Polyhedral Computation and their Applications

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1 Introduction

It is indeniable that convex polyhedral geometry is an important tool of modern mathematics. For example, it is well-known that understanding the facets of the TSP polytope has played a role on the solution of large-scale traveling salesman problems and in the study of cutting planes. In the past fiften years, there have been new developments on the understanding of the structure of polyhedra and their lattice points that have produced amazing new algorithmic ideas to solve mixed integer programs.

These notes intend to introduce smart undergraduates to this topic. The method used is the famous Moore method. Where students are expected to think like grown-up mathematicians and discover and prove theorems on their own! The assumption is that the reader is smart but a novice to polyhedral geometry, who is willing to think hard about what is written here. By solving a sequence of problems the students will be introduced to the subject. This is learning by doing!!!

1. Basic Polyhedral Theory
   (a) Weyl-Minkowski, Farkas Duality
   (b) Faces and Graphs of Polytopes
2. The Simplex Method.
3. Fourier-Motzkin Elimination
4. Reverse-Search Enumeration.
5. Lattice points in Polyhedra
   (a) Lattices and Minkowski's Geometry of numbers.
   (b) Lattice Basis Reduction

2 Basic Polyhedral Theory

We begin with the key definition of this notes:

Definition 2.1 *The set of solutions of a system of linear inequalities is called a* polyhedron. *In its general form a polyhedron is then a set of the type*

$$P = \{x \in \mathbb{R}^d :< c_i, x > \leq \beta_i\}$$

for some non-zero vectors c_i in $\mathbb{R}^d$ and some real numbers β_i.

Recall from linear algebra that a linear function $f : \mathbb{R}^d \to \mathbb{R}$ is given by a vector $c \in \mathbb{R}^d, c \neq 0$. For a number $\alpha \in \mathbb{R}$ we say that $H_\alpha = \{x \in \mathbb{R}^d : f(x) = \alpha\}$ is an *affine hyperplane* or *hyperplane* for short. Note that a hyperplane divides $\mathbb{R}^d$ into two halfspaces $H_\alpha^+ = \{x \in \mathbb{R}^d : f(x) \geq \alpha\}$ and $H_\alpha^- = \{x \in \mathbb{R}^d : f(x) \leq \alpha\}$. Halfspaces are convex sets. In other words A *polyhedron* in $\mathbb{R}^d$ is the intersection of finitely many halfspaces.

By the way, the plural of the word *polyhedron* is *polyhedra*. Although everybody has seen pictures or models of two and three dimensional polyhedra such as cubes and triangles and most people may have a mental picture of what edges, ridges, or facets for these objects are, we will formally introduce them later on. This is a very short introduction and there are excellent books that can help the reader to the beautiful of polyhedra, but you are NOT allowed to consult a book until after you tried hard on your own!!

Everything we do takes place inside Euclidean d-dimensional space $\mathbb{R}^d$. We have the traditional Euclidean distance between two points x, y defined by $\sqrt{(x_1 - y_1)^2 + \ldots (x_2 - y_2)^2}$. Given two points x, y. We will use the common fact that $\mathbb{R}^d$ is a real vector space and thus we know how to add or scale its points. The set of points $[x, y] := \{\alpha x + (1 - \alpha)y : 0 \leq \alpha \leq 1\}$ is called *the interval* between x and y. The points x and y are the endpoints of the interval.

Definition 2.2 *A subset S of $\mathbb{R}^n$ is called convex if for any two distinct points x_1, x_2 in S the line segment joining x_1, x_2, lies completely in S. This is equivalent to saying $x = \lambda x_1 + (1 - \lambda)x_2$ belongs to S for all choices of λ between* 0 *and* 1*.*

Lemma 2.3 *Let $Ax \subseteq b$, $Cx \supseteq d$, be a system of inequalities. The set of solutions is a convex set.*

Write a proof!

We will assume that the empty set is also convex. Observe that the intersection of convex sets is convex too. Let $A \subset \mathbb{R}^d$, the *convex hull* of A, denoted by *conv*(A), is the intersection of all the convex sets containing A. In other words, A is the smallest convex set containing A. The reader can check that the image of a convex set under a linear transformation is again a convex set. An important definition

Definition 2.4 *A* polytope *is the convex hull of a finite set of points in $\mathbb{R}^d$.*

Lemma 2.5 *For a finite set of points in $\mathbb{R}^d$ $A := \{a_1, a_2, \ldots, a_n\}$ we have that conv(A) equals*

$$\{\sum_{i=1}^{n} \gamma_i a_i : \gamma_i \geq 0 \text{ and } \gamma_1 + \ldots \gamma_n = 1\}$$

Write a proof!

Write a proof!

Now it is easier to speak about examples of polytopes! Find more on your own!

1. **Standard Simplex** Let $e_1, e_2, \ldots, e_{d+1}$ be the standard unit vectors in $\mathbb{R}^{d+1}$. The *standard* d-dimensional simplex Δ_d is $conv(\{e_1, \ldots, e_{d+1}\})$. From the above lemma we see that the set is precisely

   $\Delta_d = \{x = (x_1, \ldots, x_{d+1}) : x_i \geq 0 \text{ and } x_1 + x_2 + \cdots + x_{d+1} = 1\}$.

   Note that for a polytope $P = conv(\{a_1, \ldots, a_m\})$ we can define a linear map $f : \Delta_{m-1} \to P$ by the formula $f(\lambda_1, \ldots, \lambda_m) = \lambda_1 a_1 + \cdots + \lambda_m a_m$. Lemma 2.5 implies that $f(\Delta_{m-1}) = P$. Hence, every polytope is the image of the standard simplex under a linear transformation. A lot of the properties of the standard simplex are then shared by all polytopes.

2. **Standard Cube** Let $\{u_i : i \in I\}$ be the set of all 2^d vectors in $\mathbb{R}^d$ whose coordinates are either 1 or -1. The polytope $I_d = conv(\{u_i : i \in I\}$ is called the *standard* d-dimensional cube. The images of a cube under linear transformations receive the name of *zonotopes*.

   Clearly $I^d = \{x = (x_1, \ldots, x_d) : -1 \leq x_i \leq 1\}$

3. **Standard Crosspolytope** This is the convex hull of the $2d$ vectors $e_1, -e_1, e_2, -e_2, \ldots, e_d, -e_d$. The 3-dimensional crosspolytope is simply an octahedron.

