MATH 17C: CALCULUS FOR BIOLOGY AND MEDICINE

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- Several biological and chemical models require us to use functions of several variables!
- **Example 1:** Chemicals (e.g., sugar) dissolve in fluids (e.g. water) and move randomly in the fluid. This is called *diffusion*
- Say we have a hypothetical sugar block at point zero (one-dimensional movement) the amount of sugar in the interval $[x_1, x_2)$ is given by

$$N_{[x_1,x_2)}(t) = \int_{x_1}^{x_2} c(x,t)dx$$

where the function $c(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp(-\frac{x}{4Dt})$. D is the *diffusion constant* The larger D, the faster the concentration spreads out.

- Challenge: what amount of sugar remains at the origin at time $t = 2$?

- **Example 2** An ecologists studies 3 species of frogs in a region. How diverse is this population? What is the level of uncertainty on predicting the identity of a sample frog taken at random?
- If p_i equals the proportion of frogs of type i, then $p_1 + p_2 + p_3 = 1$ This is given by

$$D(p) = -(p_1 \ln(p_1) + p_2 \ln(p_2) + p_3 \ln(p_3))$$

.

- Challenge: For which values of (p_1, p_2) does this function make sense?
- Challenge: What is the maximum value for diversity $D(p)$? And which values of p_1, p_2, p_3 attain it?
- **GOAL** To answer these questions using geometry and calculus!!

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- $f(x, y, z) = x + 2y - 3z$ (function of 3 variables)

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- **Definition:** Let D be a set (or region) of points (x, y). If for each point (x, y) in D, there is a unique value $z = f(x, y)$, then f is a **function** of x and y.
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- **Question:** Find the domain and range of the function $f(x, y) = \sqrt{x^2 + y^2 - 4}$?

Answer:

Since $f(x, y) = \sqrt{x^2 + y^2 - 4}$ only makes sense when $x^2 + y^2 - 4 \geq 0$. The domain is the set $D = \{(x, y) : x^2 + y^2 - 4 \geq 0\}$, which is the set of all points (x, y) such that $x^2 + y^2 - 4 \geq 0$. The range of f is $[0, \infty)$.

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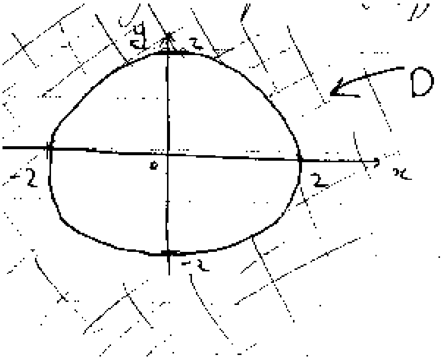

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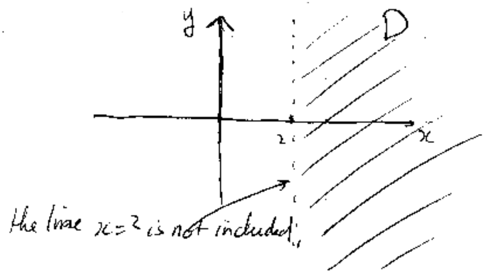
Question: What is the domain and range of $f(x,y) = ln(x-2)$?

Answer: Note that the logarithm is defined only for positive numbers. So, the domain is $D = \{(x,y) : x - 2 > 0\}$. The range is $(-\infty, \infty)$.

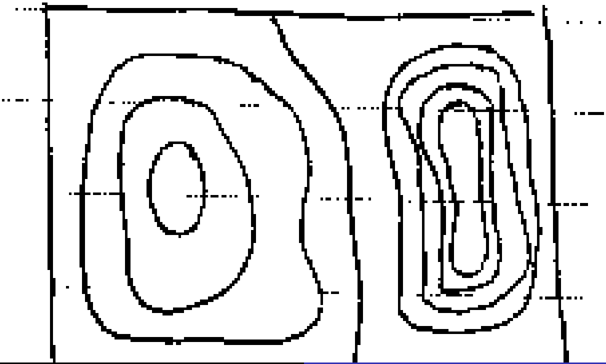


Contour maps and Level curves

Examples of contour maps are topographical maps and weather maps.

For topographical maps, contour lines, also called level curves, are lines of equal elevation.

For Weather maps, isobars, also called level curves, are lines of equal pressure.



