1. (8 points) A 200-gallon tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 pounds of concentrate per gallon enters the tank at a rate of 5 gallons per minute, and the well-stirred mixture is withdrawn at the same rate. Find the amount Q of concentrate in the tank after 30 minutes.

Q = amount of concentrate (in lbs)

+1 $\frac{dQ}{dt}$ = (rate of conc. IN in lbs/min) − (rate of conc. OUT in lbs/min) *

$$= (0.5 \text{ lbs/gal})(5 \text{ gal/min}) - 5 \cdot \frac{Q(t)}{100}$$

+2 $$= \frac{5}{2} - \frac{Q(t)}{20}$$

Thus, $\frac{dQ}{dt} + \frac{1}{20}Q = \frac{5}{2}$. Using 1st order linear D.E. method,

$$u = e^{\int \frac{dt}{20}} = e^{t/20}, \text{ thus}$$

+1 $$Q(t) = \frac{1}{e^{t/20}} \int \frac{5}{2} e^{t/20}\, dt = \frac{1}{e^{t/20}} \left[\frac{5}{2}\left(\frac{1}{1/20}\right) e^{t/20} + C \right] = 50 + Ce^{-t/20}$$ +1

Initial condition: Distilled water, so <u>no</u> lbs of concentrate @ $t=0$ in the tank.
+2 In other words, when $t=0$, $Q(t)=0$. Solve by substitution in here to get $C=-50$.

So $Q(t) = 50 - 50e^{-t/20}$.

After 30 minutes ($t=30$), $Q(30) = 50 - 50e^{-30/20}$. +1

> After half an hour, $(50 - 50e^{-3/2})$ lbs of concentrate in the tank.

2. (7 points. This is Hmw 2 #44 A38) An infectious disease spreads through a large population according to the model:

$$\frac{dy}{dt} = \frac{1-y}{4}$$

where y is the percentage of the population exposed to the disease, and t is the time in years.

a • Solve this differential equation assuming that $y(0) = 0$

b • Find the number of years it takes for half of the population to have been exposed to the disease.

c • Find the percentage of the population that has been to the disease after 4 years.

+1 Rewrite diff. eqn. in standard form: $\frac{dy}{dt} + \underbrace{\frac{1}{4}}_{P(t)} y = \underbrace{\frac{1}{4}}_{Q(t)}$ or $\int \frac{1}{1-y}\,dy = \int \frac{1}{4}\,dt$

So $P(t)$ and $Q(t)$ are both $\frac{1}{4}$.

Integrating factor $= u(t) = e^{\int P(t)\,dt} = e^{\int \frac{1}{4}\,dt} = e^{\frac{1}{4}t}$.

+2 General solution is $y = \frac{1}{u(t)} \int Q(t) u(t)\,dt = \frac{1}{e^{\frac{1}{4}t}} \int \frac{1}{4} e^{\frac{1}{4}t}\,dt = \frac{1}{e^{\frac{1}{4}t}}\left(e^{\frac{1}{4}t} + C\right)$

$= 1 + Ce^{-t/4}$.

Initial condition $y(0)=0$ is given, so $0 = 1 + Ce^{-0/4}$, meaning $C = -1$. +2

Thus, our particular solution is $\boxed{y = 1 - e^{-t/4}}$.

Half the population: $y = \frac{1}{2}$

$$\frac{1}{2} = 1 - e^{-t/4}$$
$$\frac{1}{2} = e^{-t/4}$$
$$\ln(1/2) = -t/4$$
$$t = -4\ln(1/2)$$

+1

$\boxed{\text{In } -4\ln(1/2) \text{ years, half the population is exposed.}}$

In 4 years: $t=4$ $\quad y(4) = 1 - e^{-4/4} = 1 - e^{-1}$.

$\boxed{\text{In 4 years, } 1-e^{-1} \text{ of the population (on a scale of 0 to 1) is exposed.}}$

+1

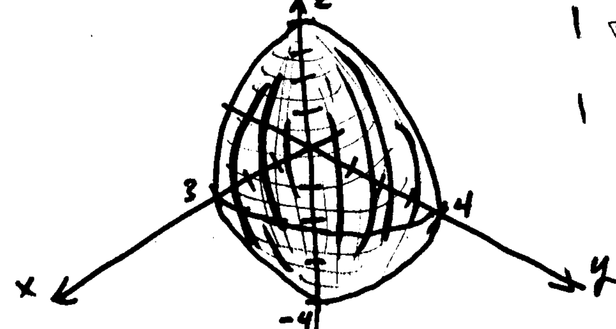
3. (7 points)

(a) (Hmw 2, page 473 #44) Identify the quadric surface $\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{16} = 1$. Use either words or a picture, but give a short reason for your answer. Your drawings don't have to be very precise.

