ACTUALLY DOING IT: POLYHEDRAL COMPUTATION AND ITS APPLICATIONS

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Chapter 1

PART I: WHAT THIS BOOK IS ABOUT

Dear Reader:

Disclaimer: These notes are still work in progress. There are still plenty of errors and typos. Please proceed with caution!

I am convinced that what I present to you in these notes is a charming beautiful and useful subject. I am sure of its beauty because most people, even non-mathematicians, recognize polyhedra as amazingly gorgeous objects. People buy polyhedra to decorate homes and offices at IKEA because of their symmetry and elegance. Sadly, the public does not know that these beauties are also useful in applications and pop-up in various areas of advanced mathematics. It is my mission to show evidence of their utility (not just pretty but useful too!). If you are not convinced of these fact after reading this introductory chapter you probably should not buy this book! I hope that even if you are not a geometry lover you will find enough compelling examples to show you polyhedra are not just beautiful from outside, but the inside too!

These lectures have the short title "Actually doing it", for two important reasons. First, the lectures leave many propositions and theorems unproved with the idea that the reader will jump in and provide their own argument of truth (don't worry, we give you a hand and hint). We think that learning mathematics is best done by doing mathematics and that, in the spirit of the Moore method, a dedicated student who seeks to find a proof of her own finds great joy and learning doing so. A second reason for the title is our focus on the computation and hands-on manipulation of polyhedra. Computers are helping us create new mathematics, and polyhedra are no exception! We are interested in actually finding the explicit numbers or detecting properties that are asked about using a computer. We need the answer now not tomorrow! For this reason, most of the lectures focus on algorithms to compute various properties of polyhedra at the level that an advance undergraduate can understand. We do not assume advance computer science knowledge either.

The book has several “laboratory” activities to exercise this hands-on philosophy we hope to guide the reader through the basics of using software to play with polyhedra. Even if you are a novice, you will find it very easy to compute. In the laboratories, we will talk about software, of course. The main software we are going to use is POLYMAKE. It was a project started in 1997 by Ewgenij Gawrilow and Michael Joswig. It is done in Germany. This is a collection of all possible software so that you can do millions of calculations. The best part is that all of the software is absolutely free for you to download. You won’t have to pay a single dime or Euro.

1.1 An exciting adventure with many hidden treasures...

Let us begin our adventure with a very quick bird-view of the subject. The inmediate goal is to introduce you to the heroes of this adventure. In this chapter we do not worry to be very formal but we show the wealth of the topic, to get you excited! Polyhedra have been around for thousands of years (the Greeks? the Babylonians?), thus one may get the wrong impression there is nothing unknown about them. In general, the public sadly thinks math is a dead subject used to torture young people. I wish more people knew that mathematics research is alive and exciting thus I use Polyhedra as cheerleaders for this noble cause. I list five easy-to-state questions which no expert can answer! Go for it, try to think about them!

This lectures are about **convex polyhedra**. We are sure you have seen pictures such as those in Figure 1.1

Figure 1.1: All are polyhedra, except the one with hair

but convex polyhedra are not like those in Figure 1.2

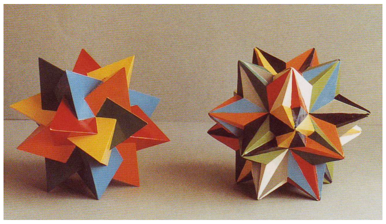


Figure 1.2: Polyhedra but not convex!

We will only talk about convex polyhedra in Euclidean space. A *convex* set S is one for which between any pair of points, the entire line segment is contained in S.

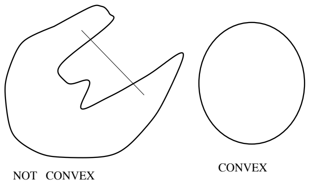


Figure 1.3: Thus a convex set does not look like a croissant!

A hyperplane is given by a linear equation breaks Euclidean space into two pieces called *halfspaces* (see Figure 1.4). A convex polyhedron is a bounded subset of Euclidean space that you obtain by intersecting a finite number of half-spaces.

