MATH 16C: MULTIVARIATE CALCULUS

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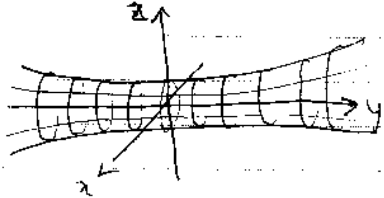
Summary of types of quadric surfaces.

Surface	Type	Basic Equation
Ellipsoid		$x^2 + y^2 + z^2 = 1$
Paraboloid	Elliptic	$z = x^2 + y^2$
	Hyperbolic	$z = x^2 - y^2$
Hyperboloid	Elliptic Cone	$x^2 + y^2 - z^2 = 0$
	one sheet	$x^2 + y^2 - z^2 = 1$
	two sheets	$x^2 + y^2 - z^2 = -1$

PROBLEM:

Classify the surface $4x^2 - y^2 + z^2 - 4z = 0$.
You must first complete the square!!!



$$
\begin{aligned}
4x^2 - y^2 + z^2 - 4z &= 0 \\
\Leftrightarrow 4x^2 - y^2 + (z-2)^2 - 4 &= 0 \\
\Leftrightarrow x^2 - \frac{y^2}{2^2} + \frac{(z-2)^2}{2^2} - 1 &= 0 \\
\Leftrightarrow x^2 + \frac{(z-2)^2}{2^2} - \frac{y^2}{2^2} &= 1
\end{aligned}
$$

Surface is a hyperboloid of one sheet with axis the y axis. The axis

Section 7.3: Functions of more than one variable.

Below are some examples of functions of more than one variable.

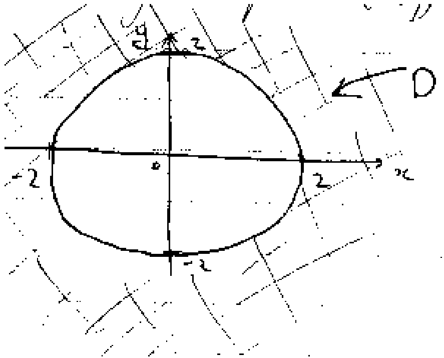
- $f(x, y) = x^2 + xy$ (function of 2 variables)
- $g(x, y) = e^{(x+y)}$ (function of 2 variables)
- $f(x, y, z) = x + 2y - 3z$ (function of 3 variables)

pause

- **Definition:** Let D be a set (or region) of points (x, y). If for each point (x, y) in D, there is a unique value $z = f(x, y)$, then f is a **function** of x and y.
  Functions of three or more variables are defined similarly.
- The set D is called the **domain** of f, and the corresponding set of all possible values of $f(x, y)$ for every point (x, y) in D is the **range** of f.
- If D is not specified, then it is assumed to be the set of all points for which $f(x, y)$ makes sense. For example, what is the domain of $f(x) = \ln(x)$, $f(x) = \sqrt{x}$, and $f(x) = \frac{1}{x}$?
- **Question:** Find the domain and range of the function $f(x, y) = \sqrt{x^2 + y^2 - 4}$?

Answer:

Since $f(x, y) = \sqrt{x^2 + y^2 - 4}$ only makes sense when $x^2 + y^2 - 4 \geq 0$. The domain is the set $D = \{(x, y) : x^2 + y^2 - 4 \geq 0\}$, which is the set of all points (x, y) such that $x^2 + y^2 - 4 \geq 0$. The range of f is $[0, \infty)$.



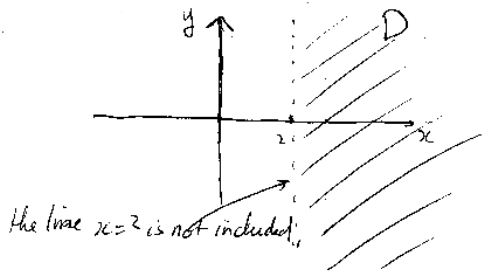
Question:

What is the domain and range of $f(x, y) = \frac{1}{\sqrt{x^2+y^2-4}}$??

Answer: The domain is $D = \{(x, y) : x^2 + y^2 - 4 > 0\}$ and the range is $(0, \infty)$.

Question: What is the domain and range of $f(x, y) = ln(x - 2)$?

Answer: Note that the logarithm is defined only for positive numbers. So, the domain is $D = \{(x, y) : x - 2 > 0\}$. The range is $(-\infty, \infty)$.

