Homework

Solve as many problems as you can!

1. Let A, B be two sets inside R^d. We define the *Minkowski sum* of A and B to be $A + B = \{x + y : x \in A, y \in B\}$. Show that if A, B are convex the Minkowski sum is a convex set too. The scaling of a set A by a scalar α is $\alpha A = \{\alpha : x \in A\}$. Show that αA is convex when A is convex. Is it always true that $(\alpha + \beta)A = \alpha A + \beta A$? Explain your answer. (I suggest you draw some pictures of Minkowski sums of sets in the plane)

2. Let A be a finite set of $d+2$ or more points inside R^d. Prove that one can represent A as the disjoint union of sets B, C such that $conv(B) \cap conv(C)$ is not empty.

3. Let $A_1, \dots A_m$ be a family of convex sets inside R^d. Suppose that the intersection of any $d+1$ of these sets is not empty. Then the intersection $\cap_{i=1}^{m} A_i$ is not empty.

4. We defined the convex hull of a set S to be the intersection of all convex sets that contain S. Prove that the convex hull of a set of point $a_1, \dots, a_m$ is the same as

$$\{\sum_{i \in I} \gamma_i a_i : \gamma_i \geq 0, \sum \gamma_i = 1, I \subset \{1..m\}|I| = d+1\}$$

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5. A^o denotes the polar set to A. Suppose that $A^o = A$. Prove that A is the unit ball. What is the polar of the standard cube? Is it true that every polyhedron is the polar of a polytope? Prove or disprove: $(A \cup B)^o = A^o \cap B^o$.

6. 6 For any $n \geq 4$ construct a 3-dimensional polytope with n vertices which is dual to itself. Describe as thoroughly as you can the dual polytope to the cyclic polytope $C(3, 8)$.

7. 7 Given a convex polytope P and a point $b \notin P$, there exists a linear function f and a number α such that $f(b) > \alpha > f(x)$ for every x in P. This means geometrically that a hyperplane separates P and b.

8. 8 Let P be a polytope in R^d. Consider an inclusion of R^d into R^{d+1} and a point $p \in R^{d+1} - R^d$. Describe the face lattice of $conv(p, P)$ in terms of the face lattice of P.

9. 9 Prove that there are only five Platonic solids.