Math 16C (De Loera)
Final Exam
June 8th 2010

Name: Solution Key
Student ID:
Signature:

DO NOT TURN OVER THIS PAGE
UNTIL INSTRUCTED TO DO SO!

Write your name, student ID, and signature NOW!

NO NOTES, CALCULATORS, OR BOOKS ARE ALLOWED.
NO ASSISTANCE FROM CLASSMATES IS ALLOWED EITHER.

Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. Be organized and neat, and use notation appropriately. You will be graded on the proper use of derivative and integral notation.

Please write legibly!

#	Student's Score	Maximum possible Score
1		5
2		7
3		5
4		5
5		5
6		5
7		7
8		6
Total points		45

(5 points)

1. Problems about differential equations:

   (a) Show that the equation $y = \frac{1}{x}$ is a solution of the differential equation

   $$xy'' + 2y' = 0.$$

   (b) A ball is dropped off a bridge. Let v be the velocity of the ball towards the ground at time t. The air resistance on the falling ball is proportional to the velocity of the ball. By Newton's Second Law of Motion, v satisfies the famous equation

   $$m\frac{dv}{dt} = mg - kv$$

   where $m = 10$ is the mass of the ball, $g = 9.8$ is the gravitational constant, and $k = 1$ is the constant of proportionality for the air resistance.

   (a) Find the particular solution to the differential equation.

   (b) What is the velocity v of the ball as $t \to \infty$?

(a) $y' = -\frac{1}{x^2}$ and $y'' = \frac{2}{x^3}$. $xy''+2y' = x\left(\frac{2}{x^3}\right) + 2\left(-\frac{1}{x^2}\right) = \frac{2}{x^2} - \frac{2}{x^2} = 0.$ +1 for verification

---

$m\frac{dv}{dt} = mg - kv \Rightarrow 10\frac{dv}{dt} = (9.8)(10) - (1)v = 98 - v$

(a) So $\frac{dv}{dt} + \frac{1}{10}v = 9.8$ Let $u(t) = e^{\int \frac{1}{10}dt} = e^{t/10}$. +1 for int. factor

Then $v = e^{-\frac{t}{10}}\int 9.8e^{\frac{t}{10}}dt = e^{-\frac{t}{10}}(98e^{\frac{t}{10}} + C)$ +1 for general solution v

Since at $t=0, v=0$ we have $0 = e^{-\frac{0}{10}}(98e^{\frac{0}{10}} + C)$, solve for C:

$C = -98$

Therefore, the particular solution is $\boxed{v = 98 - 98e^{-\frac{t}{10}}}$ +1 for particular solution v

---

(b) $\lim_{t\to\infty} v(t) = \lim_{t\to\infty}(98 - 98e^{-\frac{t}{10}}) = 98$

The terminal velocity is $\boxed{98}$.

+1 for setting up and evaluating limit of previous answer

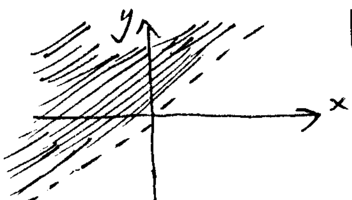
(7 points)

2. (a) Consider the function $f(x) = 1 + \ln(y - x)$. Find the *domain* and *range* of f and sketch the domain.

(b) Sketch the plane in 3 dimensions given by the equation $2y + z = 2$.

(c) For each of the following 2 equations describe by name or by a sketch what kind of surface is supposed to describe. Make sure to give the most details possible (e.g. show a few level curves): (a) $3x^2 + y^2 - z = 0$. (b) $x^2 - 2y^2 + z = 0$.