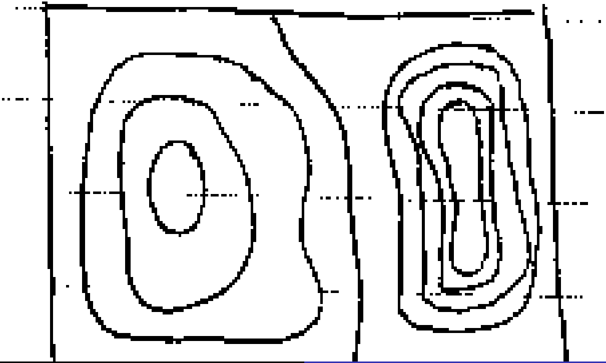


Contour maps and Level curves

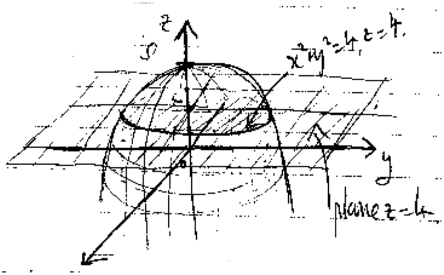
Examples of contour maps are topographical maps and weather maps.

For topographical maps, contour lines, also called level curves, are lines of equal elevation.

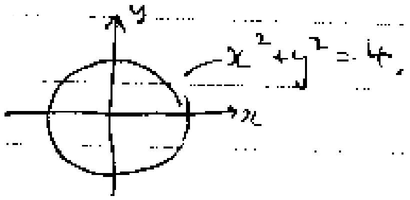
For Weather maps, isobars, also called level curves, are lines of equal pressure.



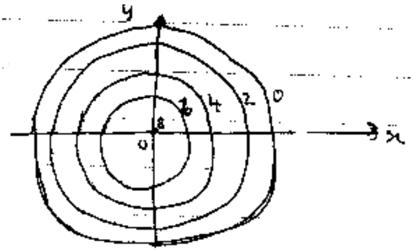
Contour maps are 2-dimensional depictions of surfaces in 3-dimensions.
EXAMPLE: Consider the surface $z = f(x, y) = 8 - x^2 - y^2$.
It is an elliptic paraboloid (upside down).
The intersection of the paraboloid with the plane $z = 4$, is the circle $4 = 8 - x^2 - y^2 \Rightarrow x^2 + y^2 = 4$.



The projection of the trace onto the xy-plane is the circle $x^2 + y^2 = 4$. This projection is called a **level curve.**



The level curves for $z = 0, 2, 4, 6, 8$ are as follows.



Definition

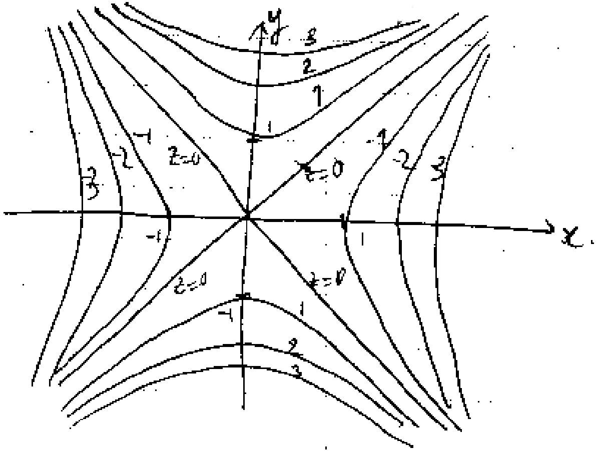
A **level curve** of a surface $z = f(x, y)$ is the curve $c = f(x, y)$ in the xy-plane where c is some constant.

Definition

A **contour map** of a surface $z = f(x, y)$ is a collection of level curves $c = f(x, y)$ for many different evenly spaced constants c.

EXAMPLE: Consider the function $z = x^2 - y^2$. What are the level curves for $z = -3, -2, -1, 0, 1, 2, 3$.

Note that for $z = c$, the level curve is $c = y^2 - x^2$, which is a hyperbola for $c \neq 0$ and it is a cross for $c = 0$.



Section 7.4: Partial derivatives.

