-colorable. Moreover, the number of k-colorings is equal to the number of solutions divided by $k!$.

Example: Petersen Graph Polynomial System of Equations

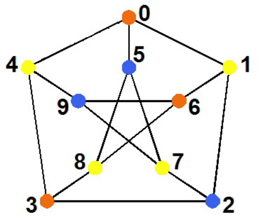

Figure: Decision Question: Is the Petersen graph 3-colorable?

$$x_1^3 - 1 = 0, x_2^3 - 1 = 0, \quad x_1^2 + x_1x_2 + x_2^2 = 0, x_1^2 + x_1x_5 + x_5^2 = 0$$
$$x_3^3 - 1 = 0, x_4^3 - 1 = 0, \quad x_1^2 + x_1x_6 + x_6^2 = 0, x_2^2 + x_2x_3 + x_3^2 = 0$$
$$x_5^3 - 1 = 0, x_6^3 - 1 = 0, \quad x_2^2 + x_2x_7 + x_7^2 = 0, x_3^2 + x_3x_8 + x_8^2 = 0$$
$$x_7^3 - 1 = 0, x_8^3 - 1 = 0, \quad \ldots\ldots \quad \ldots\ldots$$
$$x_9^3 - 1 = 0, x_{10}^3 - 1 = 0, \quad x_7^2 + x_7x_9 + x_9^2 = 0, x_8^2 + x_8x_{10} + x_{10}^2 = 0$$

Other algebraic ways to think about colorability

Definition: Let G be a graph with vertices $V = \{1, \ldots, n\}$ and edges E. The *graph polynomial* of G is

$$f_G = \prod_{\{i,j\} \in E,\ i<j} (x_i - x_j).$$

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Theorem: (1990 Kleitman Lovász) Let $\mathcal{H}(n,k)$ be the set of all graphs with n vertices consisting of a clique of size $k+1$ and all other $n-k+1$ vertices isolated. The graph G on n vertices is not k-colorable if and only if

$$f_G = \sum_{H \in \mathcal{H}(n,k)} \alpha_H f_H$$

where α_H are polynomials.

Polynomials are expressive: Largest k-colorable subgraph

A graph G has a k-colorable subgraph with R edges if and only if the following system of equations has a solution:

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- For each edge $\{i,j\} \in E(G)$:
$$y_{ij}^2 - y_{ij} = 0, \quad y_{ij}\left(x_i^{k-1} + x_i^{k-2}x_j + \cdots + x_j^{k-1}\right) = 0.$$

Many other interesting encodings: e.g., existence of k-cycle in a graph, largest planar subgraph, graph isomorphism problem, etc.

Applications: Proving theorems and characterizations

- (Lovász-Schrijver 1990) A graph is t-**perfect**: A linear form $f(z) \geq 0$ for all incidence vectors of stable sets if and only if there exist polynomials g_i, of degree $\leq t$, such that

$$f = g_1^2 + \ldots g_k^2 + \sum a_{ij} x_i x_j + \sum b_i (x_i^2 - x_i)$$

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- (Hillar-Windfeldt 2008) An algebraic characterization for when a graph is uniquely k-colorable. The number of k-colorings equals dimension of quotient ring.

The Combinatorial Nullstellensatz

(Alon-Tarsi 1989) If a graph G has an orientation D such that max outdegree is d and

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Theorem Let F be an arbitrary field, and let $f(x_1,\dots,x_n)$ be polynomial in $F[x_1,\dots,x_n]$. Suppose the degree $deg(f)$ is $\sum_{i=1} t_i$, where each t_i is a nonnegative integer and suppose the coefficient of the monomial $x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}$ is non-zero inside f. Then, if $S_1,\dots S_n$ are subsets of F with $|S_i| > t_i$, there are $(s_1,\dots,s_n) \in S_1 \times S_2 \times \dots S_n$.

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This has been used in many other applications: Graph factorizations, additive number theory, hamiltonian cycles, others...

A big important issue...