(a) Domain: $y - x > 0$, so $y > x$.
+1 for this or this
+1 for sketch



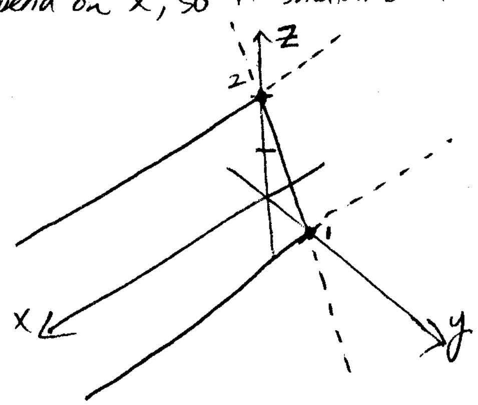
[N.B. Up to one point taken away for $\geq$ and including boundary line in sketch.]

Range: all real numbers are possible. Why? If you want the output to be p, set $x = 0$ and $y = e^{p-1}$. Then, $f(x,y) = f(0, e^{p-1}) = 1 + \ln(e^{p-1} - 0) = 1 + \ln e^{p-1} = 1 + p - 1 = p$.

+2 for identifying the correct range
~~+1 for justification~~

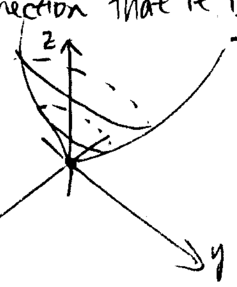
(b) Note: doesn't depend on x, so it shouldn't "tilt" in that direction

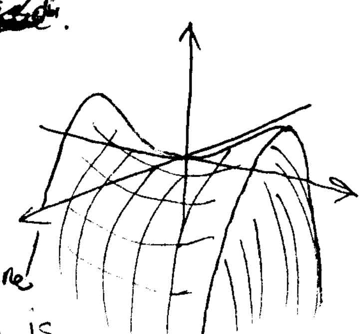
+1 for sketch



(c) The surface defined by $3x^2 + y^2 - z = 0$ can be rewritten $z = 3x^2 + y^2$. For $z < 0$, no solution. For a fixed $z \geq 0$, you get an ellipse, which is $\sqrt{3}$ times wider in the y direction that it is the x direction. It's an "elliptic [illegible]".

+1 for words or picture





+1 for words or picture

The second equation can be rewritten $z = -x^2 + 2y^2$, which is a hyperbola for each fixed value of z. This gives a "hyperbolic paraboloid". [N.B. If words are correct but not sufficent, award one point for both parts of (c).]

5 pts

3. (a) Find all of the critical points of the function $f(x, y) = 3x^2y - 4xy + y^2$. (Hint: the function has three critical points.)

   (b) The function $g(x, y) = 2x^3 + y^2 - 6xy$ has two critical points: (0,0) and (3, 9). Classify each of the critical points of g.

(a) $f_x = 3 \cdot 2xy - 4y = 6xy - 4y = y(6x-4)$ +1 finding partials correctly and setting them equal to zero

$f_y = 3x^2 - 4x + 2y$

$\underbrace{y(6x-4)=0} \text{ and } 3x^2-4x+2y=0$

$[y=0 \text{ or } x=\frac{2}{3}] \text{ and } 3x^2-4x+2y=0.$ +1 finding all crit pts

If $y=0$, $3x^2-4x+0=0 \xrightarrow{\text{rewrite}} x(3x-4)=0$, so $x=0$ or $x=\frac{4}{3}$.

If $x=\frac{2}{3}$, the second equation becomes $\frac{4}{3}-\frac{8}{3}+2y=0$, so $y=\frac{2}{3}$.

