Math 16C (De Loera)
Mid-term exam 1
April 26 2007

Name: SOLUTION KEY
Student ID#
Section number

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Show your work on every problem. Correct answers with no support work will not receive full credit. Be organized and use the notation appropiately. No calculators are allowed, nor is any assistance from classmates, notes, or books. You should only have a writing and an erasing implement on your desk.

Please write legibly!!

#	Student's Score	Maximum possible Score
1		8
2		7
3		7
4		8
Total points		30

1. (8 points) A 200-gallon tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 pounds of concentrate per gallon enters the tank at a rate of 5 gallons per minute, and the well-stirred mixture is withdrawn at the same rate. Find the amount Q of concentrate in the tank after 30 minutes.

Q = amount of concentrate (in lbs)

$$\frac{dQ}{dt} = (\text{rate of conc. IN in lbs/min}) - (\text{rate of conc. OUT in lbs/min})$$

$$= (0.5 \text{ lbs/gal})(5 \text{ gal/min}) - 5 \cdot \frac{Q(t)}{100}$$

$$= \frac{5}{2} - \frac{Q(t)}{20}$$

Thus, $\frac{dQ}{dt} + \frac{1}{20}Q = \frac{5}{2}$. Using 1st order linear D.E. method,

$$u = e^{\int \frac{dt}{20}} = e^{t/20}, \text{ thus}$$

$$Q(t) = \frac{1}{e^{t/20}} \int \frac{5}{2} e^{t/20}\, dt = \frac{1}{e^{t/20}} \left[\frac{5}{2}\left(\frac{1}{1/20}\right) e^{t/20} + C \right] = 50 + Ce^{-t/20}$$

Initial condition: Distilled water, so no lbs of concentrate @ $t=0$ in the tank. In other words, when $t=0$, $Q(t)=0$. Solve by substitution in here to get $C=-50$.

So $Q(t) = 50 - 50e^{-t/20}$.

After 30 minutes ($t=30$), $Q(30) = 50 - 50e^{-30/20}$.

$\boxed{\text{After half an hour, } (50 - 50e^{-3/2}) \text{ lbs of concentrate in the tank.}}$

2. (7 points. This is Hmw 2 #44 A38) An infectious disease spreads through a large population according to the model:

$$\frac{dy}{dt} = \frac{1-y}{4}$$

where y is the percentage of the population exposed to the disease, and t is the time in years.

- Solve this differential equation assuming that $y(0) = 0$
- Find the number of years it takes for half of the population to have been exposed to the disease.
- Find the percentage of the population that has been to the disease after 4 years.

Rewrite diff. eqn. in standard form: $\frac{dy}{dt} + \underbrace{\frac{1}{4}}_{P(t)} y = \underbrace{\frac{1}{4}}_{Q(t)}$

So $P(t)$ and $Q(t)$ are both $\frac{1}{4}$. $\longrightarrow$ $P(t)$ $Q(t)$

Integrating factor $= u(t) = e^{\int P(t)\,dt} = e^{\int \frac{1}{4}dt} = e^{\frac{1}{4}t}$.

General solution is $y = \frac{1}{u(t)} \int Q(t)u(t)\,dt = \frac{1}{e^{\frac{1}{4}t}} \int \frac{1}{4} e^{\frac{1}{4}t} dt = \frac{1}{e^{\frac{1}{4}t}}\left(e^{\frac{1}{4}t} + C\right)$

$= 1 + Ce^{-t/4}$.

Initial condition $y(0) = 0$ is given, so $0 = 1 + Ce^{-0/4}$, meaning $C = -1$.

Thus, our particular solution is $\boxed{y = 1 - e^{-t/4}}$.

Half the population: $y = \frac{1}{2}$

$$\frac{1}{2} = 1 - e^{-t/4}$$
$$\frac{1}{2} = e^{-t/4}$$
$$\ln(1/2) = -t/4$$
$$t = -4\ln(1/2)$$

$\boxed{\text{In } -4\ln(1/2) \text{ years, half the population is exposed.}}$

In 4 years: $t = 4$ $\quad y(4) = 1 - e^{-4/4} = 1 - e^{-1}$.

$\boxed{\text{In 4 years, } 1-e^{-1} \text{ of the population (on a scale of 0 to 1) is exposed.}}$

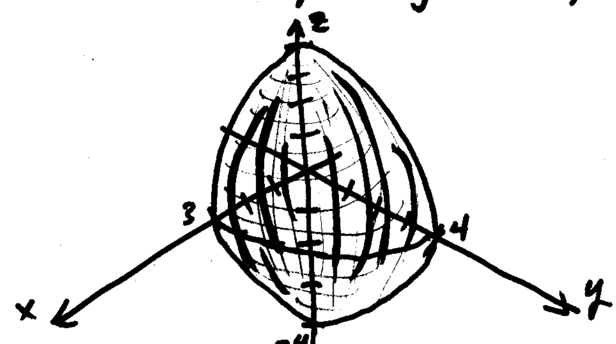
3. (7 points)

(a) (Hmw 2, page 473 #44) Identify the quadric surface $\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{16} = 1$. Use either words or a picture, but give a short reason for your answer. Your drawings don't have to be very precise.