Noga Alon 2000: *"Is it possible to modify the algebraic proofs given here so that they yield efficient ways of solving the corresponding algorithmic problems? It seems likely that such algorithms do exists. "*

To answer this let us go back 120 years!!!

Part II

David Hilbert c. 1900

Hilbert's Nullstellensatz and Combinatorial Infeasibility

Hilbert's Nullstellensatz

- **Theorem:** Let $\mathbb{K}$ be a field and $\bar{\mathbb{K}}$ its algebraic closure field. Let $f_1, \ldots, f_s$ be polynomials in $\mathbb{K}[x_1, \ldots, x_n]$. The system of equations $f_1 = f_2 = \cdots = f_s = 0$ has **no** solution over $\bar{\mathbb{K}}$ if and only if there exist polynomials $\alpha_1, \ldots, \alpha_s \in \mathbb{K}[x_1, \ldots, x_n]$ such that

$$1 = \sum_{i=1}^{s} \alpha_i f_i$$

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- Let $d = \max\{\deg(\alpha_1), \deg(\alpha_2), \ldots, \deg(\alpha_s)\}$. Then d is the degree of the Nullstellensatz certificate.
- **Remark:** Nullstellensatz certificates are certificates for the *infeasibility* of a given system of polynomial equations.

Key Point: For fixed degree this is a linear algebra Problem!!

- **Example:** Consider system of polynomial equations

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1. Assume Nullstellensatz certificate has degree 1

$$1 = (c_0x_1 + c_1x_2 + c_2x_3 + c_3)(x_1^2 - 1) + (c_4x_1 + c_5x_2 + c_6x_3 + c_7)(x_1 + x_2)$$
$$+ (c_8x_1 + c_9x_2 + c_{10}x_3 + c_{11})(x_1 + x_3) + (c_{12}x_1 + c_{13}x_2 + c_{14}x_3 + c_{15})(x_2 + x_3)$$

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2. Expand the Nullstellensatz certificate, group by monomials

$$c_0x_1^3 + c_1x_1^2x_2 + c_2x_1^2x_3 + (c_3 + c_4 + c_8)x_1^2 + (c_5 + c_{13})x_2^2 + (c_{10} + c_{14})x_3^2 + (c_4 + c_5 + c_9 + c_{12})x_1x_2 + (c_6 + c_8 + c_{10} + c_{12})x_1x_3 + (c_6 + c_9 + c_{13} + c_{14})x_2x_3 + (c_7 + c_{11} - c_0)x_1 + (c_7 + c_{15} - c_1)x_2 + (c_{11} + c_{15} - c_2)x_3 - c_3$$

3. We extract a *linear* system of equations from expanded certificate

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4. Solve the linear system, and reconstitute the certificate

$$1 = -(x_1^2 - 1) + \frac{1}{2}x_1(x_1 + x_2) - \frac{1}{2}x_1(x_2 + x_3) + \frac{1}{2}x_1(x_1 + x_3)$$

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  But for the ideals in question we have a better bound:

- **Theorem:** (Brownawell-Lazard) The $\deg(\alpha_i)$ is bounded by $n(D-1)$.

NulLA: Nullstellensatz Linear Algebra Algorithm for checking infeasibility:

- **INPUT:** A system of polynomial equations $F = \{f_1 = 0, f_2 = 0, \ldots, f_s = 0\}$.
- While $d \leq$ HBound and no solution found for L_d
  - Construct a tentative Nullstellensatz certificate of degree d
  - Extract a *linear* system of equations from tentative Nullstellensatz certificate
  - Solve the linear system L_d.
  - If there is a solution, construct the certificate, **OUTPUT: F is Infeasible**.
  - Else, $d = d + 1$,
- If $d =$ HBound and no solution found for L_d, then **OUTPUT: F is Feasible**

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Lemma: (Razborov, Beam, Impagliazzo et al) Propositional logic statements encoded via "boolean" polynomials. Nullstellensatz degree grows linear on number of logical variables for the Pigeonhole principle.

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NEXT THE RESULTS...

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- They aim to work over the reals, but for our purposes we can work over field. Semidefinite programming is replaced by large-scale linear algebra.