The critical points are (x,y) is $\boxed{(0,0),\ (\tfrac{4}{3},0) \text{ or } (\tfrac{2}{3},\tfrac{2}{3})}$

(b) $g = 2x^3+y^2-6xy$

$g_x = 6x^2-6y$

$g_{xx} = 12x$ and $g_{xy} = -6$

$g_y = 2y - 6x$

$g_{yy} = 2$

+2 for computing discriminant and evaluating at the crit pts (w/ partial credit)

+1 correct identification

The discriminant is $d = (12x)(2) - (-6)^2 = 24x - 36$

- at $(x,y)=(0,0)$, $d = 24(0)-36<0$, so $\boxed{(0,0) \text{ is a saddle point.}}$
- at $(x,y)=(3,9)$, $d = 24(3)-36>0$ and $g_{xx}(3,9) = 12(3)>0$, so $\boxed{(3,9) \text{ is a relative/local minimum.}}$

Note: +1 for any correct id i.e. if the evaluation of disc. is incorrect and id is good then still +1, but no pt. for evaluation

5 pts

4. Use the method of Lagrange multipliers to maximize the function

$$f(x, y, z) = xyz$$

subject to the constraints $3x + y = 3$ and $2x - z = 0$.

$g(x,y,z) = 3x+y-3$

$h(x,y,z) = 2x-z$

3 pts. for subst, algebra, & interpretation ~~of~~ x, y, z values as the solns (not req. to plug into f coordinates... some partial credit

$F(x,y,z,\lambda,\mu) = xyz - \lambda(3x+y-3) - \mu(2x-z)$

2 pts for finding partials & setting equal to zero

Set partial derivatives equal to zero:

$F_x = yz - 3\lambda - 2\mu = 0$

$F_y = xz - \lambda = 0 \longrightarrow \lambda = xz$

$F_z = xy + \mu = 0 \longrightarrow \mu = -xy$

$F_\lambda = -3x - y + 3 = 0$

$F_\mu = -2x + z = 0$

$yz - 3xz + 2xy = 0$

$z = 2x \qquad y = 3 - 3x$

~~The pts for the substitutions, algebra and partial credit for some correct work~~

$(3-3x)(2x) - 3x(2x) + 2x(3-3x) = 0$

$6x - 6x^2 - 6x^2 + 6x - 6x^2 = 0$

$2x - 3x^2 = 0$

$x(2-3x) = 0$

$x = 0$ or $x = \frac{2}{3}$ not a minimum, so disqualified for consideration

So $z = 2x = 2 \cdot \frac{2}{3} = \frac{4}{3}$ and $y = 3 - 3x = 1$

The point at which f is maximum is the input $\boxed{\left(\frac{2}{3}, 1, \frac{4}{3}\right)}$.

The maximum value of f is $f\left(\frac{2}{3}, 1, \frac{4}{3}\right)$, which is

$f\left(\frac{2}{3}, 1, \frac{4}{3}\right) = \left(\frac{2}{3}\right)(1)\left(\frac{4}{3}\right) = \boxed{\frac{8}{9}}$.

~~1 pt for the actual answer, i.e. appropriate interpretation & evaluation (plug into f)~~

5. Consider the following double integral:

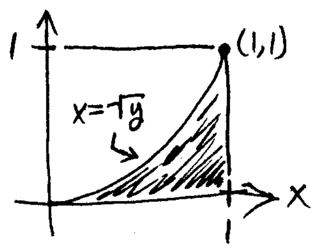
$$\int_0^1 \int_{\sqrt{y}}^1 \frac{1}{x^3+2} dx dy.$$

(a) Sketch the region R over which we are integrating.

(b) Change the order of integration of the double integral.

(c) Evaluate the double integral.

(a)



1 point (must have right convexity).

(b) $$\int_0^1 \int_{\sqrt{y}}^1 \frac{1}{x^3+2} dx dy = \boxed{\int_0^1 \int_0^{x^2} \frac{1}{x^3+2} dy dx}$$ 1 point

(c) $$\int_0^1 \int_0^{x^2} \frac{1}{x^3+2} dy dx = \int_0^1 \left(\frac{y}{x^3+2}\right)\Bigg]_0^{x^2} dx = \int_0^1 \frac{x^2}{x^3+2} dx$$

$$= \left(\frac{1}{3} \ln|x^3+2|\right)\Bigg|_0^1 = \boxed{\frac{1}{3}(\ln(3) - \ln(2))}$$

1 point for intermediate step $\frac{y}{x^3+2}$

1 point for $\frac{1}{3}\ln|x^3+2|$

1 point for final answer

6. (a) Determine whether the sequence $a_n = \frac{\sqrt{n^2-5}}{2n+3}$ converges or diverges. You must justify your answer.