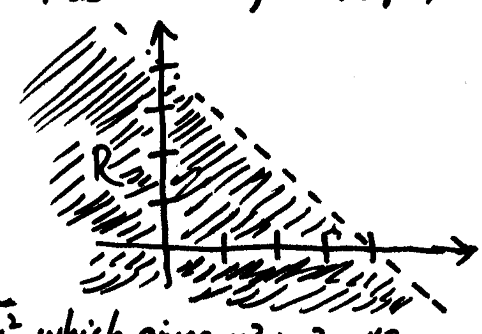
(b) (Hmw 3, sec 7.3, #27) Describe the region R in the xy-coordinate plane that corresponds to the domain of the function: $ln(4 - x - y)$.

(c) (Hmw 3, sec 7.3, #35) Describe the level curves of the function $z = \sqrt{16 - x^2 - y^2}$ for $c = 0, 1, 2, 3, 4$.

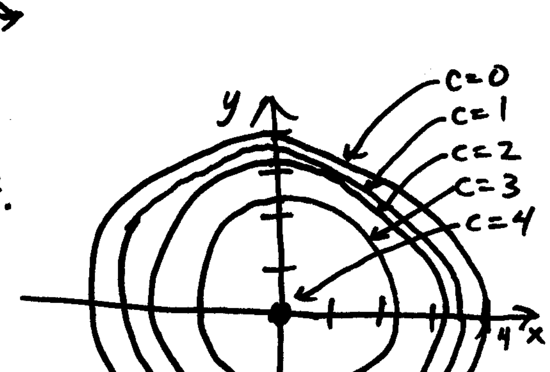
(a) The surface equation can be rewritten $\frac{x^2}{3^2} + \frac{y^2}{4^2} + \frac{z^2}{4^2} = 1$. The surface is an ellipsoid centered at the origin with axes of lengths 3, 4, and 4 in the x, y, and z directions (respectively). In pictures,



(b) Natural log can't take negative inputs, so valid inputs for the function $f(x,y) = ln(4-x-y)$ occur when $4-x-y>0$. What does the region R defined by $4-x-y>0$ look like? Since $4-x-y=0$ describes a line (call it L), $4-x-y>0$ is everything on one side of L (but not including L itself). The line L has 4 as x- and y-intercepts, and the origin satisfies the inequality, thus:



(c) For $c=0$ $0=\sqrt{16-x^2-y^2}$ which gives $x^2+y^2=4^2$. It's a circle centered @ origin with radius 4.
Plug in $c=1$ to get $x^2+y^2=15$, a circle of radius $\sqrt{15}$.
Similarly $\left.\begin{array}{l} c=2 \\ \ \ =3 \\ \ \ =4 \end{array}\right\}$ are circles of radius $\left\{\begin{array}{l} \sqrt{12} \\ \sqrt{7} \\ 0 \end{array}\right.$



A circle of radius 0, so just the origin alone.

4. (8 points) A corporation manufactures a product at two locations. The cost function for producing x_1 units at location 1 and x_2 units at location 2 are given by

$$C_1 = 0.03x_1^2 + 4x_1 + 300$$

and

$$C_2 = 0.05x_2^2 + 7x_2 + 175$$

respectively. If the product sells for \$10 per unit, find the production levels x_1, x_2 such that the profit is maximized.

Profit function: $P(x_1,x_2) = \text{Earnings} - \text{Cost}$

$$= 10(x_1+x_2) - C_1 - C_2$$

$$= -0.03x_1^2 + 6x_1 - 0.05x_2^2 + 3x_2 - 475.$$

$$\left.\begin{array}{l} \dfrac{\partial P}{\partial x_1} = -0.06x_1 + 6 = 0 \\ \dfrac{\partial P}{\partial x_2} = -0.10x_2 + 3 = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x_1 = \dfrac{6}{0.06} = 100 \\ x_2 = \dfrac{3}{0.10} = 30 \end{array}\right\}$$

$$\frac{\partial^2 P}{\partial x_1^2} = -0.06 \qquad \frac{\partial^2 P}{\partial x_2^2} = -0.10 \qquad \frac{\partial^2 P}{\partial x_1 \partial x_2} = \frac{\partial^2 P}{\partial x_2 \partial x_1} = 0$$

$D = (-0.06)(-0.10) - 0^2 > 0$ and $\dfrac{\partial^2 P}{\partial x_1^2} < 0$ so this critical point is a relative maximum of P.

Tell the boss to produce $x_1 = 100$ units at location 1 and $x_2 = 30$ units at location 2.