Part III

The NulLA algorithm and Beyond

- **Theorem:** For a graph G, a minimum-degree Nullstellensatz certificate for the non-existence of a stable set of size greater than $\alpha(G)$ has degree equal to $\alpha(G)$ and contains at least one term for every stable set in G.

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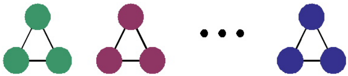
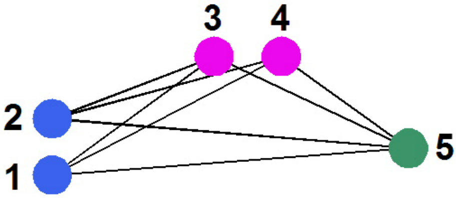
- **Example:** The disjoint union of triangles has a minimum-degree Nullstellensatz of degree $n/3$ and at least $4^{n/3-1}$ terms.



Turán Graph $T(5,3)$: Reduced Certificate Example



$$1 = \left(\frac{x_1x_2 + x_3x_4}{12} - \frac{x_1 + x_3 + x_5 + x_2 + x_4}{12} - \frac{1}{4}\right)(x_1 + x_3 + x_5 + x_2 + x_4 - 4) +$$

$$\left(\frac{x_4}{12} + \frac{x_2}{12} + \frac{1}{6}\right)x_1x_3 + \left(\frac{x_2}{12} + \frac{1}{6}\right)x_1x_4 + \left(\frac{x_2}{12} + \frac{1}{6}\right)x_1x_5 + \left(\frac{x_4}{12} + \frac{1}{6}\right)x_2x_3 +$$

$$\frac{x_2x_4}{6} + \frac{x_2x_5}{6} + \left(\frac{x_4}{12} + \frac{1}{6}\right)x_3x_5 + \frac{x_4x_5}{6} + \left(\frac{x_2}{12} + \frac{1}{12}\right)(x_1^2 - x_1) +$$

$$\left(\frac{x_1}{12} + \frac{1}{12}\right)(x_2^2 - x_2) + \left(\frac{x_4}{12} + \frac{1}{12}\right)(x_3^2 - x_3) + \left(\frac{x_3}{12} + \frac{1}{12}\right)(x_4^2 - x_4) + \frac{x_5^2 - x_5}{12}$$

Nullstellensatz certificates for non-3-colorability

Theorem Every Nullstellensatz certificate for non-3-colorability of a graph has degree at least four. Moreover, in the case of a graph containing an odd-wheel or a clique as a subgraph, a minimum-degree Nullstellensatz certificate for non-3-colorability has degree exactly four.

So far all has used fields of characteristic zero...

We tried it with finite fields...

Graph 3-Coloring as a System of Polynomial Equations over $\overline{\mathbb{F}_2}$ (inspired by Bayer)

- one **variable** per **vertex**
- **vertex polynomials:** For every vertex $i = 1, \ldots, n$,

$$x_i^3 + 1 = 0$$

- **edge polynomials:** For every edge $(i, j) \in E(G)$,

$$x_i^2 + x_i x_j + x_j^2 = 0$$

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$$x_i^2 + x_i x_j + x_j^2 = 0$$

- **Theorem:** Let G be a graph encoded as the above $(n + m)$ system of equations. Then this system has a solution if and only if G is 3-colorable.

Experimental results for NulLA 3-colorability

Graph	*vertices*	*edges*	*rows*	*cols*	*deg*	*sec*
Mycielski 7	95	755	64,281	71,726	1	.46
Mycielski 9	383	7,271	2,477,931	2,784,794	1	268.78
Mycielski 10	767	22,196	15,270,943	17,024,333	1	14835
$(8,3)$-Kneser	56	280	15,737	15,681	1	.07
$(10,4)$-Kneser	210	1,575	349,651	330,751	1	3.92
$(12,5)$-Kneser	792	8,316	7,030,585	6,586,273	1	466.47
$(13,5)$-Kneser	1,287	36,036	45,980,650	46,378,333	1	216105
1-Insertions_5	202	1,227	268,049	247,855	1	1.69
2-Insertions_5	597	3,936	2,628,805	2,349,793	1	18.23
3-Insertions_5	1,406	9,695	15,392,209	13,631,171	1	83.45
ash331GPIA	662	4,185	3,147,007	2,770,471	1	13.71
ash608GPIA	1,216	7,844	10,904,642	9,538,305	1	34.65
ash958GPIA	1,916	12,506	27,450,965	23,961,497	1	90.41