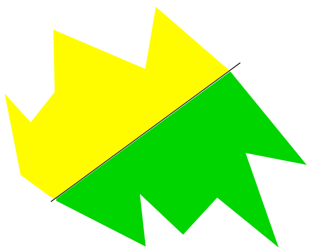


Figure 1.4: A halfspace

What about an infinite number of halfspaces? Can those give polytopes? Well, yes, but in fact the same set would have been easily defined with finitely many half-spaces, thus why waste so much? On the other hand, all other convex figures such as a circle or an ellipse can also be written as intersection of halfspaces, but one requires infinitely many of them for sure and thus they are not polyhedra. One can be very explicit. A polyhedron has a representation as the set of solutions of a system of linear inequalities.

$$\begin{gathered} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,d}x_d \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,d}x_d \leq b_2 \\ \vdots \\ a_{k,1}x_1 + a_{k,2}x_2 + \cdots + a_{k,d}x_d \leq b_k \end{gathered}$$

Each inequality represents one halfspace (chosen implicitly by the direction of the inequality). Note that this allows the possibility of using some equations too. We will use the standard matrix notation $Ax \leq b$ to denote the above system. This is a natural extension of linear algebra. We teach our students how to solve systems of equalities, so why not inequalities? Here's a little exercise that will make the transition from linear algebra to polyhedra geometry more natural: Convince yourself that one can represent a polyhedron as a system of linear inequalities with only non-negative variables $\{x : Ax \leq b \ x \geq 0\}$. Now, not all polyhedra are bounded (why?), but we focus most of our attention on bounded polyhedra, which will receive the name of *polytopes*. We will revisit this with more detail later.

The most evident feature on the "anatomy" of a polytope are its *faces*. What is a face? Essentially, the moment you approach with a hyperplane, at some point the plane touches the polyhedron. That's what you call a face. So, a "corner" is a face. A triangle, of an icosahedron, is a face too. An edge is a face. See Figure 1.5.

From the system of inequalities one can recover the list of all the faces of different dimensions. Already this is so simple, but we are going to be asking some good questions about them!

The great swiss mathematician Leonard Euler (Figure 1.6) was one of the first to think about the possible numbers of faces of a polytope and the numeric relations between them. Today we know this question is strongly related to other areas of mathematics such as topology, algebraic geometry and combinatorics. You may have heard of Euler's surprising equation $f_0 - f_1 + f_2 = 2$, where f_0 is the number of 0-dimensional faces (or *vertices*), f_1 is the number of 1-dimensional faces (known as *edges*), and f_2 is the number of 2-dimensional faces (or *facets*). You can ask the question, if you give me three numbers, and assign them to the numbers f_1, f_2, f_3, is this triple of numbers coming from a polyhedron? It turns out we know the answer to this question. In 1906, Steinitz (another Swiss mathematician) proved

Theorem 1.1.1 *A vector of non-negative integers (f_0, f_1, f_2) is the f-vector of a 3-dimensional polytope if and only if*