Let P be a polytope in $\mathbb{R}^d$. A linear inequality $f(x) \leq \alpha$ is said to be *valid* on P if every point in P satisfies it. A set $F \subset P$ is a face of P if and only there exists a linear inequality $f(x) \leq \alpha$ which is valid on P and such that $F = \{x \in P : f(x) = \alpha\}$. In this case f is called a *supporting function* of F and the hyperplane defined by f is a *supporting hyperplane* of F. For a face F consider the smallest affine subspace $aff(F)$ in $\mathbb{R}^d$ generated by F. Its dimension is called the dimension of F. Similarly we define the dimension of the polytope P.

A face of dimension 0 is called a *vertex*. A face of dimension 1 is called an *edge*, and a face of dimension $dim(P) - 1$ is called a *facet*. The empty set is defined to be a face of P of dimension -1. Faces that are not the empty set or P itself are called proper.

Definition 2.6 *A point x is a convex set S is an* extreme point *of S if it is not an interior point of any line segment in S. This is equivalent to saying that when $x = \lambda x_1 + (1 - \lambda)x_2$, then either $\lambda = 1$ or $\lambda = 0$.*

Lemma 2.7 *Every vertex of a polyhedron is an extreme point.*

Write a proof!

Theorem 2.8 *Consider the linear program* $\min cx$ *subject to* $Ax = b, x \geq 0$. *Suppose the* m *columns* $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$ *of the* $m \times n$ *matrix* A *are linearly independent and there exist non-negative numbers* x_{i_j} *such that*

$$x_{i_1}A_{i_1} + x_{i_2}A_{i_2} + \cdots + x_{i_m}A_{i_m} = b.$$

Then the points with entry x_{i_j} *in position* i_j *and zero elsewhere is an extreme point of the polyhedron* $P = \{x : Ax = b, x \geq 0\}$.

Write a proof!

Theorem 2.9 *Suppose* $x = (x_1, \ldots, x_n)$ *is an extreme point of a polyhedron* $P = \{x : Ax = b,\ x \geq 0\}$. *Then*

1) the columns of A *which correspond to positive entries of* x *form a linearly independent set of vectors in* $\mathbb{R}^m$

2) At most m *of the entries of* x *can be positive, the rest are zero.*

Write a proof!

Proof: Suppose the columns are linearly dependent. Thus there are coefficients, not all zero, such that $c_{i_1}A_{i_1} + c_{i_2}A_{i_2} + \cdots + c_{i_m}A_{i_m} = 0$

Thus we can form points

$(x_{i_1} - dc_{i_1})A_{i_1} + (x_{i_2} - dc_{i_2})A_{i_2} + \cdots + (x_{i_m} - dc_{i_m})A_{i_m} = d$

$(x_{i_1} + dc_{i_1})A_{i_1} + (x_{i_2} + dc_{i_2})A_{i_2} + \cdots + (x_{i_m} + dc_{i_m})A_{i_m} = d$

Since d is any scalar, we may choose d less than the minimum of $x_j/|c_j|$ for those $c_j \neq 0$.

We have reached a contradiction! Since $x = 1/2(u) + 1/2(v)$ and both u, v are inside the polyhedron. For part (2) simply observe that there cannot be more than m linearly independent vectors inside $\mathbb{R}^m$.

Definition 2.10 *In any basic solution, the* $n - m$ *variables which are set equal to zero are called* nonbasic *variables and the* m *variables we solved for are called the* basic *variables. A* basic solution *is a a solution of the system* $Ax = b$ *where* $n - m$ *variables are set to zero. If in addition the solutions happens to have* $x \geq 0$ *then we say is* basic feasible solution.

2.1 Weyl-Minkowski, Polarity

It makes sense to study the relation between polytopes and polyhedra. Clearly standard cubes,simplices and crosspolytopes are also polyhedra, but is this the case in general? What one expects is really true:

Theorem 2.11 (Weyl-Minkowski theorem) *Every polytope is a polyhedron. Every bounded polyhedron is a polytope.*

This theorem is very important. Having this double way of representing a polytope allows you to prove, using either the vertex representation or the inequality representation representation, what would be hard to prove using a single representation. For example, every intersection of a polytope with an affine subspace is a polytope. Similarly, the intersection of finitely many polytopes is a polytope. Both statements are rather easy to prove if one knows that polytopes are just given by systems of linear inequalities, since then the intersection of polytopes is just adding new equations. On the other hand, It is known that every linear projection of a bounded polyhedron is a bounded polyhedron. To prove this from the inequality representation is difficult, but it is easy when one observes that the projection of convex hull is the convex hull of the projection of the vertices of the polytope. In addition, the Weyl-Minkowski theorem is very useful in applications! its existence is key in the field of combinatorial and linear optimization.

Before we discuss a proof of Weyl-Minkowski theorem we need to introduce a useful operation. To every subset of Euclidean space we wish to associate a convex set. Given a subset A of $\mathbb{R}^d$ the *polar* of A is the set A^o in $\mathbb{R}^d$ defined as the linear functionals whose value on A is not greater than 1, in other words:

$$A^o = \{x \in \mathbb{R}^d :< x, a >\leq 1 \ for \ every \ a \in A\}$$

Another way of thinking of the polar is as the intersection of the halfspaces, one for each element $a \in A$, of the form

$$\{x \in \mathbb{R}^d :< x, a >\leq 1\}$$

Here are two little useful examples: Take L a line in $\mathbb{R}^2$ passing through the origin, what is L^0? Well the answer is the perpendicular line that passes through the origin. If the line does not pass through the origin the answer is different. What is it? Answer: it is a clipped line orthogonal to the given line that passes through the origin. To see without loss of generality rotate the line until it is of the form $x = c$ (because the calculation of the polar boils down to checking angles and lengths between vectors we must get the same answer up to rotation).

What happens with a circle of radius one with center at the origin? Its polar set is the disk of radius one with center at the origin. Next take $B(0, r)$. What is $B(0, r)^o$? The concept of polar is rather useful. We use the following lemma:

Lemma 2.12 *1. If P is a polytope and $0 \in P$, then $(P^o)^o = P$.*

2. Let $P \subset \mathbb{R}^d$ be a polytope. Then P^o is a polyhedron.

Write a proof!

Now we are ready to prove the Weyl-Minkowski theorem:

Proof: **Weyl-Minkowski** First we verify that a bounded polyhedron is a polytope: Let P be $\{x \in \mathbb{R}^d :< x, c_i > \leq b_i\}$.