Let $z = f(x, y)$ be a function defined at each point in some region D.
The **partial derivative** of f with respect to x, written as $\frac{\partial f}{\partial x}$ or written as f_x, is the ordinary derivative of f with respect to x as in the one variable case where we treat the y variable as if it were a constant.
The **partial derivative** of f with respect to y, written as $\frac{\partial f}{\partial y}$ or f_y, is the ordinary derivative of f with respect to y as in the one variable case where we treat the x variable as if it were a constant.
EXAMPLE: Consider the function of two variables $f(x, y) = x^2 + y + x^3y^4$. Then, the partial derivative of f with respect to x and the partial derivative of f with respect to y is as follows respectively:

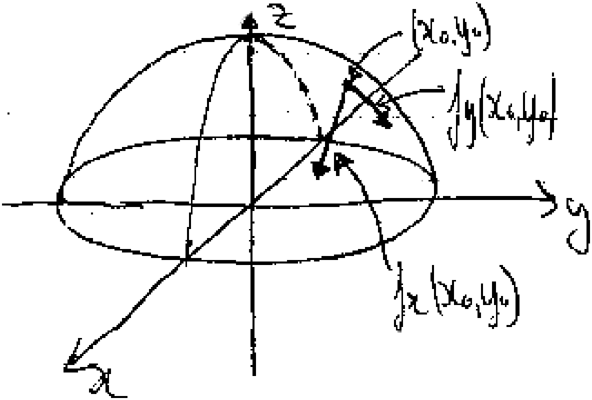
$$\frac{\partial f}{\partial x} = f_x = 2x + 3x^2y^4 \text{ and } \frac{\partial f}{\partial y} = f_y = 1 + 4x^3y^3.$$

- A partial derivative of a function of two variables, f_x or f_y, is again a function of two variables.
- We can evaluate the partial derivative at a point (x_0, y_0). The notation for evaluation a partial derivative at a point (x_0, y_0) is the following: $\frac{\partial f}{\partial x}|_{(x_0,y_0)} = f_x(x_0, y_0)$ and $\frac{\partial f}{\partial y}|_{(x_0,y_0)} = f_y(x_0, y_0)$.
- EXAMPLE consider again the function of two variables $f(x, y) = x^2 + y + x^3y^4$.
  We can evaluate the partial derivatives of f with respect to x at the point $(1, 2)$ as follows:

  $f_x(1, 2) = 2(1)+3(1)^2(2)^4 = 50$ and $f_y(1, 2) = 1+4(1)^3(2)^3 = 33$.

- WHAT IS THIS GOOD FOR???
  FINDING MAXIMAL/MINIMAL SOLUTIONS!!

- What is the meaning of the partial derivative??
- The partial derivative $\frac{\partial f}{\partial x}|_{(x_0,y_0)} = f_x(x_0, y_0)$, is the slope of the surface $z = f(x, y)$ at the point (x_0, y_0) in the x-direction.



- $\frac{\partial f}{\partial y}|_{(x_0,y_0)} = f_y(x_0, y_0)$, is the slope of the surface $z = f(x, y)$ at the point (x_0, y_0) in the y-direction.

- We can easily extend this concept of partial derivatives of functions of two variables to functions of three or more variables.
- EXAMPLE: Consider the function of three variables $f(x, y, z) = xe^{xy+2z}$. It has three first order derivatives, one for each variable.

$$\frac{\partial f}{\partial x} = e^{xy+2z} + xye^{xy+2z}$$
$$\frac{\partial f}{\partial y} = x^2 e^{xy+2z}$$
$$\frac{\partial f}{\partial z} = 2xe^{xy+2z}$$

Higher Order Partial Derivatives

- We can find partial derivatives of higher order. The partial derivative of $f(x, y)$ with respect to either x or y is again a function of x and y, so we can again take the partial derivative of the partial derivative.
- Consider a function $f(x, y)$.
  The second order derivatives of f are as follows.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} = (f_x)_x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy} = (f_y)_y$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} = (f_x)_y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx} = (f_y)_x$$

- EXAMPLE Consider the function of two variables $f(x,y) = 3xy^2 - 2y + 5x^2y^2$. The first order partial derivatives are as follows:

$$\frac{\partial f}{\partial x} = 3y^2 + 10xy^2 \text{ and } \frac{\partial f}{\partial y} = 6xy - 2 + 10x^2y.$$

- The second order partial derivatives are as follows:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial f}{\partial x}\left[3y^2 + 10xy^2\right] = 10y^2$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial y}\left[6xy - 2 + 10x^2y\right] = 6x + 10x^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial f}{\partial y}\left[3y^2 + 10xy^2\right] = 6y + 20xy$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial x}\left[6xy - 2 + 10x^2y\right] = 6y + 20xy$$

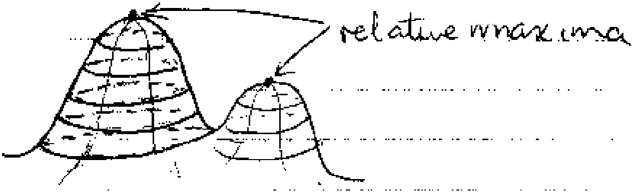
- Note that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. This is true for function $f(x,y)$ IF it has continuous second order partial derivatives.

