MATH 16C: APPLICATIONS OF DIFFERENTIAL EQUATIONS

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- You study a population of mayflies to determine how quickly a particular characteristic will pass from one generation to the next.
- At the start, half of the population has the characteristic. After four generations, 75% of the population has the characteristic.
- In genetics, key model is $\frac{dy}{dt} = ky(1-y)(a-by)$ where y is the proportion of the population with the characteristic, t is the number of generations, and a and b are constants that depend on genetics.
- We will assume that $a = 2$ and $b = 1$.
- PROBLEM:Find the proportion of the mayflies with the characteristic after 10 generations.

LAST EPISODE...

We solved the differential equation using separation of variables (and partial fractions integration)

$$\int \frac{1}{y(1-y)(2-y)}dy = \int kdt \Rightarrow \int \frac{\frac{1}{2}}{y} + \frac{1}{1-y} + \frac{-\frac{1}{2}}{2-y}dy = \int kdt$$

$$\frac{1}{2}\int \frac{dy}{y} - \int \frac{-dy}{1-y} + \frac{1}{2}\int \frac{-dy}{2-y} = kt + C \ (C \text{ is a constant})$$

$$\frac{1}{2}\ln(y) - \ln(1-y) + \frac{1}{2}\ln(2-y) = kt + C$$

$$\ln(y) - 2\ln(1-y) + \ln(2-y) = 2kt + 2C$$

From properties of the Logarithm we have:

$$\Rightarrow \ln\left(\frac{y(2-y)}{(1-y)^2}\right) = 2kt + 2C$$

$$\Rightarrow \frac{y(2-y)}{(1-y)^2} = e^{2kt+2C}$$

$$\Rightarrow \frac{y(2-y)}{(1-y)^2} = Ae^{2kt} \text{ (A is a constant)}$$

Next, use the initial condition that $y = \frac{1}{2}$ when $t = 0$ to find A.

$$\frac{\frac{1}{2}(2-\frac{1}{2})}{(1-\frac{1}{2})^2} = \frac{(\frac{1}{2})(\frac{3}{2})}{\frac{1}{4}} = Ae^0 \Rightarrow A = 3.$$

Next use the initial condition that $t = \frac{3}{4}$ when $t = 4$ to find k.

$$\frac{\frac{3}{4}(2-\frac{3}{4})}{(1-\frac{3}{4})^2} = \frac{(\frac{3}{4})(\frac{5}{4})}{\frac{1}{16}} = 3e^{8k} \Rightarrow 15 = 3e^{8k} \Rightarrow k = \frac{1}{8}\ln(5).$$

- Therefore, the particular solution is $\frac{y(2-y)}{(1-y)^2} = 3e^{\frac{1}{4}\ln(5)t}$.
- Lastly, we solve for y when $t = 10$.

$$\begin{aligned}
\frac{y(2-y)}{(1-y)^2} = 3e^{\frac{10}{4}\ln(5)} &\Rightarrow \frac{y(2-y)}{(1-y)^2} \approx 168(\text{using calculator}) \\
&\Rightarrow y(2-y) \approx 168(1-y)^2 \\
&\Rightarrow 2y - y^2 \approx 168 - 336y + 168y^2 \\
&\Rightarrow 0 \approx 168 - 338y + 169y^2
\end{aligned}$$

- So, we have a quadratic equation involving y. Solving this gives $y \approx 0.92$. Thus 92% of the population has the characteristic.

Modeling chemical mixture

- A solution containing $\frac{1}{2}$lb of salt per gallon flows into a tank at the rate of 2 gallons per minute.
- The well-stirred mixture flows out of the tank at the same rate.
- The tank initially holds 100 gallons of solution containing 5lbs of salt.

- Question: How much salt is in the tank after 30 minutes? And, how much salt is in the tank as $t \to \infty$?

- Our Aim: to model S(t)=amount of salt with respect to time.

Answer:

Step 1 (Let us find the differential equation!!!):

- Salt flowing IN $= 2$ gallons/min $\times$ $\frac{1}{2}$lbs/gallon $=1$lbs/min.
- Salt flowing OUT $= 2$ gallons/min $\times$ $\frac{S}{100}$lbs/gallon $=\frac{S}{50}$lbs/min.

$$\frac{dS}{dt} = IN - OUT = 1 - \frac{S}{50}.$$

Step 2 (find the general solution):

$$\frac{dS}{dt} = 1 - \frac{S}{50} \text{ a linear first order DE}$$

In standard form, it is $\frac{dS}{dt} + \frac{S}{50} = 1$. So, $P(t) = \frac{1}{50}$ and $Q(t) = 1$.