(b) Determine whether each series converges or diverges. You must justify your answer and state which test(s) you are using.

i. $\sum_{n=1}^{\infty} 1000n^{-\frac{5}{2}}$.

ii. $\sum_{n=0}^{\infty} \frac{n^5+3^n}{4^n}$.

(a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{n^2-5}}{2n+3} = \lim_{n\to\infty} \frac{\sqrt{1-\frac{5}{n^2}}}{2+\frac{3}{n}} = \frac{1}{2}$.

$\boxed{\text{The sequence converges to } \tfrac{1}{2}.}$ 1 point ($\frac{1}{2}$ not required).

(b) The series $\sum_{n=1}^{\infty} 1000n^{-\frac{5}{2}}$ is a p-series with $p=\frac{5}{2}$.

Since $p>1$, $\boxed{\text{the series } \cancel{\text{diverges}} \text{ converges, by the p-series test.}}$ 1 point for convergence

1 point for p-series.

For the second series, rewrite as $\underbrace{\left(\sum_{n=0}^{\infty} \frac{n^5}{4^n}\right)} + \underbrace{\left(\sum_{n=0}^{\infty} \frac{3^n}{4^n}\right)}$

This series converges by the ratio test, since $\lim_{n\to\infty} \left| \frac{(n+1)^5/4^{n+1}}{n^5/4^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^5}{n^5} \cdot \frac{4^n}{4^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(1+\frac{1}{n})^5}{1} \cdot \frac{1}{4} \right| = \frac{1}{4} < 1$

This series converges by the geometric series test, with common ratio $|r| = \frac{3}{4} < 1$

Since both series converge, the sum converges: that is,

$$\left(\sum_{n=0}^{\infty} \frac{n^5}{4^n}\right) + \left(\sum_{n=0}^{\infty} \frac{3^n}{4^n}\right) = \boxed{\sum_{n=0}^{\infty} \frac{n^5+3^n}{4^n} \text{ converges.}}$$

1 point for convergent.

1 point for both ratio and geometric series.

7. (a) Find the exact value of the series $\sum_{n=0}^{\infty} \frac{3^{n+1}}{5^{2n-1}}$.

(b) A company releases a new laptop. The company estimates that it will sell 1000 laptops every year. Suppose that, in any given year, 10% of all old laptops in use break, so assume a laptop will not break during the year in which it was sold.

i. How many laptops are in use after n years?

ii. How many laptops are in use as $n \to \infty$?

This portion was 4 points see grading on the back

(a) $$\sum_{n=0}^{\infty} \frac{3^{n+1}}{5^{2n-1}} = \sum_{n=0}^{\infty} \frac{3 \cdot 3^n}{5^{-1} \cdot 5^{2n}} = \sum_{n=0}^{\infty} \frac{5 \cdot 3 \cdot 3^n}{25^n} = \sum_{n=0}^{\infty} 15 \cdot \left(\frac{3}{25}\right)^n.$$

This is a geometric series w/ first term $a = 15$ and $r = \frac{3}{25}$. Since $|r| < 1$, the series converges.

The exact value, by the geometric series formula, is $\frac{a}{1-r} = \frac{15}{1 - \frac{3}{25}} = \frac{15}{\left(\frac{22}{25}\right)}$

$$= \frac{15}{1} \cdot \frac{25}{22} = \boxed{\frac{375}{22}}.$$

(b) Let L_n be the number of laptops in use after n years.