1. $f_0 - f_1 + f_2 = 2$

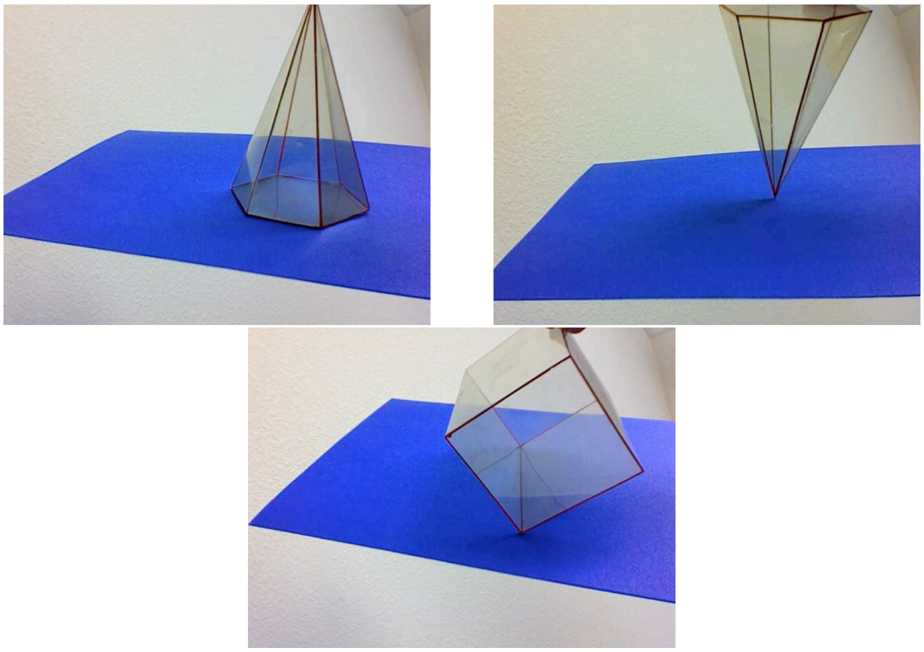


Figure 1.5: faces of a pyramid, described as it is supported on a plane

2. $2f_1 \geq 3f_0$
3. $2f_1 \geq 3f_2$

These inequalities make sense, right? The smallest 2-dimensional face must look like a triangle. So, the third inequality indicates if each edge gets counted twice we must have at least three times the number of facets. One can actually find a polytope for each possible triple of non-negative integers vector that satisfies the inequalities. The proof is very long and elaborate. Steinitz demonstrated that if you start chopping off edges and corners, you can get other polytope starting with a tetrahedron. Nevertheless, here we are in the year 2010 and we don't know a similar set of conditions in dimension four. Here is a first concrete question where mathematicians don't know the answer.

OPEN PROBLEM 1: Can one find similar complete set of conditions characterizing f-vectors of 4-dimensional polytopes? In this case the vectors have 4 components (f_0, f_1, f_2, f_3).

A way to visualize 4-dimensional polyhedra is through *Schlegel diagrams*. This is similar to what you may know as stereographic projection. I take my triangular prism and a light source. When I shine the light through the polytope, then I have a projection to the floor (see Figure 1.7 we see this construction for a prism). In Figure 1.8 we see the Schlegel diagrams of the five Platonic solids.

This is a methodology to go from d dimensions to $d-1$ dimensions. This is one way to see four dimensions. Let's look at the Schlegel diagram of the 4-cube.

Figure 1.6: Leonard Euler

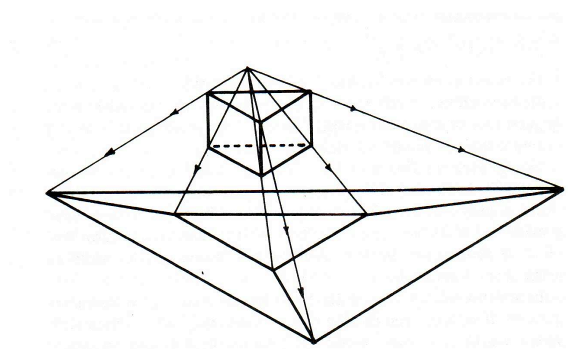


Figure 1.7: Schlegel diagram of a prism

You see it in Figure 1.9, why is the outside bounding box is a 3-dimensional cube?

Schlegel diagrams are not the only way to visualize polyhedra in high dimensions. Imagine that you have scissors. The natural thing is to take it and start cutting it. Cut, cut, cut along the edges (of a 3-polytope). You open it, and you might obtain a *net*. See Figure 1.10.

There are computer programs that you can use to print these unfoldings or nets. The unfolding is not unique. See Figure 1.11 unfolding of cubes. Instead of having scissors, imagine you have meta-scissors. If anybody is familiar with the art of Salvador Dalí, he used such unfoldings in surrealistic version of the crucifixion of Jesus. See Figure 1.11.