Section 7.5: Extrema of functions of two variables

Extrema of functions of two variables

Definition

We say $f(x_0, y_0)$ is a <u>relative minimum</u> of f if there exists a circular region R with center (x_0, y_0) such that $f(x, y) \geq f(x_0, y_0)$ for all points (x, y) in R.
We say (x_0, y_0) is a <u>relative maximum</u> of f if there exists a circular region R with center (x_0, y_0) such that $f(x, y)\ lef(x_0, y_0)$ for all points (x, y) in R.



WARNING: a relative maximum/minimum is NOT always global!!

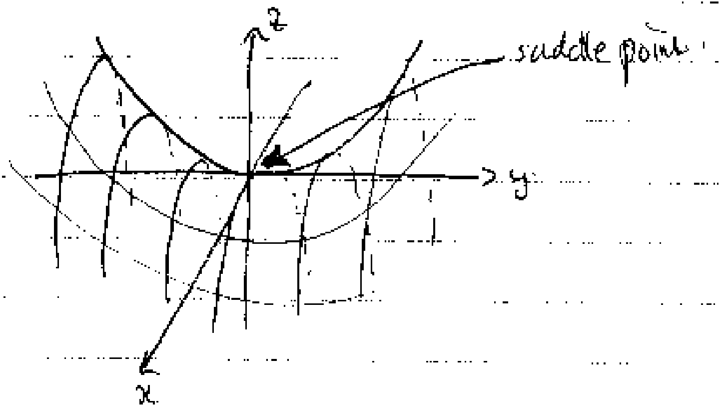
THEOREM: If (x_0, y_0) is a relative minimum or relative maximum and if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ are finite, then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

Definition

A point (x_0, y_0) is a **critical point** of the function $f(x, y)$ if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ OR if either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ is undefined (e.g. division by 0).

All relative maxima and relative minima are critical points, but not all critical points are relative maxima or relative minima.

EXAMPLE: consider $f(x, y) = x^2 - y^2$ (saddle shape). The point $(0, 0)$ is called a **saddle point.** It is a critical point, but it is not a relative minimum or relative maximum.



Finding relative minima and relative maxima has many applications.

THEOREM:

(the 2nd partial derivatives test) Let $z = f(x, y)$ have continuous partial derivatives f_x and f_y and continuous 2nd order partial derivatives f_{xx}, f_{yy}, f_{xy}, and f_{yx} in a region R containing (a, b) which is a critical point of f.

Let

$$d(a, b) = f_{xx}(a, b) \cdot f_{yy}(a, b) - (f_{xy}(a, b))^2,$$

Then

1. if $d(a, b) > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a relative minimum.
2. if $d(a, b) > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a relative maximum.
3. if $d(a, b) < 0$, then (a, b) is a saddle point.
4. if $d(a, b) = 0$, then inconclusive (try other methods).

PROBLEM:

Find and classify the critical points of the function
$f(x, y) = x^2 + 10x + 3y^2 - 18y + 5$.
Answer: (1) find the critical points.

$f_x(x, y) = 2x + 10 = 0 \Leftrightarrow x = -5 \quad f_y(x, y) = 6y - 18 = 0 \Leftrightarrow y = 3.$

So, there is one and only one critical point: $(-5, 3)$.
(2) Classify the critical point: Compute the 2nd order partial derivatives:

$$f_{xx}(x, y) = 2, f_{yy}(x, y) = 6, f_{xy}(x, y) = 0 = f_{yx}(x, y).$$

Then,

$$d(-5, 3) = f_{xx}(-5, 3) \cdot f_{yy}(-5, 3) - (f_{xy}(-5, 3))^2 = (2)(6) - (0)^2 = 12 > 0.$$

Since $f_{xx}(-5, 3) = 2 > 0$, the critical point is a relative minimum.

PROBLEM:

Find and classify the critical points of the function
$f(x, y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$.