- The integrating factor is

$$u(t) = e^{\int P(t)dt} = e^{\int \frac{1}{50}dt} = e^{\frac{1}{50}t}.$$

Then,

- $$\begin{aligned} S &= \frac{1}{u(t)} \int u(t)Q(t)dt \\ &= \frac{1}{e^{\frac{1}{50}t}} \int e^{\frac{1}{50}t} dt \\ &= e^{-\frac{1}{50}t} \left(50e^{\frac{1}{50}t} + C \right) \text{ (} C \text{ is a constant)} \\ &= 50 + Ce^{-\frac{1}{50}t}. \end{aligned}$$

- Therefore, the general solution is

$$S = 50 + Ce^{-\frac{1}{50}t}.$$

Step 3 (find the particular solution):

- The initial condition is that $S = 5$ when $t = 0$. So, $5 = 50 + Ce^0 \Rightarrow C = 45$.
- Therefore, the particular solution is

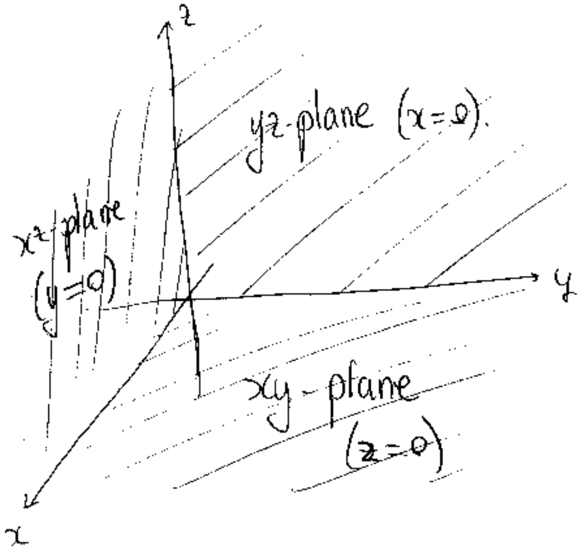
$$S = 50 - 45e^{-\frac{1}{50}t}.$$

Step 4 (solve for given values):

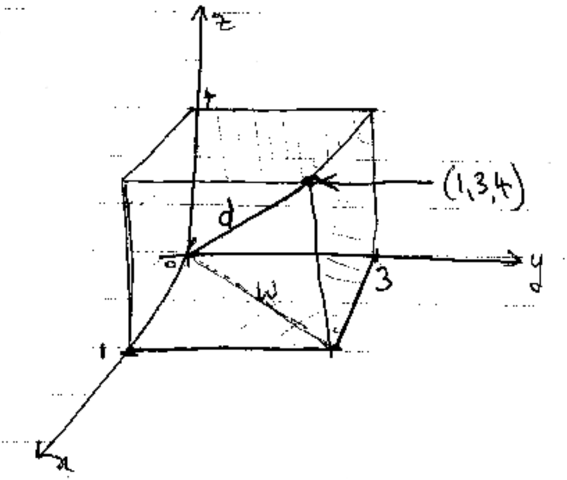
- When $t = 30$, we have $S = 50 - 45e^{-\frac{1}{50}30} \approx 25.30$lbs.
- Also, as $t \to \infty$, we have $S \to 50$ since $45e^{-\frac{1}{50}30} \to 0$ as $t \to \infty$.

Chapter 7: Multivariate Calculus

7.1: The 3-dimensional coordinate system



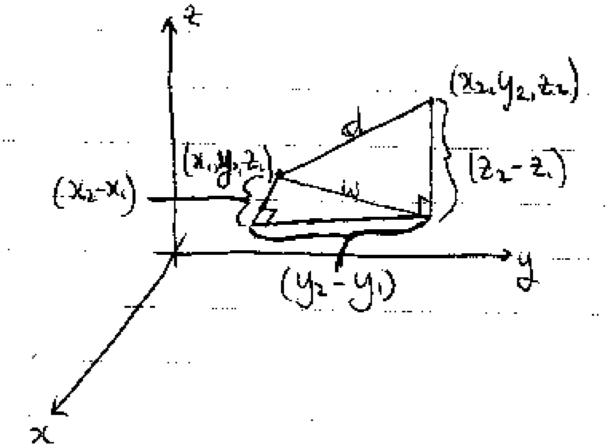
The points (1,3,4) is located at the corner of a $1 \times 3 \times 4$ box.



- We want to know the distance from $(1,3,4)$ to the origin (0,0,0), We want to know that value of d in the diagram.
- Use Pythagoras' theorem twice: $d = \sqrt{4^2 + w^2}$ and $w = \sqrt{1^2 + 3^2}$, so

$$d = \sqrt{1^2 + 3^2 + 4^2} = \sqrt{26}.$$

- In general, the distance of a point (x, y, z) to the origin is $\sqrt{x^2 + y^2 + z^2}$ using the same technique as above.
- The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2). Then, $d = \sqrt{(z_2 - z_1)^2 + w^2}$ and $w = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.



Thus, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
EXAMPLE: Find the distance between $(-3, 2, 5)$ and $(4, -1, 2)$.

$$d = \sqrt{((-3) - 4)^2 + (2 - (-1))^2 + (5 - 2)^2} = \sqrt{49 + 9 + 9} = \sqrt{67}.$$