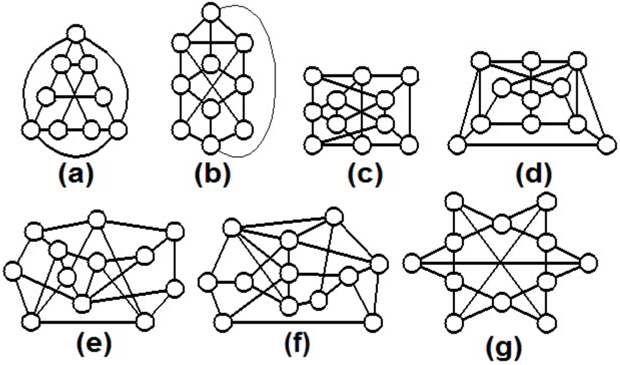
Table: Graphs without 4-cliques

Comparison with graph coloring heuristics

- *A Branch-and-Cut algorithm for graph coloring* by Isabel Méndez-Díaz and Paula Zabala (2006)

			B&C		DSATUR		NulLA	
Graph	*n*	*m*	*lb*	*up*	*lb*	*up*	deg	*sec*
4-Insertions_3.col	79	156	3	4	2	4	1	0
3-Insertions_4.col	281	1046	3	5	2	5	1	2
4-Insertions_4.col	475	1795	3	5	2	5	1	6
2-Insertions_5.col	597	3936	3	6	2	6	1	19
3-Insertions_5.col	1,406	9695	3	6	2	6	1	169

What are the ugliest examples?



near-4-clique free 4-critical graphs by Nishihara-Mizuno

Growth in Nullstellensatz degree

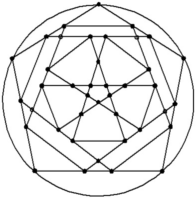
G_i	n	m	*row*	*col*	deg	*sec*	*max terms*
G_0	10	18	336	319	1	0	3
G_1	20	37	401,699	626,934	4	5	563
G_2	30	55	3,073,952	4,081,088	4	58	1961
G_3	39	72	11,703,170	14,192,150	4	287	2272
G_4	49	90	–	–	≥ 6	–	–

Comparison with Gröbner bases

Wheels	*n*	*m*	*GB*	*NulLA*
17	18	34	0	0
151	152	302	2.21	.21
501	502	1,002	126.83	15.58
1001	1,002	2,002	1706.69	622.73
2001	2,002	4,002	–	12905.6

NOTE: Lower bounds for the Nullstellensatz translate in lower bounds for Gröbner!!!!

Appending auxiliary equations helps!!

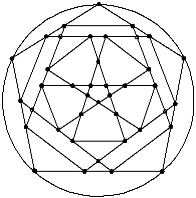


degree 4 certificate

$7,585,826 \times 9,887,481$

over 4 hours

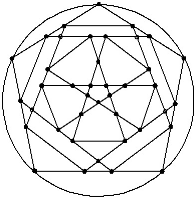
Appending auxiliary equations helps!!



degree 4 certificate
$7,585,826 \times 9,887,481$
over 4 hours

$\Longrightarrow$ 25 triangles

Appending auxiliary equations helps!!



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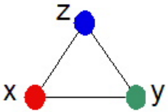
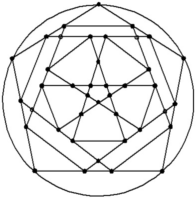


Appending auxiliary equations helps!!