Answer:

- Find the critical points.

  $$f_x = -6xy - 6x = -6x(y+1) \quad f_y = 3y^2 - 3x^2 - 6y = 3(y^2 - x^2 - 2y).$$

  The first partial derivatives are defined for all points (x, y), so the only critical points are when $f_x = 0$ and $f_y = 0$.
  $f_x = -6x(y + 1) = 0 \Rightarrow x = 0$ or $y = -1$.
- Consider the cases when $x = 0$ and when $y = 1$ separately.
  First, if $x = 0$, then
  $f_y = 3(y^2 - x^2 - 2y) = 0 \Rightarrow 3(y^2 - 2y) = 0 \Rightarrow y = 0$ or $y = 2$.
  We have two critical points: $(0, 0)$ and $(0, 2)$.
  Second, if $y = -1$, then $f_y = 3(y^2 - x^2 - 2y) = 0 \Rightarrow$
  $3(1 - x^2 + 2) = 0 \Rightarrow x^3 - 3 = 0 \Rightarrow x = \sqrt{3}$ or $x = -\sqrt{3}$.
  We have critical points: $(\sqrt{3}, -1)$ and $(-\sqrt{3}, -1)$.

Next, we classify the critical points using second derivatives.

$$f_{xx} = -6y - 6, f_{yy} = 6y - 6, \text{ and } f_{xy} = f_{yx} = -6x.$$

So, $d(x, y) = (-6y - 6)(6y - 6) - (-6x)^2 = 36 - 36y^2 - 36x^2 = 36(1 - y^2 - x^2)$.

Critical Point	$d = 36(1 - y^2 - x^2)$	f_{xx}	type
$(0, 0)$	$36(1 - 0 - 0) = 36 > 0$	< 0	rel. max
$(0, 2)$	$36(1 - 2^2 - 0) = -108 < 0$	-	saddle
$(\sqrt{3}, -1)$	$36(1 - (-1)^2 - (\sqrt{3})^2) = -108 < 0$	-	saddle
$(-\sqrt{3}, -1)$	$36(1 - (-1)^2 - (-\sqrt{3})^2) = -108 < 0$	-	saddle

Question:

Find and classify the critical points of the function
$f(x,y) = xy - \frac{1}{4}x^4 - \frac{1}{4}y^4$.
Answer: Find the critical points.

$$f_x(x,y) = y - x^3 \text{ and } f_y(x,y) = x - y^3.$$

The critical points occur when $f_x(x,y) = y - x^3 = 0$ and
$f_y(x,y) = x - y^3 = 0$. Then,

$$y - x^3 = 0 \Leftrightarrow y = x^3.$$

Putting this into $x - y^3 = 0$, we have

$$x - y^3 = 0 \Leftrightarrow x - (x^3)^3 = 0 \Leftrightarrow x - x^9 = 0 \Leftrightarrow x(1 - x^8) = 0,$$

so $x = 0$ or $x = 1$ or $x = -1$.
Then, using the fact that $y = x^3$, we have that if $x = 0$, then
$y = 0$, and if $x = 1$, then $y = 1$, and if $x = -1$, then $y = -1$. So,
there are three critical points $(0,0)$, $(1,1)$, and $(-1,-1)$.
NEXT, **classify the critical points!!**

$$f_{xx}(x,y) = -3x^2, f_{yy}(x,y) = -3y^2, \text{ and } f_{xy}(x,y) = f_{yx}(x,y) = 1.$$

We classify a critical point according to the value of

$$d = f_{xx}(x,y) \cdot f_{yy}(x,y) - (f_{xy}(x,y))^2 = (-3x^2)(-3y^2) - 1.$$

Critical Pt.	$d = f_{xx}(x,y) \cdot f_{yy}(x,y) - (f_{xy}(x,y))^2$	f_{xx}	type
(0,0)	$d = (-3(0)^2)(-3(0)^2) - 1 = -1 < 0$	-	saddle poi
(1,1)	$d = (-3(1)^2)(-3(1)^2) - 1 = 8 > 0$	< 0	rel. max.
(-1,-1)	$d = (-3(-1)^2)(-3(-1)^2) - 1 = 8 > 0$	< 0	rel. max.

PROBLEM:

Find and classify the critical points of $f(x, y) = (x^2 + y^2)^{\frac{1}{3}}$.

Answer: Find the critical points.

$$f_x = \frac{1}{3} \cdot 2x \cdot (x^2 + y^2)^{-\frac{2}{3}} \text{ and } f_y = \frac{1}{3} \cdot 2y \cdot (x^2 + y^2)^{-\frac{2}{3}}.$$

WARNING: The partial derivatives are undefined at the point $(0, 0)$. So, $(0, 0)$ is a critical point.

We cannot use the 2nd partial derivative test since f_{xx}, f_{yy}, f_{xy}, and f_{yx} are discontinuous at the point $(0, 0)$.

Use the graph of the function!! $(0, 0)$ is a relative minimum!!