Now, you don't want to do a stupid unfolding. You cannot print out overlapping unfoldings because the overlaps will prevent you from reconstructing . You want to avoid these self-intersections. Is it always possible to cut in a

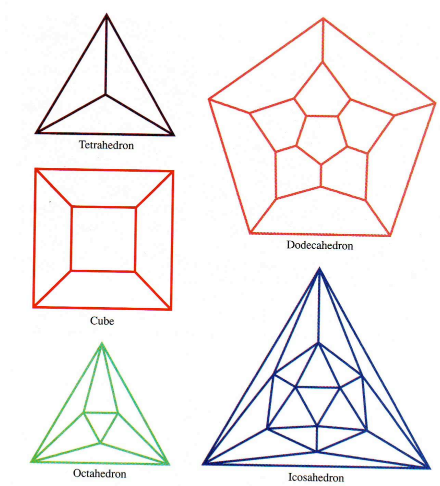


Figure 1.8: Schlegel diagrams of the Platonic solids

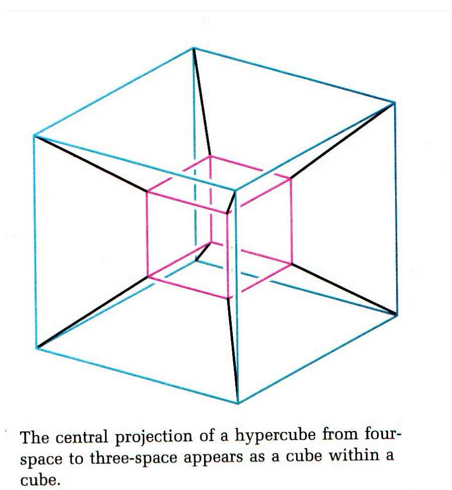


Figure 1.9: Schlegel diagram of a 4-dimensional cube

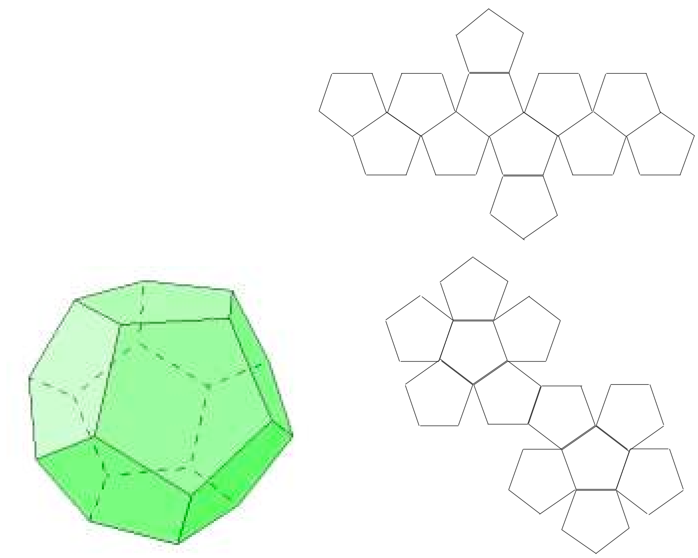


Figure 1.10: A dodecahedron and two of its unfoldings

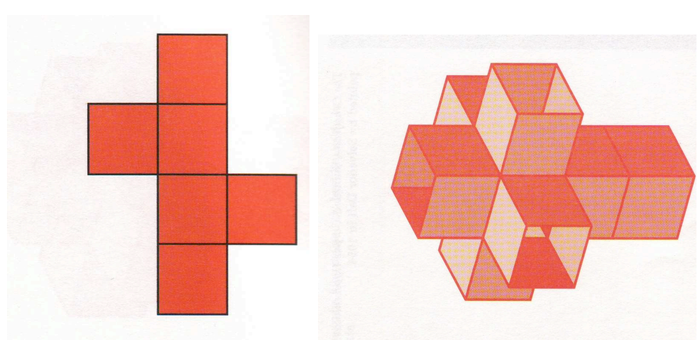


Figure 1.11: Net unfoldings of cubes

nice organized way to get a non-self-intersecting unfolding? Well, my friends, nobody knows the answer! This is a simple question for a elementary school kid, and we're all on the same knowledge for the answer.

OPEN PROBLEM 2: Can one always find an unfolding that has no self-overlappings?

