Math and Computers, Math 165
New Problem set

1. Using Descartes' rule of signs find as much information as you can about the possible number of roots (counting multiplicities) of each of the following polynomials:

   a) $x^4 - x^2 + x - 2$

   b) $x^9 - x^5 + x^2 + 2$

   c) $x^5 + 2x^3 - x^2 + x - 1$

2. Apply Sturm's sequences and find out exactly how many distinct roots are there for each of the polynomials of problem one.

3. Is the polynomial $x^2 - 4$ in the ideal generated by the polynomials $x^3 + x^2 - 4x - 4, x^3 - x^2 - 4x + 4, x^3 - 2x^2 - x + 2$?

4. Explain why $GCD(f, g, h) = GCD(GCD(f, g), h)$. Also explain why for univariate polynomials the ideal $\langle f_1, f_2, \ldots, f_k \rangle$ is equal to $\langle GCD(f_1, f_2, \ldots, f_k) \rangle$.

5. Sketch the following affine varieties (or at least the real parts of it!). in $\mathbb{R}^2$: a) $V(x^2 - y^2)$, b) $V(x^2 + 4y^2 + 2x - 16y + 1)$ in $\mathbb{R}^3$: c) $V(xz^2 - xy)$, d) $V(x^4 - zx, x^3 - yz)$.

6. Consider the set $\{(x, x) : x \in \mathbb{R}, x \neq 1\} \subset \mathbb{R}^2$. This is a straight line minimus a point. Show that this set is not an affine variety (Hint: Prove that if a polynomial vanishes at the set must also vanish at $(1, 1)$.

7. The basis of an ideal is different from a basis in linear algebra in that we do not care about linear independence! As a consequence when we write an element $f \in \langle f_1, \ldots, f_s \rangle$ as $f = \sum h_i f_i$ the coefficients h_i are not always unique. As an example, write $x^2 + xy + y^2 \in \langle x, y \rangle$ in two different ways.

8. Each of the following polynomials is written with its monomials ordered according to exactly one of the monomial orders: Lex, graded lex, or graded reverse lex. Determine which monomial order was used in each case.

   (a) $7x^2y^4z - 2xy^6 + x^2y^2$ (b) $xy^3z + xy^2z^2 + x^2z^3$ (c) $x^4y^5z + 2x^3y^2z - 4xy^2z^4$

9. show that graded reverse lexicographic order is indeed a monomial order.

10. Let $>$ be a monomial order in $S = \mathbb{C}[x_1, \ldots, x_n]$.

    (a) Let $f \in S$ and let m be a monomial. Show that $LT(m \cdot f) = m \cdot LT(f)$.

    (b) Let $f, g \in S$. Is $LT(f \cdot g)$ necessarily the same as $LT(f) \cdot LT(g)$?