Consider the set of points E in P that are the unique intersection of d or more of the defining hyperplanes. The cardinality of E is at most $\binom{m}{d}$ so it is clearly a finite set and all its element are on the boundary of P. Denote by Q the convex hull of all elements of E. Clearly Q is a polytope and $Q \subset P$. We claim that $Q = P$. Suppose there is a $y \in P - Q$. Since Q is closed and bounded (bounded) we can find a linear functional f with the property that $f(y) > f(x)$ for all $x \in Q$. Now P is compact too, hence f attains its maximum on the boundary moreover we claim it must reach it in a point of E. The reason is that a boundary point that is not in E is in the solution set

We verify next that a polytope is indeed a polyhedron: We can assume that the polytope contains the origin in its interior (otherwise translate). So for a sufficiently small ball centered at the origin we have $B(0, r) \subset P$. Hence $P^o \subset B(0, r)^o = B(0, 1/r)$. This implies that P^o is a bounded polyhedron. But we saw in the first part that bounded polyhedra are polytopes. Then P^o is a polytope. We are done because we know from the above lemma that $(P^o)^o = P$ and polar of polytopes are polyhedra.

2.2 The Face Poset of a Polytope

Now is time to look carefully at the partially ordered set of faces of a polytope.

Proposition 2.13 *Let $P = conv(a_1, \dots, a_n)$. and $F \subset P$ a face. Then $F = conv(a_i, a_i \in F)$.*

Proof: Let $f(x) = \alpha$ be the suporting hyperplane. That $Q = conv(a_i, a_i \in F)$ is contained in F is clear. For the converse take $x \in F - Q$. We can still write x as $\lambda_1 a_1 + \cdots + \lambda_n a_n$ with the lambdas as usual. Applying f we get that if $\lambda_j > 0$ for an index not in Q, then we get $f(x) < \alpha$ because $f(a_j) < \alpha$ thus $\lambda_1 f(a_1) + \cdots + \lambda_n f(a_n) < \lambda_1 \alpha + \dots \lambda_n \alpha = \alpha$. nd thus we arrive to a contradiction

Corollary 2.14 *A Polytope has a finite number of faces, in particular a finite number of vertices and facets.*

For a polytope with vertex set $V = \{v_1, v_2, \dots, v_n\}$ the *graph of P* is the abstract graph with vertex set V and the set of edges $E = \{(v_i, v_j) : [v_i, v_j] \textit{is an edge of } P\}$. You can have a very entertaining day by drawing the graphs of polytopes. Later on we will prove a lot of cute properties about the graph of a polytope. Now there is a serious problem. We still don't have a formal verification that the graph of a polytope under our definition is non-empty! we must verify that there is always at least a vertex in a polytope. Such a seemingly obvious fact requires a proof. From looking at models of polyhedra one is certain that there is a containment relation among faces: a vertex of an edge that lies on the boundary of several facets, etc. Here is a first step to understand the

Lemma 2.15 *Let P be a d-polytope and $F \subset P$ be a face. Let $G \subset F$ be a face of F. Then G is a face of P as well.*

Write a proof!

Corollary 2.16 *Every non-empty polytope has at least one vertex.*

Write a proof!

Lemma 2.17 *Every basic feasible of a polyhedron $\{x : Ax = b, x \geq 0\}$ is a vertex. Thus the sets of basic feasible solutions, vertices, and extreme points are identical.*

Write a proof!

Theorem 2.18 *Every polytope is the convex hull of the set of its vertices.*

Write a proof!

2.3 Polar Polytopes and Duality of Face Posets

Now we know that a polytope has a canonical representation as the convex hull of its vertices. The results above establishes that the set of all faces of a polytope form a partially ordered set by the order given by containment. This poset receives the name of the *face poset* of a polytope. We say that two polytopes are *combinatorially equivalent* or *combinatorially isomorphic* if their face posets are the same. In particular, two polytopes P, Q are isomorphic if they have the same number of vertices and there is a one-to-one correspondence p_i to q_i between the vertices such that $conv(p_i : i \in I)$ is a face of P if and only if $conv(q_i : i \in I)$ is a face of Q. The bijection is called an isomorphism.

A property that can guess from looking at the Platonic solids is that there is a duality relation where two polytopes are matched to each other by paring the vertices of one with the facets of the other and vice versa. We want now to make this intuition precise. We will establish a bijection between the faces of P and the faces of P^o. Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope containing the origin as its interior point. For a non-empty face F of P define

$$\hat{F} = \{x \in P^o :< x, y >= 1 for\ all\ y \in F\}$$

and for the empty face define $\hat{} = Q$.

Theorem 2.19 *The hat operation applied to faces of a d-polytope P satisfies*

1. *The set $\hat{F}$ is a face of P^o*
2. $dim(F) + dim(\hat{F}) = d - 1$.
3. *The hat operation is involutory:* $(\hat{\hat{F}}) = F$.
4. *If $F, G \subset P$ are faces and $F \subset G \subset P$, then $\hat{G}, \hat{F}$ are faces of P^o and $\hat{G} \subset \hat{F}$.*

Proof: To set up notation we take $P = conv(a_1, a_2, \ldots, a_m)$ and $F = conv(a_i : i \in I)$.

(1) Define $v := 1/|I| \sum_{i \in I} a_i$. We claim that in fact, $\hat{F} = \{x \in P^o :< x, v >= 1\}$. It is clear that $\hat{F} \subset \{x \in P^o :< x, v >= 1\}$ The reasons for the other containment are: we already know that $< x, a_i > \leq 1$ and $< x, v >= 1$ implies then that $< x, a_i >= 1$ for all $i \in I$. Since all other elements of F are convex linear combinations of a_i's we are done.

Now that the set $\hat{F}$ is a face of P^o is clear because the supporting hyperplane to the face is the linear functional $< x, v >= 1$. Warning! the $\hat{F}$ could be still empty face!!

(2) Now we convince ourselves that if F is a non-empty face, then $\hat{F}$ is non-empty and moreover the sum of their dimensions is equal to $d - 1$.

By definition of face $F = \{x :< x, c >= \alpha\}$ and for other points in P we have $< y, c >< \alpha$. Because the origin is in P we have that $\alpha > 0$, which means that $b = c/\alpha \in \hat{F}$ because 1) $< b, a_i >= 1$ for $i \in I$ and 2) $< b, a_i > \leq 1$ (this second observation is a reality check: b is in P^o). Hence $\hat{F}$ is not empty.

Suppose $dim(F) = k$ and let $h_1, \ldots, h_{d-k-1} \in \mathbb{R}^d$ be linear independent vectors orthogonal to the linear span of F. The orthogonality means that $< h_i, a_j >= 0$ for $j \in I$ and all h_i. We complete to a basis!

For all sufficiently small values $\epsilon_1, \ldots, \epsilon_{d-k-1}$ we have that $r := b + \epsilon_1 h_1 + \epsilon_2 h_2 + \cdots + \epsilon_{d-k-1} h_{d-k-1}$ satisfies $< r, a_i >= 1$ for $i \in I$ and $< r, a_j >< 1$ for other indices. Hence r is in $\hat{F}$ proving that $dim(\hat{F}) \geq d - 1 - dim(F)$.

