Math 67
Homework 6 Due March 2nd

IMPORTANT 1: The Final exam is Tuesday March 17nd. On Monday March 16th I will hold an special review session (class time) as well as special office hours (TBA).

Some tips:

1) Make sure you read and understand the materials in chapter 1-9 and 12. in the book. These are the topics that we will cover in the exam.

2) Learn all the definitions and key theorems/formulas in those chapters.

3) There will be 8 questions in the final exam. At least four problems will come from some of the homeworks or exams (but possibly with numbers changed).

IMPORTANT 2: Please hand-in calculational exercises separated from proof-writing exercises (there will be two piles). PRACTICE exercises don't have to be handed-in they are supposed to help you prepare for the final. I separated the list of exercises in the three categories for your convenience.

——————CALCULATION——————

1. Solve calculational exercises 1,3,4,6 in Chapter 8.
2. Solve all calculational exercises 1,3,5 in Chapter 9.
3. Using the determinant of the matrix $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 4 & 1 & 1 \end{bmatrix}$ decide whether it is invertible or not.
4. Use the cofactor expansion of the determinant and its adjoint to compute the inverse of the matrices:

   a) $A := \begin{bmatrix} 1 & 1 & 1 \\ 2 & -6 & -1 \\ 3 & 4 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 3 & 2 \\ 6 & 0 & 3 \\ 4 & 1 & -1 \end{bmatrix}$
5. Use Cramer's rule to solve the following linear systems of equations

   a) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -6 & -1 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \\ 0 \end{bmatrix}$

   b) $\begin{bmatrix} -2 & 3 & 2 \\ 6 & 0 & 3 \\ 4 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ 6 \\ 0 \end{bmatrix}$

6. Compute the determinant of the matrix $A = \begin{bmatrix} x & x & x & x \\ x & 1 & 0 & 0 \\ x & 0 & 1 & 0 \\ x & 0 & 0 & 1 \end{bmatrix}$ Which values of x give $det(A) = 0$?

7. Let $<,>$ denote an inner product for $\mathbb{R}^2$. Let $a = (1,2)$ and $b = (-1,1)$. If g is a vector such that $< a, g >= -1$ and $< b, g >= 3$, find g. More generally show that for each vector $a \in \mathbb{R}^2$ one has $a =< a, e_1 > e_1 + < a, e_2 > e_2$.

8. Consider $\mathbb{R}^4$ with the usual inner product. Let W be the subspace consisting of all vectors orthogonal to $(1, 0, -1, 1)$ and $(2, 3, -1, 2)$. Find a basis for W.

9. Find and orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by the vectors $x_1 = (1, 1, 0)$, $x_2 = (2, 0, 1)$ and $x_3 = (2, 2, 1)$.

——————PROOF-WRITING————————

10. Solve proof writing exercises 1,2,3 in Chapter 8.

11. Solve proof-writing exercises 1,4,10 in Chapter 9.

12. Find a compact formula for the determinant of $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$

13. A matrix A is orthogonal if $AA^T = I$ (i.e. the inverse of A is its own transpose). Show that if A is orthogonal then $det(A)$ equals $1 or - 1$. Give an example of an orthogonal matrix for which determinant is 1 and one for which determinant is -1.

14. Given an $n \times n$ matrix A over $\mathbb{C}$, show that there exist no more than n distinct numbers c such that $det(A - cI) = 0$

15. Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 1 & 2 & 3 & -1 \\ 3 & 3 & 3 & 3 \end{bmatrix}. \text{ Namely, } T(X) = AX.$$

a) Find a basis for $nullspace(T)$ and the $range(T)$

b) Is T injective or surjective?

c) What is the matrix for T with respect to the canonical bases (i.e. the unit vectors e_i)?

d) Find a matrix representing T with respect to the bases $(1, 1, 1, 1)$, $(1, 1, 1, 0)$,$(1, 1, 0, 0)$, and $(1, 0, 0, 0)$ and $(2, 0, 0)$, $(0, 2, 0)$, $(0, 0, 2)$.

16. Given the matrix $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 6 & -11 & 6 \end{bmatrix}$ Find its eigenvalues, its characteristic polynomial and if possible diagonalize it over the complex numbers (find explicit S such that $S^{-1}AS$ is diagonal).

17. Given the matrix $A = \begin{bmatrix} 21 & 6 & -12 \\ 6 & 12 & -6 \\ -12 & -6 & 21 \end{bmatrix}$

Find the eigenvalues over $\mathbb{C}$ and an orthonormal basis for each of the eigenspaces. Is this matrix diagonalizable over the reals?

18. Find an orthonormal basis for the subspace W of $\mathbb{R}^4$ consisting of all vectors (a, b, c, d) such that $a - b - 2c + d = 0$. Find the distance of the vector $(1, 2, 0, 0)$ to W.

19. Is the following matrix singular? $\begin{bmatrix} 0 & 2 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ -2 & 1 & 5 & 2 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

20. Decide whether the following statements are true or false and justify your answer:

a) Suppose $A = SBS^{-1}$ for A, B, S $n \times n$ matrices (S invertible). Then A, B have the same characteristic polynomial and same eigenvalues.

b) If two matrices A, B have the same eigenvalues they must have the same characteristic polynomial.

c) The inverse of an upper triangular matrix is upper triangular.

d) Let W_1, W_2 be subspaces of V. Then $W_1 \cap W_2$ is a subspace too.

e) Let W_1, W_2 be subspaces of V. Then $W_1 \cup W_2$ is a subspace too.

f) Let A be a $k \times n$ matrix. $dim(range(A)) + dim(Kernel(A)) = n$

g) Let $A \in M_{k,n}$ and $B \in M_{n,k}$. If $tr(X)$ denotes the trace of X, then $tr(AB) = tr(BA)$.

h) Let $A \in M_n$ and x, y represent non-zero vectors such that $Ax = 3x$ and $Ay = 5y$. Then x, y are linearly independent.

i) If λ is an eigenvalue of both A, B then λ is an eigenvalue of $A + B$.

j) Matrices with the same determinant have the same dimension of range for their associated linear maps.

k) If V, W are two distinct 2-dimensional subspaces of R^3, then $V + W = R^3$.

l) Eigenspaces of a matrix are always orthogonal to each other.

m) $range(A + B) \subset range(A) + range(B)$.

n) $range(AB) \subset range(A)$.

o) The matrix for a linear transformation is the same regardless of the bases used to constructed.

p) A linear transformation P is a projection if $P^2 = P$. All projections are invertible.

q) $nullspace(A) = nullspace(A^T A)$.

r) There are real vector spaces with exactly seven vectors.

s) If A, B are $n \times n$ matrices with $det(AB)$ not equal to zero then A, B are both invertible.

t) If the rows of a 4×6 matrix A are linearly independent, then $dim(range(A)) = 4$.

u) Every orthogonal set of n vectors in a vector space of dimension n is a basis of the space.