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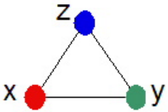
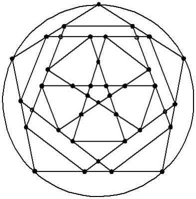


"Triangle" equation:

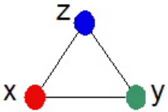
$$0 = x + y + z$$

Appending auxiliary equations helps!!



degree 4 certificate
$7,585,826 \times 9,887,481$
over 4 hours

$\Longrightarrow$ 25 triangles



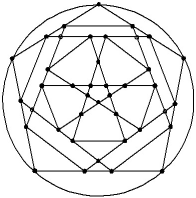
"Triangle" equation:

$$0 = x + y + z$$

Degree two triangle equation:

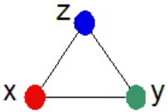
$$0 = x^2 + y^2 + z^2$$

Appending auxiliary equations helps!!



degree 4 certificate
$7,585,826 \times 9,887,481$
over 4 hours
$\Downarrow$
degree 1 certificate

$\Longrightarrow$ 25 triangles



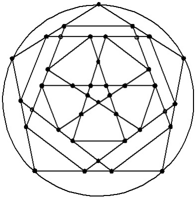
"Triangle" equation:

$$0 = x + y + z$$

Degree two triangle equation:

$$0 = x^2 + y^2 + z^2$$

Appending auxiliary equations helps!!



degree 4 certificate

$7,585,826 \times 9,887,481$

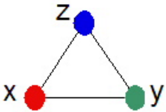
over 4 hours

$\Downarrow$

degree 1 certificate

$4,626 \times 4,3464$

$\Longrightarrow$ 25 triangles



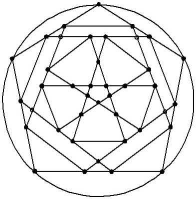
"Triangle" equation:

$$0 = x + y + z$$

Degree two triangle equation:

$$0 = x^2 + y^2 + z^2$$

Appending auxiliary equations helps!!



degree 4 certificate

$7,585,826 \times 9,887,481$

over 4 hours

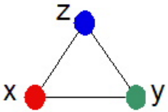
$\Downarrow$

degree 1 certificate

$4,626 \times 4,3464$

.2 seconds

$\Longrightarrow$ 25 triangles



"Triangle" equation:

$$0 = x + y + z$$

Degree two triangle equation:

$$0 = x^2 + y^2 + z^2$$

Example

Consider the complete graph K_4. A degree-one Hilbert Nullstellensatz certificate for non-3-colorability, over $\overline{\mathbb{F}}_2$ is

$$\begin{aligned}
1 = {} & c_0(x_1^3+1) \\
& + (c_{12}^1x_1 + c_{12}^2x_2 + c_{12}^3x_3 + c_{12}^4x_4)(x_1^2 + x_1x_2 + x_2^2) + (c_{13}^1x_1 + c_{13}^2x_2 + c_{13}^3x_3 + c_{13}^4x_4)(x_1^2 + x_1x_3 + x_3^2) \\
& + (c_{14}^1x_1 + c_{14}^2x_2 + c_{14}^3x_3 + c_{14}^4x_4)(x_1^2 + x_1x_4 + x_4^2) + (c_{23}^1x_1 + c_{23}^2x_2 + c_{23}^3x_3 + c_{23}^4x_4)(x_2^2 + x_2x_3 + x_3^2) \\
& + (c_{24}^1x_1 + c_{24}^2x_2 + c_{24}^3x_3 + c_{24}^4x_4)(x_2^2 + x_2x_4 + x_4^2) + (c_{34}^1x_1 + c_{34}^2x_2 + c_{34}^3x_3 + c_{34}^4x_4)(x_3^2 + x_3x_4 + x_4^2)
\end{aligned}$$