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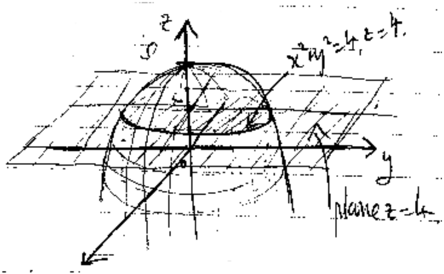
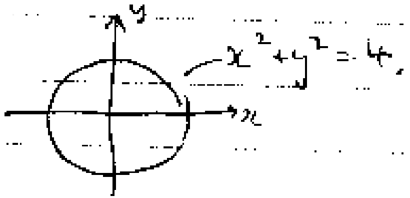
EXAMPLE: Consider the surface $z = f(x, y) = 8 - x^2 - y^2$. It is an elliptic paraboloid (upside down). The intersection of the paraboloid with the plane $z = 4$, is the circle $4 = 8 - x^2 - y^2 \Rightarrow x^2 + y^2 = 4$.



The projection of the trace onto the xy-plane is the circle $x^2 + y^2 = 4$. This projection is called a **level curve.**



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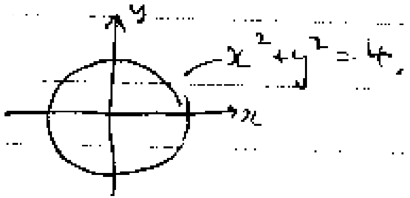
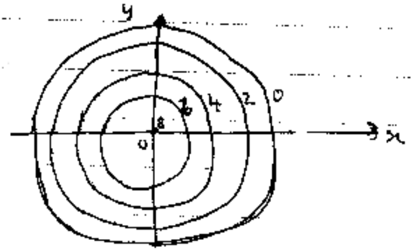


The level curves for $z = 0, 2, 4, 6, 8$ are as follows.



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A **contour map** of a surface $z = f(x, y)$ is a collection of level curves $c = f(x, y)$ for many different evenly spaced constants c.

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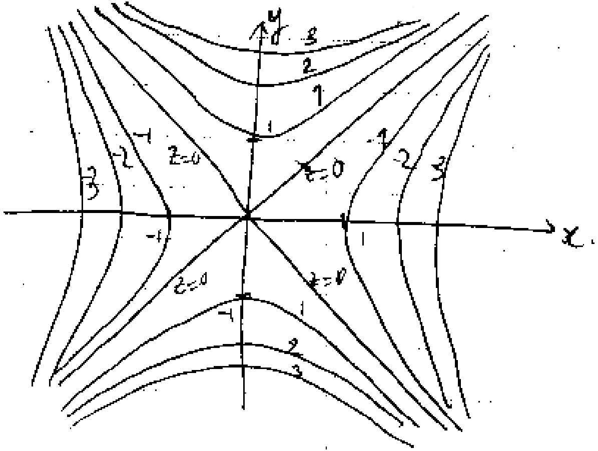
Note that for $z = c$, the level curve is $c = y^2 - x^2$, which is a hyperbola for $c \neq 0$ and it is a cross for $c = 0$.



Limits and Continuity

- Calculus needs to notion of **limit** of a function. Why? Continuity!!
- In one-variable calculus it was easy to calculate limits and continuity. Left and right limit.
- Computing limits in multi-variable calculus is trickier! Many directions! Take

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

  The limit depends on where you approach!! Try x-axis, y-axis.
- Limit exists only when it exists along all curves approaching the point.
- Limit rules from one-variable extend to many variables: sum rule, product rule, quotient rule.

Using the partial derivatives.

Let $z = f(x, y)$ be a function defined at each point in some region D.

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EXAMPLE: Consider the function of two variables $f(x, y) = x^2 + y + x^3y^4$. Then, the partial derivative of f with respect to x and the partial derivative of f with respect to y is as follows respectively:

$$\frac{\partial f}{\partial x} = f_x = 2x + 3x^2y^4 \text{ and } \frac{\partial f}{\partial y} = f_y = 1 + 4x^3y^3.$$

- A partial derivative of a function of two variables, f_x or f_y, is again a function of two variables.
- We can evaluate the partial derivative at a point (x_0, y_0). The notation for evaluation a partial derivative at a point (x_0, y_0) is the following: $\frac{\partial f}{\partial x}|_{(x_0,y_0)} = f_x(x_0, y_0)$ and $\frac{\partial f}{\partial y}|_{(x_0,y_0)} = f_y(x_0, y_0)$.
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- EXAMPLE consider again the function of two variables $f(x, y) = x^2 + y + x^3y^4$. We can evaluate the partial derivatives of f with respect to x at the point $(1, 2)$ as follows:

  $$f_x(1, 2) = 2(1)+3(1)^2(2)^4 = 50 \text{ and } f_y(1, 2) = 1+4(1)^3(2)^3 = 33.$$

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- WHAT IS THIS GOOD FOR???
  FINDING MAXIMAL/MINIMAL SOLUTIONS!
  APPROXIMATING COMPLICATED FUNCTIONS!