One natural question is the number of unfoldings. It is not known if there is a bound in terms of the f-vector. The exact number of cuts needed is to find a spanning tree on the (dual) graph. For every spanning tree, you have an unfolding. The question is, is there a spanning tree corresponding to a non-overlapping? I just happened to pick the wrong unfoldings in Figure **??**.

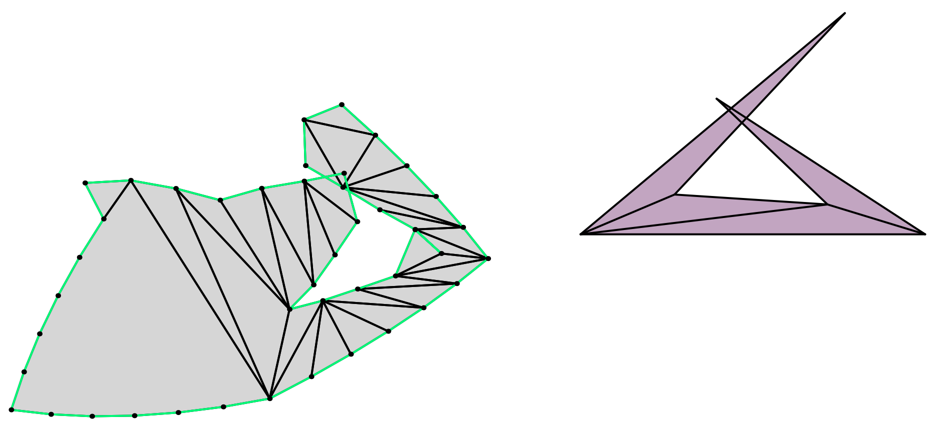


Figure 1.12: When unfoldings go wrong and overlap

The questions we discussed so far lie within pure mathematics, but Polyhedra are useful in practical calculations. I will touch briefly on some questions arising from applied mathematics. Linear Programming is a part of Optimization where you are given a polyhedron (a system of linear inequalities) in $\mathbb{R}^n$ and you want to find a point inside it that maximizes a certain linear function $C = C_1x_1 + C_2x_2 + \cdots + C_dx_d$. It turns out that a maximal solution is always found at a "corner", a vertex. Solving linear programs is a useful operation performed thousands of times in various application domains (see []) In mathematical terms, you are trying to find the best vertex, the vertex with the best value on this function C. A linear program is written as

$$\text{maximize } C_1x_1 + C_2x_2 + \cdots + C_dx_d$$

$$\text{among all } \ x_1, x_2, \ldots, x_d, \text{satisfying:}$$

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,d}x_d \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,d}x_d \leq b_2$$
$$\vdots$$
$$a_{k,1}x_1 + a_{k,2}x_2 + \cdots + a_{k,d}x_d \leq b_k$$

George Dantzig invented the simplex algorithm to solve linear programs. It goes as follows: We already know the optimal solution is found at a vertex. So, let's start at a vertex. Then, let's try to find an adjacent vertex that is better. If that is not already optimal, then move again. Do this again and again. Then you reach a vertex where all neighbors are worse. The union of the vertices and edges of a polytope define its graph. The simplex method *walks* along the graph of the polytope, each time moving to a better and better cost. following beautiful question: How many steps do I need to arrive to an optimal solution?

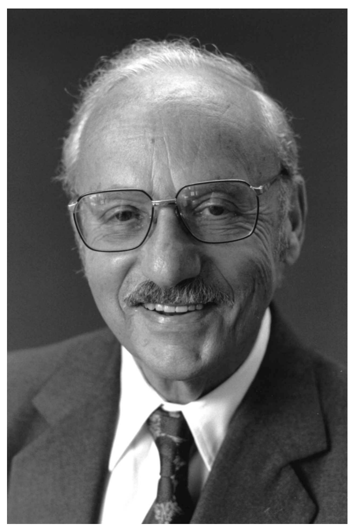


Figure 1.13: George Dantzig

Performance of the simplex method depends on the *diameter* of the graph of the polytope, i.e., the largest distance between any pair of nodes. The diameter is the largest distance between any pair of nodes. The Hirsch conjecture says that the diameter of a polytope is the number of facets[1] minus its dimension. This problem remains unsolved after more than 50 years.