On the other hand $\hat{F}$ is in the intersection of the hyperplanes $\{x \in \mathbb{R}^d :< x, a_i >= 1\}$ therefore $dim(\hat{F}) \leq d - 1 - dim(F)$. We are done.

(3) Denote by G the set $\{x \in P :< x, y >= 1 for\ y \in P\}$. We know from the previous two parts that G is a face of P and it has the same dimension as F and F is contained in G! who else can it be? why?

(4) Suppose again that $F \subset G$. If $x \in \hat{G}$ then $< x, y >= 1$ for all $y \in G$ in particular for all members of F hence $x \in \hat{F}$.

From now on, by the *dual* of a polytope $P \subset \mathbb{R}^d$ we mean the following: Consider the smallest affine subspace containing P, so that P has full dimension. Move the origin to be inside P and apply the polar operation to P. Regardless of where exactly you put the origin you will get the same combinatorial type of polytope!! thus if we are only interested on the combinatorial structure of polytopes the dual will be unique.

Definition 2.20 *Consider the* moment curve *which is parametrized as follows:* $\gamma(t) = (t, t^2, t^3, \ldots, t^d)$ *Take n different values for t. That gives n different points in the curve. The cyclic polytope $C(n,d)$ is the convex hull of such points.*

Firt of all we would need to convince ourselves that when we say "the" cyclic polytope it makes sense! what we really mean is that no matter which choice of values you make the same kind of polytope will appear!! That means the same number of faces, number of vertices, and their adjacencies are preserved everytime. Can you make a good picture of C(6,2)? Observe what happens when the choice of points different. This is not so trivial but is true! Here is a first step toward the understanding of this:

Proposition 2.21 *Every hyperplane intersects the moment curve $\gamma(t) = (t, t^2, t^3, \ldots, t^d)$ in no more than d points.*

Proof: Key idea: think what it means to cut the curve and think on polynomials....What kind of polytope is this?

Theorem 2.22 *The largest possible number of i-dimensional faces of a d-polytope with n vertices is achieved by the cyclic polytope $C(n,d)$.*

3 Feasibility of Polyhedra and the Simplex Method

4 Polyhedra: Solving Systems of Linear Inequalities

When is a polytope empty? Can this be decided algorithmically? How can one solve a system of linear inequalities $Ax \leq b$? We start this topic looking back on the already familiar problem of how to solve systems of linear equations. It is a crucial algorithmic step in many areas of mathematics and also would help us better understand the new problem of solving systems of linear inequalities. Recall the fundamental problem of linear algebra is

Problem: Given an $m \times n$ matrix A with rational coefficients, and a rational vector $b \in \mathbb{Q}^m$, is there a solution of $Ax = b$? If there is solution we want to find one, else, can one produce a proof that no solution exist?

I am sure you are well-aware of the Gaussian elimination algorithm to solve such systems. Thanks to this and other algorithms we can answer the first question. Something that is usually not stressed in linear algebra courses is that when the system is infeasible (this is a fancy word to mean no solution exists) Gaussian elimination can provide a *proof* that the system is indeed infeasible! This is summarized in the following theorem:

Theorem 4.1 (Fredholm's theorem of the Alternative) *The system of linear equations Ax=b has a solution if and only for each y with the property that $yA = 0$, then $yb = 0$ as well.*

In other words, one and only one of the following things can occur: Either $Ax = b$ has solution or there exist a vector y with the property that $yA = 0$ but $yb \neq 0$.

Write a proof!

Thus when $Ax = b$ has no solution we get a certificate, a proof that the system is infeasible. But, how does one compute this special certificate vector y? With care, it can be carefully extracted from the Gaussian elimination. Here is how: The system $Ax = b$ can be written as an extended matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

We perform *row operations* to eliminate the first variable from the second, third rows. Say a_{11} is non-zero (otherwise reorder the equations). Substract multiples of the first row from the second row, third row, etc. Note that this is the same as multiplying the extended matrix, on the left, by elementary lower triangular matrices. After no more than m steps the new extended matrix looks like.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ 0 & a'_{22} & \dots & a'_{2n} & b'_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a'_{m2} & \dots & a'_{mn} & b'_m \end{bmatrix}$$

Now the last $m-1$ rows have one less column. Recursively solve the system of $m-1$ equations. What happens is the variables have to be eliminated in all but one of the equations creating eventually a row-echelon shaped matrix B. Again all these row operations are the same as multiplying A on the left by a certain matrix U. If there is a solution of this smaller system, then to obtain the solution value for the variable x_1 can be done by substituing the values in the first equation. When there is no solution we detect this because one of the

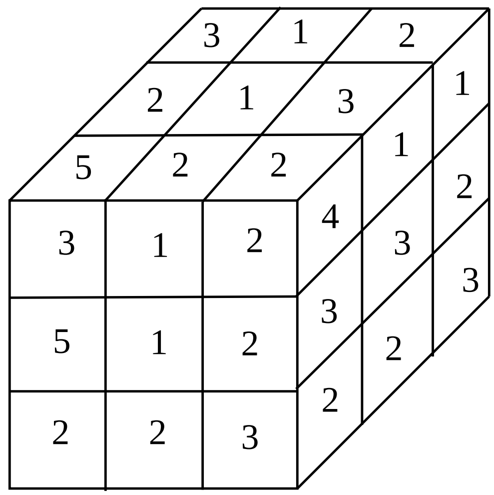

Figure 1: A cubical array of 27 seven numbers and the 27 line sums

rows, say the i-th row, in the row-echelon shaped matrix B has zeros until the last column where it is non-zero. The certificate vector y is given then by the i-th row of the matrix U which is the one producing a contradition $0 = c \neq 0$.

If you are familiar with the concerns of numerical analysis, you may be concerned about believing the vector y is an exact proof of infeasibility. "What if there are round of errors? Can one trust the existence of y?" you will say. Well, you are right! It is good time to stress a fundamental difference in this lecture from what you learned in a numerical analysis course: Computer operations are performed using *exact arithmetic* not *floating point arithmetic*. We can trust the identities discovered as exact.

Unfortunately, in many situations finding just *any* solution might not be enough. Consider the following situations:

Suppose a friend of yours claims to have a $3 \times 3 \times 3$ array of numbers, with the property that when adding 3 of the numbers along vertical lines or any horizontal row or column you get the numbers shown in Figure 1:

The challenge is to figure out whether your friend is telling the truth or not? Clearly because the numbers in the figure are in fact integer numbers one can hope for an *integral* solution, or even for a *nonnegative integral* solution because the numbers are non-negative integers. This suggests three interesting variations of linear algebra. We present them more or less in order of difficulty here below. We begin now studying an algorithm to solve problem A. We will encounter problems B and C later on. Can you guess which of the three problems is harder in practice?

