1. (8 points) A 200-gallon tank is half full of distilled water. At time \( t = 0 \), a solution containing 0.5 pounds of concentrate per gallon enters the tank at a rate of 5 gallons per minute, and the well-stirred mixture is withdrawn at the same rate. Find the amount \( Q \) of concentrate in the tank after 30 minutes.

\[
\frac{dQ}{dt} = (\text{rate of conc. IN in lbs/min}) - (\text{rate of conc. OUT in lbs/min})
\]

\[
= (0.5 \text{ lbs/gal})(5 \text{ gal/min}) - 5 \cdot \frac{Q(t)}{100}
\]

\[
\Rightarrow \frac{dQ}{dt} = \frac{5}{2} - \frac{Q(t)}{20}
\]

Thus, \( \frac{dQ}{dt} + \frac{1}{20}Q = \frac{5}{2} \). Using 1st order linear DE method,

\[
u = e^{\int \frac{1}{20} dt} = e^{\frac{t}{20}}, \text{ thus}
\]

\[
Q(t) = e^{\frac{-t}{20}} \int \frac{5}{2} e^{\frac{t}{20}} dt = e^{\frac{-t}{20}} \left[ \frac{5}{2} \left( \frac{1}{\frac{1}{20}} \right) e^{\frac{t}{20}} + C \right] = 50 + Ce^{-\frac{t}{20}}
\]

Initial condition: Distilled water, so no lbs of concentrate at \( t = 0 \) in the tank.

In other words, when \( t = 0 \), \( Q(t) = 0 \). Solve by substitution in here to get \( C = -50 \).

So \( Q(t) = 50 - 50e^{-\frac{t}{20}} \).

After 30 minutes (\( t = 30 \)), \( Q(30) = 50 - 50e^{-\frac{30}{20}} \). +1

After half an hour, \( 50 - 50e^{-\frac{3}{2}} \) lbs of concentrate in the tank.
2. (7 points. This is Hnw 2 #44 A38) An infectious disease spreads through a large population according to the model:

\[
\frac{dy}{dt} = \frac{1 - y}{4}
\]

where \( y \) is the percentage of the population exposed to the disease, and \( t \) is the time in years.

\( a. \) Solve this differential equation assuming that \( y(0) = 0 \)

\( b. \) Find the number of years it takes for half of the population to have been exposed to the disease.

\( c. \) Find the percentage of the population that has been to the disease after 4 years.

**Rewrite diff. eqn. in standard form:**

\[
\frac{dy}{dt} + \frac{1}{4} y = \frac{1}{4}
\]

or \( \int \frac{1}{1-y} \, dy = \int \frac{1}{4} \, dt \)

So \( P(t) \) and \( Q(t) \) are both \( \frac{1}{4} \):

Integrating factor \( u(t) = e^{\int P(t) \, dt} = e^{\int \frac{1}{4} \, dt} = e^{\frac{1}{4} t} \)

General solution is

\[
y = \frac{1}{u(t)} \int Q(t) u(t) \, dt = \frac{1}{e^{\frac{1}{4} t}} \int \frac{1}{4} e^{\frac{1}{4} t} \, dt = \frac{1}{4} \left( e^{\frac{1}{4} t} + C \right)
\]

Initial condition \( y(0) = 0 \) is given, so \( 0 = 1 + Ce^{0/4} \), meaning \( C = -1 \).

Thus, our particular solution is

\[
y = 1 - e^{-t/4}
\]

Half the population: \( y = \frac{1}{2} \)

\[
\frac{1}{2} = 1 - e^{-t/4}
\]

\[
\frac{1}{2} = e^{-t/4}
\]

\[
\ln(\frac{1}{2}) = -\frac{t}{4}
\]

\[
t = -4 \ln(\frac{1}{2})
\]

In \(-4 \ln(\frac{1}{2})\) years, half the population is exposed.

In 4 years: \( t = 4 \)

\[
y(4) = 1 - e^{-4/4} = 1 - e^{-1}
\]

In 4 years, \( 1 - e^{-1} \) of the population (on a scale of 0 to 1) is exposed.
3. (7 points)

(a) (Hmw 2, page 473 #44) Identify the quadric surface \( \frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{16} = 1 \). Use either words or a picture, but give a short reason for your answer. Your drawings don’t have to be very precise.