[1]Facets are the faces of highest dimension

OPEN PROBLEM 3: ((the Hirsch conjecture) The diameter of a polytope P is at most # of facets$(P) - dim(P)$.

One of the practical reasons people started to look at polytopes is their volume (or more general integration of functions over polyhedral regions). The Egyptian pyramid is one of the most famous early polytopes. One of the computations we will look at is how to efficiently compute the volume of polytopes. We pretend that we teach this to calculus students. I'm going convince you that there's much more to it than you can imagine.

For instance, one of the interesting things about computing volumes of polyhedra is whether using limits is necessary. We tell students to use calculus (and thus use a limit) in integration, but perhaps you can avoid using limits. Here's one case where we dont limits. Modify the pyramid to have its apex over one of the base vertices (see Figure **??**). If I take three copies of this "distorted" Egyptian pyramid, then I would get a cube. So, three times the volume of the bad Egyptian pyramid is the volume of the cube. That is a special case of the formula we know for all pyramids: one third of the area of the base times the height.

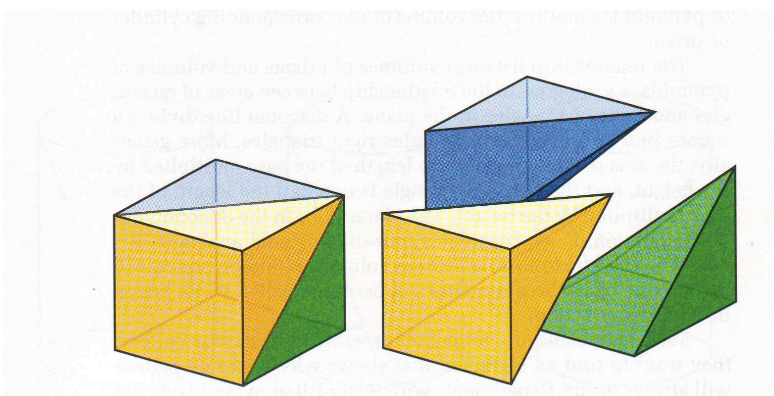


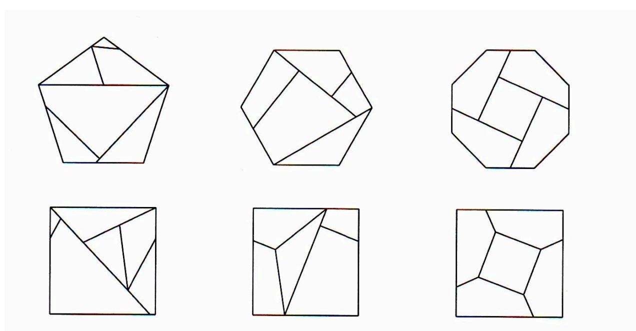
Figure 1.14: distorted Egyptian pyramid

In dimension two, one can show that no limits are necessary either. If you give me two polygons of equal area, I can use scissors to cut and rearrange the pieces to transform one into the other. In two dimensions, any two polytopes of the same area are *equidecomposable* (See Figure 1.1). This theorem was proved by Bolyai, the father of Janos Bolyai, co-inventor of non-Euclidean geometries. A finite algorithm exists to find the transformation, and this implies you don't need limits to compute areas of polygons because we already know the area of a rectangle. The famous 20th century mathematician David Hilbert asked if any two convex 3-dimensional polytopes of the same volume are also equidecomposable. The answer was found less than a year later by Max Dehn.

He proved that there are two 3-polytopes that have the same volume but yet are not equidecomposable! Dehn showed limits are already necessary in the

Figure 1.15: David Hilbert

computation of In some sense, the bad news indicates that you need to study calculus after all (darn!). Nevertheless, we will look at new ways to compute that won't look like we need to draw the symbol of integration. It's a new technology that's not in textbooks yet.