Problem A: Given a rational matrix $A \in Q^{m \times n}$ and a rational vector $b \in Q^m$.

Is there a solution for the system $Ax = b,\ x \geq 0$, i.e. a solution with all non-negative entries? If yes, find one, otherwise give a proof of infeasibility.

Problem B: Given an integral matrix $A \in \mathbb{Z}^{m\times n}$ and an integral vector $b \in \mathbb{Z}^m$. Is there a solution for the system $Ax = b$, with x an integral vector? If yes, find a solution, otherwise, find a proof of infeasibility.

Problem C: Given an integral matrix $A \in \mathbb{Z}^{m\times n}$ and an integral vector $b \in \mathbb{Z}^m$. Is there a solution for the system $Ax = b,\ x \geq 0$? i.e. a solution x using only non-negative integer numbers? If yes, find a solution, otherwise, find a proof of infeasibility.

We will use a simple iterative algorithm. The key idea was introduced by Robert Bland in [3] and developed in this form by Avis and Kaluzny [1].

Algorithm: *B-Rule Algorithm*

input: $A \in Q^{m\times n}$ of full row rank and $b \in Q^m$.

output: Either a nonnegative vector x with $Ax = b$ or a vector y certifying infeasibility.

Step 1: Find an invertible $m \times m$ submatrix B of A. Rewrite the system $Ax = b$ leaving the variables associated to B in left

Step 2: Set the non-basic variables to zero. Find the *smallest index* of a basic variable with negative solution.

If there is none, we have found a feasible solution x stop.

Else, select the equation corresponding to that basic variable.

Step 3: Find the non-basic variable in the equation chosen in step 2, that has smallest index and a positive coefficient.

If there is none, then the problem is infeasible.

Else, solve this equation for the non-basic variable and substitute the result in all other equations. Go to step 2.

Example 4.2 *Solve the next system for $x_i \geq 0,\ i = 1, 2, ..., 7$.*

$$\begin{aligned}
&2x_1 + x_2 + 3x_3 + x_4 + x_5 = 8\\
&2x_1 + 3x_2 + 4x_4 + x_6 = 12\\
&3x_1 + 2x_2 + 2x_3 + x_7 = 18.
\end{aligned}$$

Before to run the B-Rule Algorithm we have to write our problem in a more convenient form: $Ax = b$, it is a good practice for the student to do it in order to avoid making common mistakes. According to the first step of the B-Rule Algorithm, we have to find a basis in the matrix A, in this example we choose the easiest-elegible basis, which is given by the 5th, 6th and 7th columns of A, let us denote the basis by $B = \{5, 6, 7\}$ and the set of the remaining vectors by $NB = \{1, 2, 3, 4\}$. Next we solve the equation $Ax = b$ for the basic variables $X_B = \{x_5, x_6, x_7\}$.

$$X_B = B^{-1}b - B^{-1}CX_{NB} = \begin{bmatrix} 8 \\ 12 \\ 18 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 3 & 1 \\ 2 & 3 & 0 & 4 \\ 3 & 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The second step in the B-Rule Algorithm says that we have to set all non-basic variables equal to zero. So we obtain the non-negative values $x_5 = 8$, $x_6 = 12$, $x_7 = 18$. Therefore, and according to third step, this problem is feasible, and its solution is given by $x_1 = x_2 = x_3 = x_4 = 0$, $x_5 = 8$, $x_6 = 12$ and $x_7 = 18$.

Now, suppose that we choose a different basis from the above, say $B' = (1, 4, 7)$, and solve the problem keeping this election. It is not difficult for the student to obtain solution $x_1 = 10/3$, $x_2 = x_3 = 0$, $x_4 = 10/3$, $x_5 = x_6 = 0$, $x_7 = 8$.

We can easily observe that the two solutions are completely different. In general, the solution always depends on the election of the basis, it means that the solution is not unique.

Example 4.3 *Solve the system $Ax = b$ for $x_i \geq 0$, $i = 1, 2, ..., 6.$, where A and b are given by*

$$A = \begin{bmatrix} -1 & -2 & 1 & 1 & 0 & 0 \\ 1 & -3 & -1 & 0 & 1 & 0 \\ -1 & -2 & 2 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

Let us to choose a basis from A, say $B = \{4, 5, 6\}$. Next we solve the system for the basic variables $X_B = \{x_4, x_5, x_6\}$.

$$\begin{aligned} x_4 &= -1 + x_1 + 2x_2 - x_3 \\ x_5 &= 2 - x_1 + 3x_2 + x_3 \\ x_6 &= -2 + x_1 + 2x_2 - 2x_3 \end{aligned}$$

Setting all non-basic variables equal to zero, as the second setp indicates, we get $x_4 = -1$, $x_5 = 2$ and $x_6 = -2$. Note that x_4 and x_6 are basic variables with negative solution, then we choose the equation that corresponds to x_4 in the equation above, which is called Dictionary. Now, we must find the non-basic variable in the equation that has smallest index and a positive coefficient, in this case it is clear that x_1 is such a variable. Next, according to step three, we have to solve the first equation for that non-basic variable, taking from now x_1 as basic variable and coming back to step two of the algorithm.

$$\begin{aligned}
x_1 &= 1 - 2x_2 + x_3 + x_4 && = 1 - 2x_2 + x_3 + x_4 \\
x_5 &= 2 - (1 - 2x_2 + x_3 + x_4) + 3x_2 + x_3 && = 1 + 5x_2 - x_4 \\
x_6 &= -2 + (1 - 2x_2 + x_3 + x_4) + 2x_2 - 2x_3 && = -1 - x_3 + x_4
\end{aligned}$$

Setting again all non-basic variables equal to zero we obtain $x_1 = 1$, $x_5 = 1$ and $x_6 = -1$. In this case x_6 has negative solution so we choose the corresponding equation to that basic variable and look for the non-basic variable in the equation that has smallest index and a positive coefficient, in this case x_4. Next we solve that equation for x_4 and rewrite the system as follow.

$$\begin{aligned}
x_1 &= 1 - 2x_2 + x_3 + (1 + x_3 + x_6) && = 2 - 2x_2 + 2x_3 + x_6 \\
x_4 &= 1 + x_3 + x_6 && = 1 + x_3 + x_6 \\
x_5 &= 1 + 5x_2 - (1 + x_3 + x_6) && = 0 + 5x_2 - x_3 - x_6
\end{aligned}$$

Notice that in the above system one equation changed position. Back again to step two, but now with x_1, x_4 and x_5 as basic variables, we set all non-basic variables equal to zero obtaining non-negative solutions for the basic variables. So we have found that one solution to the problem is $x1 = 2$, $x_2 = x_3 = 0$ $x_4 = 1$ and $x_5 = x_6 = 0$.