Matrix $M_{F,1}$

	c_0	c_{12}^1	c_{12}^2	c_{12}^3	c_{12}^4	c_{13}^1	c_{13}^2	c_{13}^3	c_{13}^4	c_{14}^1	c_{14}^2	c_{14}^3	c_{14}^4	c_{23}^1	c_{23}^2	c_{23}^3	c_{23}^4	c_{24}^1	c_{24}^2	c_{24}^3	c_{24}^4	c_{34}^1	c_{34}^2	c_{34}^3	c_{34}^4
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_1^3	1	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$x_1^2x_2$	0	1	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$x_1^2x_3$	0	0	0	1	0	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$x_1^2x_4$	0	0	0	0	1	0	0	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$x_1x_2^2$	0	1	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0
$x_1x_2x_3$	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
$x_1x_2x_4$	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
$x_1x_3^2$	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0
$x_1x_3x_4$	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0
$x_1x_4^2$	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0
x_2^3	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0
$x_2^2x_3$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	0	0	0	0	0
$x_2^2x_4$	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0
$x_2x_3^2$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	0	0
$x_2x_3x_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0
$x_2x_4^2$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1	0	1	0	0
x_3^3	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0
$x_3^2x_4$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	1
$x_3x_4^2$	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	1
x_4^3	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1

Suppose we have a group acting...

Suppose a finite permutation group G acts on the variables $x_1, \ldots, x_n$. Assume that the set F of polynomials is invariant under the action of G, i.e., $g(f_i) \in F$ for each $f_i \in F$.

We wish to shrink the matrix using the group!!!

Example, Part 2, action of Z_3 by (2,3,4)

	c_0	c^1_{12}	c^1_{13}	c^1_{14}	c^2_{12}	c^3_{13}	c^4_{14}	c^3_{12}	c^4_{13}	c^2_{14}	c^4_{12}	c^2_{13}	c^3_{14}	c^1_{23}	c^1_{34}	c^1_{24}	c^2_{23}	c^3_{34}	c^4_{24}	c^2_{24}	c^3_{23}	c^4_{34}	c^2_{34}	c^3_{24}	c^4_{23}
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_1^3	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$x_1^2x_2$	0	1	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$x_1^2x_3$	0	0	1	0	0	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$x_1^2x_4$	0	0	0	1	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$x_1x_2^2$	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
$x_1x_3^2$	0	0	1	0	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
$x_1x_4^2$	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
$x_1x_2x_3$	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0
$x_1x_2x_4$	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0
$x_1x_3x_4$	0	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0
x_2^3	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0
x_3^3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0
x_4^3	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0
$x_2^2x_3$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	1	0
$x_3^2x_4$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	1
$x_2x_4^2$	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	1	0	0	1	0	0
$x_2^2x_4$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	1
$x_2x_3^2$	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1	0	1	0	0
$x_3x_4^2$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1	0	1	0
$x_2x_3x_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1

The Matrix $M_{F,1,G}$

	$\bar{c}_0$	$\bar{c}^1_{12}$	$\bar{c}^2_{12}$	$\bar{c}^3_{12}$	$\bar{c}^4_{12}$	$\bar{c}^1_{23}$	$\bar{c}^2_{23}$	$\bar{c}^2_{24}$	$\bar{c}^2_{34}$
$Orb(1)$	1	0	0	0	0	0	0	0	0
$Orb(x_1^3)$	1	3	0	0	0	0	0	0	0
$Orb(x_1^2 x_2)$	0	1	1	1	1	0	0	0	0
$Orb(x_1 x_2^2)$	0	1	1	0	0	2	0	0	0
$Orb(x_1 x_2 x_3)$	0	0	0	1	1	1	0	0	0
$Orb(x_2^3)$	0	0	1	0	0	0	1	1	0
$Orb(x_2^2 x_3)$	0	0	0	1	0	0	1	1	1
$Orb(x_2^2 x_4)$	0	0	0	0	1	0	1	1	1
$Orb(x_2 x_3 x_4)$	0	0	0	0	0	0	0	0	3