Then $L_1 = 1000$. $L_2 = 1000 + 1000(0.9)$

$L_3 = 1000 + 1000(0.9)L_2 = 1000 + 1000(0.9) + 1000(0.9)^2$

So, $\boxed{L_n = \sum_{j=0}^{n-1} 1000(0.9)^j}$

This portion was 3 points. See explanation on the back

What happens as $n \to \infty$?

$$\lim_{n \to \infty} L_n = \sum_{j=0}^{\infty} 1000(0.9)^j = \frac{1000}{1-0.9} = \boxed{10{,}000}.$$

definition of infinite series

geometric series formula

(7a) +1 for correctly rewriting

as $\sum_{n=0}^{\infty} 15 \left(\frac{3}{25}\right)^n$

+1 for identifying this is a geometric series

+2 for using the formula $\frac{a}{1-r}$ (in this case giving $\left(15 / 1-\left(\frac{3}{25}\right)\right)$

points substracted for disorganization, conceptual mistake of algebra.

(7b) +1 for writing 2, 3 or 4 terms of the situation $L_1, L_2, L_3 \dots$

+1 for finding the right formula as a Geometric series

+1 for computing the limit using the formula

$$\frac{1000}{1-(0.9)}$$

Differential equations had nothing to do with this problem.

8. (a) Find the Maclaurin series for $f(x) = x(e^{-x^3} - 1)$ by using the Maclaurin series of e^x. Remember that the Maclaurin series is just the Taylor series centered at zero.

(b) Approximate the definite integral $\int_0^1 e^{-x^2}\,dx$ using the 4th degree Taylor polynomial for e^{-x^2} centered at 0.

(a) Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $e^{-x^3} = \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}$

The term when $n=0$ evaluates to 1, so

$$e^{-x^3} - 1 = \sum_{n=1}^{\infty} \frac{(-1)x^{3n}}{n!}$$

. Now, multiply by x:

$$x(e^{-x^3} - 1) = x\sum_{n=1}^{\infty} \frac{(-1)x^{3n}}{n!} = \boxed{\sum_{n=1}^{\infty} \frac{(-1)x^{3n+1}}{n!}}$$

This portion was worth 3 points. See explanation on back

(b) Doing similar work as above, $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$, so the 4th degree Maclaurin series for e^{-x^2} is $1 - x^2 + \frac{x^4}{2}$.

So
$$\int_0^1 e^{-x^2}\,dx \approx \int_0^1 1 - x^2 + \frac{x^4}{2}\,dx$$
$$= \left[x - \frac{x^3}{3} + \frac{x^5}{10}\right]_0^1$$
$$= 1 - \frac{1}{3} + \frac{1}{10}$$
$$= \boxed{\frac{23}{30}}$$

This portion was 3 points. See explanation on the back.

(8a) +1 for substituting $(-x^3)$ inside the expression for $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

+1 for correctly obtaining $e^{-x^3} - 1$ (common mistake the -1 was substracted infinitely many times)

+1 for correctly computing $x\left(e^{-x^3} - 1\right)$.

- A few people attempted to use differentiation but failed to give a general term (−2 points)
- Bad conceptual understanding of algebra $\left((-x^3)^n \neq -x^{3n}\right)$ or bad parenthesization (−2 points)
- Improper use of notation (e.g. $e^{-x} = \frac{x^n}{n!}$) (−1 point)

(8b) +1 for finding that $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$.

+1 for noticing that one only needs $1 - x^2 + \frac{x^4}{2}$

+1 for setting up correctly the integral

$$\int_0^1 e^{-x^2} dx \approx \int_0^1 \left(1 - x^2 + \frac{x^4}{2}\right) dx = 1 - \frac{1}{3} + \frac{1}{10}$$

- points were taken off for bad use of notation, conceptual mistakes of algebra or integration (e.g. people believing $\left(\int x^2 dx = x^2\right)$)

NO POINTS WERE TAKEN OFF FOR SIMPLE ARITHMETIC ERRORS