We have showed, so far, examples where there exists a solution, next we will cosider an example which does not have any solution.

Example 4.4 *Solve the system $Ax = b$ for $x_i \geq 0$, $i = 1, ...6$, where A and b are given as follow.*

$$A = \begin{bmatrix} -1 & 2 & 1 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ -1 & -6 & 23 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ -17 \\ 19 \end{bmatrix}$$

First choose a basis from A, say $B = \{4, 5, 6\}$, and solve the system for the basic variables x_4, x_5 and x_6.

$$\begin{aligned}
x_4 &= 3 + x_1 - 2x_2 - x_3 \\
x_5 &= -17 - 3x_1 + 2x_2 - x_3 \\
x_6 &= 19 + x_1 + 6x_2 + 23x_3
\end{aligned}$$

Next set all non-basic variables equal to zero. We obtain $x_4 = 3$, $x_5 = -17$ and $x_6 = 19$. Since x_5 has negative solution we have to find the non-basic variable in the second equation that has smallest index and a positive coefficient. Solve that equation for x_2 and after substitute the variable x_5 by the variable x_2 in the basis.

$$\begin{aligned}
x_2 &= 17/2 + 3/2x_1 + 1/2x_3 + 1/2x_5 && = 17/2 + 3/2x_1 + 1/2x_3 + 1/2x_5 \\
x_4 &= 3 + x_1 - 2(17/2 + 3/2x_1 + 1/2x_3 + 1/2x_5) - x_3 && = -14 - 2x_1 - 2x_3 - x_5 \\
x_5 &= 19 + x_1 + 6(17/2 + 3/2x_1 + 1/2x_3 + 1/2x_5) + 23x_3 && = 70 + 10x_1 + 26x_3 + 3x_5
\end{aligned}$$

Now we are in setp two again, and we set all non-basic variables equal to zero. The only solution negative is $x_4 = -14$, so we must choose the correspondig equation to x_4. We realize that all coefficients in that equation are negative, therefore, according to step three, the problem is infeasible. There is not solution to the problem.

Suppose there exists such a solution $x_i \geq 0$, $\forall i$ such that$2x_1 + 0x_2 + 2x_3 + x_4 + x_5 = -14$. Clearly this is a contradiction, because a positive number can not be equal to a negative number.

The first main theorem is the following:

Theorem 4.5 *The B-Rule Algorithm terminates*

The proof is really easy from the following lemma:

Lemma 4.6 *If x_n is the last variable, during the b-rule iterations, x_n cannot enter the basic variables and then leave neither be chosen to leave the basic variables and then later on re-enter.*

Write a proof

Proof: (of Theorem). We procede by contradiction. Suppose there is a matrix A and a vector b for which the algorithm does not terminate. Let us assume that A is an example with smallest number of rows plus columns. Since there is a finite number of bases, in fact no more than $\binom{n}{m}$, then if the algorithm does not terminate one can find a cycle of iterations. In other words, one starts at one basis B_1, then moves to $B_2, B_3, \ldots$, and after say k iterations one returns to B_1.

Now by the lemma, during this cycle of bases, the last variable x_n is either in all B_i or in none of them. In the first case x_n is the basic variable associated to an equation that we can discard without affecting the choice of variables entering or leaving the basis. Thus we have a smaller counterexample. Similarly, if x_n is always non-basic then we can set $x_n = 0$ and still the remaining equations would give a smaller counterexample. In both cases we reach a contradiction.

Theorem 4.7 *Given $Ax = b$, $A \in \mathbb{Q}^{m \times n}$ matrix, $b \in Q^m$, then either $\exists x \geq 0$, a solution, or $\exists y, yA \geq 0$ but $yb < 0$. Only one of these cases is possible.*

Proof: Say both x and y exist simultaneously. Then we see that Ax=b $\Rightarrow$ $(yA)x = yb < 0$ where (yA ≥ 0). Now using the B-Rule Algorithm, you find a solution or reach a situation where $x_i = b'_i + \sum_{j \notin B} a'_{ij} x_j$ where B is the set of basic variables. Rewriting the equation we see that $x_i + \sum_{j \notin B} (a'_{ij}) x_j = b'_i$ Contradition.$\blacksquare$

We define $Cone(A)$ as the set of all non-negative linear combinations of columns of A. A solution resides within the cone. If there is no solution then b is not inside the cone and y represents a plane that separates b from A as shown in Figure 2. However, in Figure 3 we see the case where b in fact resides within the cone which means that there is a solution.

The key is that empty polyhedra come with some kind of "emptiness certificate", similar to Fredholms alternative theorem, via Farka's lemma: There is something really profound in Farkas'lemma and we will look at another nice new proof of it.

Theorem 4.8 *For a system of equations $Ax = b$, where A is a matrix and b is a vector. One and only one of the following choices holds:*

- *There is a non-negative vector x with $Ax = b$.*
- *There exists a non-trivial vector c such that $cA \geq 0$ but $c \cdot b < 0$.*

Proof: Clearly if the second option holds there cannot be positive solution for $Ax = b$ because it gives $0 \leq (cA)x = c(Ax) = cb < 0$.

Now suppose that $yb \geq 0$ for all y such that $yA \geq 0$. We want to prove that then b is an element of the cone K generated by the non-negative linear combinations of columns of A. For every b in $\mathbb{R}^n$ there exist in the cone $K = \{Ax|x \geq 0\}$ a point a that is closes to b and $Ax = a$ for $x \geq 0$. This observation is quite easy to prove and we leave it as an exercise (there are very easy arguments when the cone K is pointed). Now using this observation we have that

$$(A_j, b - a) \leq 0, \quad j = 1 \ldots k \tag{1}$$

and

$$(-a, b - a) \leq 0. \tag{2}$$

Why? the reason is a simple inequality on dot products. If we do not have the inequalities above we get for sufficiently small $t \in (0, 1)$:

$$|b - (a + tA_j)|^2 = |(b - a) - tA_j|^2 =$$
$$|b - a|^2 - 2t(A_j, b - a) + t^2|A_j|^2 < |b - a|^2$$

or similarly we would get

$$|b - (a - ta)|^2 = |(b - a) + ta|^2 = |b - a|^2 - 2t(-a, b - a) + t^2|a|^2| < |b - a|^2$$