$\overset{\pmod 2}{\equiv}$

	$\bar{c}_0$	$\bar{c}^1_{12}$	$\bar{c}^2_{12}$	$\bar{c}^3_{12}$	$\bar{c}^4_{12}$	$\bar{c}^1_{23}$	$\bar{c}^2_{23}$	$\bar{c}^2_{24}$	$\bar{c}^2_{34}$
$Orb(1)$	1	0	0	0	0	0	0	0	0
$Orb(x_1^3)$	1	1	0	0	0	0	0	0	0
$Orb(x_1^2 x_2)$	0	1	1	1	1	0	0	0	0
$Orb(x_1 x_2^2)$	0	1	1	0	0	0	0	0	0
$Orb(x_1 x_2 x_3)$	0	0	0	1	1	1	0	0	0
$Orb(x_2^3)$	0	0	1	0	0	0	1	1	0
$Orb(x_2^2 x_3)$	0	0	0	1	0	0	1	1	1
$Orb(x_2^2 x_4)$	0	0	0	0	1	0	1	1	1
$Orb(x_2 x_3 x_4)$	0	0	0	0	0	0	0	0	1

Theorem

Let $\mathbb{K}$ be an algebraically-closed field. Let $F = \{f_1, \ldots, f_s\} \subset \mathbb{K}[x_1, \ldots, x_n]$ polynomials and suppose F is closed under the action of the group G on the variable. Suppose that the order of the group $|G|$ and the characteristic of the field $\mathbb{K}$ are relatively prime.

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Then, the degree d Nullstellensatz linear system of equations $M_{F,d}\, y = b_{F,d}$ has a solution over $\mathbb{K}$ if and only if the system of linear equations $\bar{M}_{F,d,G}\, \bar{y} = \bar{b}_{F,d,G}$ has a solution over $\mathbb{K}$.

THANK YOU!

Poset Dimension

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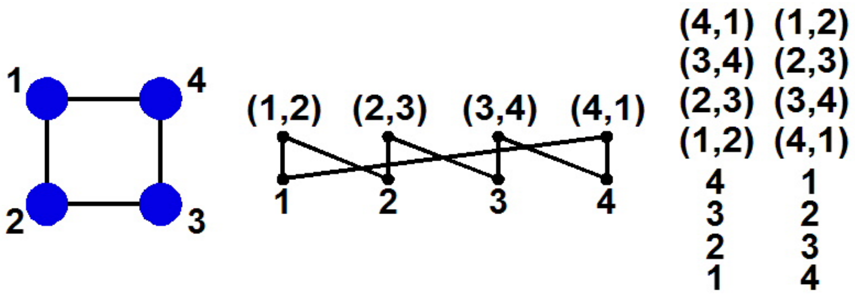
- The *incidence poset* $P(G)$ of a graph G with node set V and edge set E is the partially ordered set of height two on the union of nodes and edges, where we say $x < y$ if x is a node and y is an edge, and y is incident to x.

Example



Schnyder's theorem

- **Theorem** A graph G is planar if and only if the poset dimension of $P(G)$ is no more than three.
- Our goal is to encode the linear extensions and the poset dimension of a poset P in terms of polynomials equations.
- **Lemma** The poset $P = (E, >)$ has poset dimension at most p if and only if the following system of equations has a solution: For $k = 1, \ldots, p$:

$$\prod_{s=1}^{|E|}(x_i(k) - s) = 0, \text{ for each } i \in \{1, \ldots, |E|\},$$

$$s_k \left(\prod_{\substack{\{i,j\} \in \{1,\ldots,|E|\}, \\ i<j}} x_i(k) - x_j(k) \right) = 1.$$

For $k = 1, \ldots, p$, and each ordered pair of comparable elements $e_i > e_j$ in P:

$$\big(x_i(k) - x_j(k) - \Delta_{ij}(k)\big) = 0. \tag{1}$$

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$$\prod_{k=1}^{p} \big(x_i(k) - x_j(k) - \Delta_{ij}(k)\big) = 0, \qquad \prod_{k=1}^{p} \big(x_j(k) - x_i(k) - \Delta_{ji}(k)\big) = 0, \qquad (2)$$

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For $k = 1, \ldots, p$, and for each pair $\{i, j\} \in \{1, \ldots, |E|\}$:

$$\prod_{d=1}^{|E|-1} (\Delta_{ij}(k) - d) = 0, \qquad \prod_{d=1}^{|E|-1} (\Delta_{ji}(k) - d) = 0. \qquad (3)$$