(b) (Hmw 3, sec 7.3, #27) Describe the region R in the xy-coordinate plane that corresponds to the domain of the function: $ln(4 - x - y)$.

(c) (Hmw 3, sec 7.3, #35) Describe the level curves of the function $z = \sqrt{16 - x^2 - y^2}$ for $c = 0, 1, 2, 3, 4$.

(a) The surface equation can be rewritten $\frac{x^2}{3^2} + \frac{y^2}{4^2} + \frac{z^2}{4^2} = 1$. The surface is an ellipsoid centered at the origin with axes of lengths 3, 4, and 4 in the x, y, and z directions (respectively). In pictures,



1 point for ellipsoid

1 point for axes (3/4/4)

(b) Natural log can't take negative inputs, so valid inputs for the function $f(x,y) = ln(4-x-y)$ occur when $4-x-y>0$. What does the region R defined by $4-x-y>0$ look like? Since $4-x-y=0$ describes a line (call it L), $4-x-y>0$ is everything on one side of L (but not including L itself). The line L has 4 as x- and y-intercepts, and the origin satisfies the inequality, thus:

1 point for $4-x-y>0$

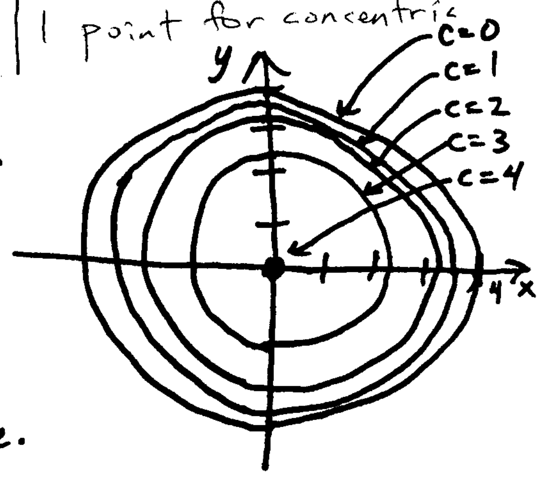
1 point for line

1 point for region

1 point for circle

1 point for concentric

(c) For $c=0$ $0 = \sqrt{16-x^2-y^2}$ which gives $x^2+y^2=4^2$.
It's a circle centered @ origin with radius 4.
Plug in $c=1$ to get $x^2+y^2=15$, a circle of radius $\sqrt{15}$.
Similarly $\left.\begin{aligned} c&=2 \\ &=3 \\ &=4 \end{aligned}\right\}$ are circles of radius $\left\{\begin{aligned} &\sqrt{12} \\ &\sqrt{7} \\ &0 \end{aligned}\right.$



A circle of radius 0, so just the origin alone.

4. (8 points) A corporation manufactures a product at two locations. The cost function for producing x_1 units at location 1 and x_2 units at location 2 are given by

$$C_1 = 0.03x_1^2 + 4x_1 + 300$$

and

$$C_2 = 0.05x_2^2 + 7x_2 + 175$$

respectively. If the product sells for \$10 per unit, find the production levels x_1, x_2 such that the profit is maximized.

Profit function: $P(x_1, x_2) = \text{Earnings} - \text{Cost}$

$$= 10(x_1 + x_2) - C_1 - C_2$$

$$= -0.03x_1^2 + 6x_1 - 0.05x_2^2 + 3x_2 - 475.$$

$$\left.\begin{aligned} \frac{\partial P}{\partial x_1} &= -0.06x_1 + 6 = 0 \\ \frac{\partial P}{\partial x_2} &= -0.10x_2 + 3 = 0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} x_1 &= \frac{6}{0.06} = 100 \\ x_2 &= \frac{3}{0.10} = 30 \end{aligned}\right\}$$

$$\frac{\partial^2 P}{\partial x_1^2} = -0.06 \qquad \frac{\partial^2 P}{\partial x_2^2} = -0.10 \qquad \frac{\partial^2 P}{\partial x_1 \partial x_2} = \frac{\partial^2 P}{\partial x_2 \partial x_1} = 0$$

$D = (-0.06)(-0.10) - 0^2 > 0$ and $\frac{\partial^2 P}{\partial x_1^2} < 0$ so this critical point is a relative maximum of P.

> Tell the boss to produce $x_1 = 100$ units at location 1 and $x_2 = 30$ units at location 2.

1 points for profit function.
2 points for first derivatives
2 points for setting $= 0$ and solving
2 points for second derivative test

-2 for wrong profit function.
(no more than 4/8 for wrong function)