Both inequalities contradict the choice of b because $a + tA_j$ is in K and the same is true for $a - ta = (1 - t)a \in K$. We have then that from the hypothesis and the equations in (ONE) that $(b, -(b-a)) \geq 0$, which is the same as $(b, b-a) \leq 0$ and this together with equation (TWO) $(-a, b-a) \leq 0$ gives $(b-a, b-a) = 0$, and in consequence $b = a$. ∎

The theorem above is equivalent to

Theorem 4.9 *For a system of inequalities $Ax \leq b$, where A is a matrix and b is a vector. One and only one of the following choices holds:*

- *There is a vector x with $Ax \leq b$.*
- *There exists a vector c such that $c \cdot b < 0$, $c \geq 0$, $\sum c_i > 0$, and $cA = 0$.*

The reason is simple, The system of inequalities $Ax \leq b$ has a solution if and only if for the matrix $A' = [I, A, -A]$ there is a non-negative solution to $A'x = b$. The rest is only a translation of the previous theorem in the second alternative. If you tried to solve the **strict** inequalities in the system $Bx < 0$, like the one we got for deciding convexity of pictures, you would run into troubles for most computer programs (e.g. MAPLE, MATHEMATICA,etc). One needs to observe that a system of strict inequalities $Bx < 0$ has a solution precisely when $Bx \leq -1$ has a solution. If the solution x gives $Bx < -1/q$ for instance px is a solution for the strict inequality and vice versa. Thus the above theorem implies the Farkas' lemma version we saw earlier.

The system of inequalities $Ax \leq b$ has a solution if and only if $yb \geq 0 yA = 0, y \geq 0$ has a solution. You have two systems that are match to each other. They have different number of variables and therefore represent polyhedra in different spaces and dimensions. But still they share this property. This is the foundation of duality in linear programming. Farkas'lemma actually implies a similar matching phenomenon between a maximization LP and a minimization LP:

Theorem 4.10 (Duality theorem of linear programming) *Let A be a matrix and b, c vectors (of adequate dimensions). The*

$$maxcx \ subject \ to Ax \leq b = minyb \ subject \ to yA = c, y \geq 0$$

.

Lemma 4.11 *Given any system of inequalities $Ax \leq b$, $Cx \geq d$, then it can be transformed into a system of the form $D\bar{x} = f$ with the property that one system has a solution $\Leftrightarrow$ the other system has a solution.*

Proof: If you have the inequality
$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$, add the variable $s_i \rightarrow \sum_{j=1}^{n} a_{ij}x_j + s_i = b_i$
Similarly, $\sum_{j=1}^{n} c_{ij}x_j \geq d_i \rightarrow \sum_{j=1}^{n} c_{ij}x_j - t_i = d_i$ ∎

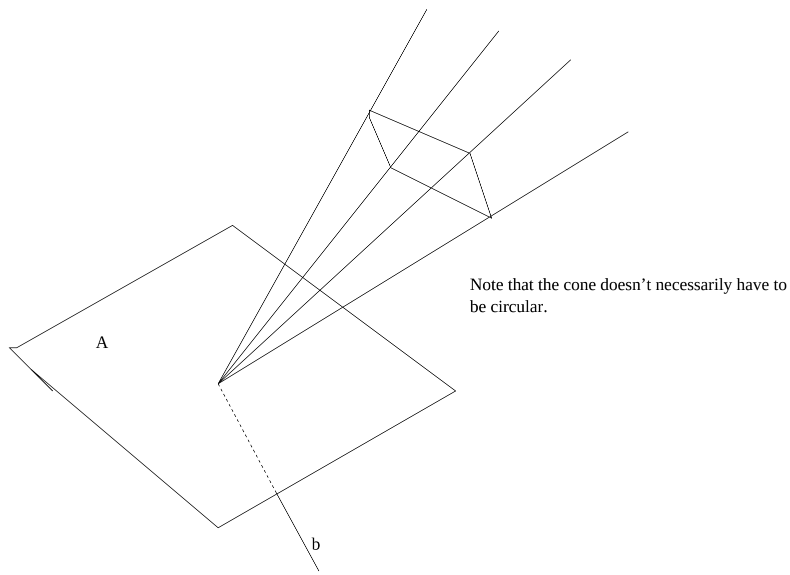

Figure 2: A problem with no solution. The plane A separates b from the cone.

Example 4.12 *Solve the system of inequalities:*

7x + 3y - 20z $\leq$ *-2*
4x - 3y + 9z $\leq$ *3*
-x + 2y - z $\geq$ *4*
11x - 2y + 2z $\geq$ *11*

Using the previous lemma, we can now modify the system:

$7x^+ - 7x^- + 3y^+ - 3y^- + 20z^+ - 20z^- + s_1 = -2$
$4x^+ - 4x^- - 3y^+ + 3y^- + 9z^+ - 9z^- + s_2 = 3$
$-x^+ + x^- + 2y^+ - 2y^- - z^+ + z^- - t_1 = 4$
$11x^+ - 11x^- - 2y^+ + 2y^- - 2z^+ + 2z^- - t_2 = 11$

where $x^\pm, y^\pm, z^\pm, t_1, t_2, S_1, s_2 \geq 0$ now we can solve by B-Rule Algorithm.

We now describe the (in very broad terms) a well-known algorithm to find the optimal solution of a linear programming problem of the form $\min cx$ subject to $x \in \{x : Ax = bx \geq 0\}$. The simplex method was developed by George Dantzig in 1947 to solve linear programs. It proceeds from a given extreme point to an adjacent extreme point in a such a way that the value of the objective function increases. The method stops when we either find an optimal solution of find the problem is in fact unbounded. We already discussed in the B-rule algorithm a way to find one basic feasible solution, if any, now we need to talk about a way to obtain a new improved basic feasible solution.

The geometric intuitive idea for solving a linear program is that, although there are infinitely many points inside a polyhedron, we already saw there are

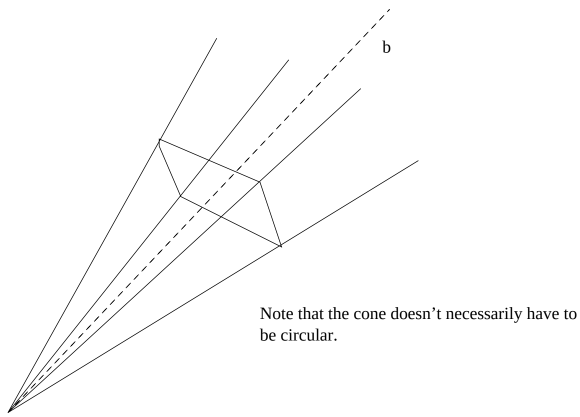

Figure 3: A problem with a solution. We see that b resides within the cone.

finitely many extreme points. Thus the search for an optimal solution can be limited to the extreme points of S. Moreover we also saw that the extreme points can be computationally identified as basic feasible solutions of the system defining a polyhedron. But rather than looking in a disorganized way we will move progressively from one vertex to the next until we find the optimal.