(b) (Hmw 3, sec 7.3, #27) Describe the region \( R \) in the \( xy \)-coordinate plane that corresponds to the domain of the function: \( \ln(4 - x - y) \).

(c) (Hmw 3, sec 7.3, #35) Describe the level curves of the function \( z = \sqrt{16 - x^2 - y^2} \) for \( c = 0, 1, 2, 3, 4 \).

(a) The surface equation can be rewritten \( \frac{x^2}{3^2} + \frac{y^2}{4^2} + \frac{z^2}{4^2} = 1 \). The surface is an ellipsoid centered at the origin with axes of lengths 3, 4, and 4 in the \( x, y, \) and \( z \) directions (respectively). In pictures,

1 point for ellipsoid
1 point for axes (3/4/4)

(b) Natural log can't take negative inputs, so valid inputs for the function \( f(x,y) = \ln(4-x-y) \) occur when \( 4-x-y>0 \). What does the region \( R \) defined by \( 4-x-y>0 \) look like? Since \( 4-x-y=0 \) describes a line (call it \( L \)), \( 4-x-y>0 \) is everything on one side of \( L \) (but not including \( L \) itself). The line \( L \) has 4 as \( x \)- and \( y \)-intercepts, and the origin satisfies the inequality, thus:

1 point for \( 4-x-y>0 \)
1 point for line
1 point for region
1 point for circle
1 point for concentric

(c) For \( c=0 \) \( 0=\sqrt{16-x^2-y^2} \) which gives \( x^2+y^2=4^2 \). It’s a circle centered at origin with radius 4.

Plug in \( c=1 \) to get \( x^2+y^2=15 \), a circle of radius \( \sqrt{15} \).

Similarly \( c=2 \) \( \begin{cases} \frac{\sqrt{12}}{2} \\ \frac{\sqrt{7}}{2} \\ 0 \end{cases} \) are circles of radius \( 3 \).

Similarly \( c=3 \) \( \begin{cases} \frac{\sqrt{12}}{3} \\ \frac{\sqrt{7}}{3} \\ 0 \end{cases} \) are circles of radius \( 2 \).

Similarly \( c=4 \) \( \begin{cases} \frac{\sqrt{12}}{4} \\ \frac{\sqrt{7}}{4} \\ 0 \end{cases} \) are circles of radius \( 1 \).

A circle of radius 0, so just the origin alone.
4. (8 points) A corporation manufactures a product at two locations. The cost function for producing \( x_1 \) units at location 1 and \( x_2 \) units at location 2 are given by

\[
C_1 = 0.03x_1^2 + 4x_1 + 300
\]

and

\[
C_2 = 0.05x_2^2 + 7x_2 + 175
\]

respectively. If the product sells for \$10 per unit, find the production levels \( x_1, x_2 \) such that the profit is maximized.

**Profit function:**

\[
P(x_1, x_2) = \text{Earnings} - \text{Cost}
\]

\[
= 10(x_1 + x_2) - C_1 - C_2
\]

\[
= -0.03x_1^2 + 6x_1 - 0.05x_2^2 + 3x_2 - 475.
\]

\[
\frac{\partial P}{\partial x_1} = -0.06x_1 + 6 = 0
\]

\[
\frac{\partial P}{\partial x_2} = -0.10x_2 + 3 = 0
\]

\[
\frac{\partial^2 P}{\partial x_1^2} = -0.06 \quad \frac{\partial^2 P}{\partial x_2^2} = -0.10 \quad \frac{\partial^2 P}{\partial x_1 \partial x_2} = \frac{\partial^2 P}{\partial x_2 \partial x_1} = 0
\]

\[
P = (-0.06)(-0.10) - 0^2 > 0 \quad \text{and} \quad \frac{\partial^2 P}{\partial x_1^2} < 0 \quad \text{so this critical point}
\]

is a relative maximum of \( P \).

Tell the boss to produce \( x_1 = 100 \) units at location 1 and \( x_2 = 30 \) units at location 2.

1 point for profit function.
2 points for first derivatives.
2 points for setting \( \partial P/\partial x = 0 \) and solving.
2 points for second derivative test.

-2 for wrong profit function.
(nore more than 4/8 for wrong function.)