WRITE DOWN AN EXAMPLE.

Definition 4.13 *Two distinct extreme points in a polyhedron P are said to be adjacent if as basic feasible solutions they have all but one basic variable in common.*

Definition 4.14 *Given a basic solution and a basis B, let c_B be the vector of basic variables. For each j define the reduced cost $\bar{c}_j$ of a variable x_j by*

$$\bar{c}_j = c_j - c_B B^{-1} A_j$$

Lemma 4.15 *Consider a basic feasible solution x associated with the basis B, and let $\bar{c}$ be the corresponding vector of reduced costs. If $\bar{c} \geq 0$ then x is optimal. Moreover if x is optimal and nondegenerate then $\bar{c} \geq 0$.*

Proof: First assume $\bar{c}$ is nonnegative. Let y be any other feasible solution and let $d = y - x$. Thus we have $Ax = Ay = b$ and thus $Ad = 0$. Thus we can rewrite this last equality as

$$Bd_B + \sum_{i \in N} A_i d_i = 0$$

Where N is the set of indices of nonbasic variables. Since B is invertible, we obtain $d_B = -\sum_{i \in N} B^{-1} A_i d_i$. Thus

$c(y-x) = cBd_B + \sum_{i \in N} c_i d_i = \sum_{i \in N}(c_i - c_B B^{-1} A_i) d_i = \sum_{i \in N} \bar{c}_i d_i \geq 0$

The last inequality is obtained since $y_i \geq 0, x_i = 0$ for all basic variablesand $d_i \geq 0$.

1. Put the linear program into standard form. This means making the linear program a minimization problem and changing inequality constraints to equality constraints.
2. Use the B-bland rule to find a first basic feasible solution.
3. Calculate the Reduced costs and test for optimality for current basis B
4. Choose the entering variable from among those that have negative reduced coset. If all are positive we have found an optimum solution else we choose j with $\bar{c_j} < 0$.
5. Test for unboundedness, for this compute $u = B^{-1} A_j$. If no component of u is positive, we have problem is unbounded. Stop.
6. Choose the leaving variable by the Min Ratio Test:
$$\min_{\{i : u_i > 0\}} x_{B_i}/u_i$$
. Let l be one variable attaining the minimum among those in the current basis.
7. Update the solution and change the basis. The new basis replaces A_l by A_j.
8. Go to Step 3.

4.1 Graph of Polytopes

The study of graphs of polytopes is a very worthy and classic endeavour. There are famous connections of graphs of 3-dimensional polytopes and the four color problem for example. We work here with graphs in the context of discrete optimization, these are abstract collections of nodes joined by edges. For several the concepts here we refer to any book in graph theory We also recommend [**?**].

In the 1960's Balinski proved a now classical theorem about the graphs of polytopes:

Theorem 4.16 *The graphs of d-dimensional polytopes are d-connected.*

before we sketch a the proof we should observe that given a linear function $l(x) : \mathbb{R}^d \to \mathbb{R}$ such that $l(v_i) \neq l(v_j)$ for any pair of vertices of a d-polytope P, we can orient or direct the edges of P from v_i to v_j whenever $l(v_i) < l(v_j)$ and v_i, v_j are adjacent. The orientation we have produced allows you to find the "winning" vertex that maximizes the linear function by simply walking along the graph!! the basic reason is the following lemma:

Lemma 4.17 *For a d-polytope P and a linear function l that is not constant in an edge of P. Then for every vertex v_0 of P either $l(v) = max\{l(x) : x \in P\}$ or there exists a neighbor u of v such that $l(u) > l(v)$.*

Proof: Denote by $v_1, v_2, \ldots, v_k$ the neighbors of v_0 in the graph of P. It is enough to be sure that any point of P, say x, can be written in the form

$$x = v_0 + \sum_{i=1}^{k} \lambda(v_i - v_0).$$

With positive coefficients. The reason this is enough is because if one such expression exists $l(v_i - v_0)$ is positive precisely when moving from v_0 to v_i increases the value of l. If no $l(v_i - v_0)$ is positive then clearly staying at v_0 is best you can do. How can we prove that equation above? Take the interval segment $[x, v_0]$. For a point r inside the interval close enough to v_0 we have that indeed

$r = t(x - v_0) = \sum_{i=1}^{k} s_i(v_i - v_0)$

By the property of being in a convex hull. So we are done. ∎

Now we prove Balinski's theorem:

Proof: **(Balinski's theorem)** We proceed by induction on the dimension of the polytope. It is clear the theorem holds for dimension one because polytopes are simply segments. Assume the theorem is true for all polytopes of dimension less than d and suppose you have a d-polytope with a vertex-cut $v_1, \ldots, v_s$ with $s < d$. This means that $s + 1 \leq d$. Because in $\mathbb{R}^d$ a hyperplane is determined by d points we can make a H hyperplane pass through the v_i's plus at least one vertex not in the set.

There are two cases: 1) H is supporting hyperplane for a face or 2) H cuts through the interior of the polytope. In the first case find the hyperplane H' that parallel to H that supports an another face. For any two points not in the cut we can move, using the previous lemma, until we reach H'. There by induction any two points are connected, so we have find a connecting path.

In the second case. We again have two hyperplanes H' and H'' parallel to H supporting the polytope from below and from above in two faces F', F''. if the two points not in the cut are in the same side of H we repeat the argument to connect via the use of F' or F'' depending on which side the points are present. If they are in opposite sides we still have the extra point along H to connect them (we can construct an increasing path from that extra point to F' or a decreasing path to F''). ∎

5 Branch-and-Bound and Integer Linear Programming

6 Convex Hull and Listing Algorithms

6.1 Reverse-Search Enumeration

References

[1] Avis, D. and Kaluzny, B. *Solving inequalities and proving Farkas's lemma made easy* American Math. Monthly, vol. 111, No.2, 2004, 152–157.

[2] Barvinok, A.I. and Pommersheim, J. *An algorithmic theory of lattice points in polyhedra*, in: *New Perspectives in Algebraic Combinatorics* (Berkeley, CA, 1996-1997), 91-147, Math. Sci. Res. Inst. Publ. 38, Cambridge Univ. Press, Cambridge, 1999.

[3] Bland, R.G *New pivot rules for the simplex method*, Math. Oper. Res. 2 (1977) 103-107.

[4] Schrijver, A. *Theory of Linear and Integer Programming.* Wiley-Interscience, 1986.