Since we know a formula to compute the volumes of tetrahedra (and pyramids) another way to compute volumes is to decompose polytopes into tetrahedra and then add the volumes of each piece. But how to decompose or *triangulate* a polytope? We will look at ways to triangulate a tetrahedron. Consider the example of the hexagonal bipyramid show in Figure **??**. You can peel it like an orange, or you can start by decomposing into two pyramids and triangulate each. Is there a nice way to find all triangulations of a polyhedron? For triangulation here, I am not allowed to add any new vertices.

Another open problem: You give me a 3-dimensional polytope. If you give me triangulations of two different sizes, is there a triangulation of every inter-

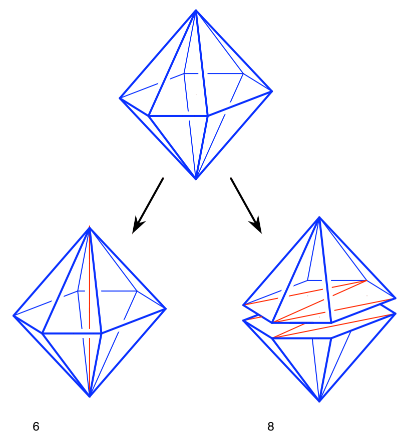


Figure 1.16: many ways to triangulate a bipyramid

mediate size? We don't know!

OPEN PROBLEM 4: If for a 3-dimensional polyhedron P we know that there is triangulation of size k_1 and triangulations of size k_2, with $k_2 > k_1$ is there a triangulation of every size k, with $k_1 < k < k_2$?

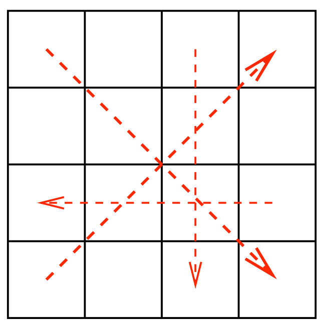
Another practical problem is to count the number of lattice points inside a polyhedron. A wide variety of topics in pure and applied mathematics involve the practical problem of counting the number of lattice points inside a a polytope. Applications range from the very pure (number theory, commutative algebra, representation theory) to the most applied (cryptography, computer science, optimization, and statistics). For example, An emerging new application of lattice point counting is computer program verification and code optimization. The systematic reasoning about a program's runtime behavior, performance, and exe cution requires specific knowledge of the number of operations and resource allocation within the program. This is importan t knowledge for the sake of checking correctness as well as for automatically detecting run-time errors, buffer overflows, null-pointer dereferen ces or memory leaks. For example, *how often is instruction I of the following computer code executed?*

```
void proc(int N, int M)
{
int i,j;
for (i=2N-M; i<= 4N+M-min(N,M), i++)
  for(j=0; j<N-2*i; j++)
    I;
}
```

Clearly, the number of times we reach instruction I depends parametrically on N,M. In terms of these parameters the set of all possible solutions is given by the number of lattice points inside of a parametrized family of polygons. In our toy example these are described by the conditions $\{(i, j) \in \mathbb{Z}^2 : i \geq 2N - M,\ i \leq 4N + M - \min(N, M),\ j \geq 0,\ j - 2i \leq N - 1\}$.

When you count lattice points, you can approximate the volume. If you dilate a polytope to be larger, it's essentially the same as making a smaller lattice. It's the same as adding tiny little cubes around the lattice points. So, it's like a Riemann Integration. Integration is another reason to look at counting lattice points.

Many objects can be counted as the lattice points in some polytope: Examples include, Sudoku configurations, routes on a network, and magic squares. These are $n \times n$ squares whose entries are non-negative integers with sums over rows columns and diagonals equal to a constant, the *magic number* (see Figure 1.17. Mathematically, the possible magic squares are non-negative integer solutions of a system of equations and inequalities: $2n + 2$ equations, one for each row sum, column sum, and diagonal sum. For example for 4×4 magic squares of magic sum 24 we have $x_{11} + x_{12} + x_{13} + x_{14} = 24$, **first row**$x_{13} + x_{23} + x_{33} + x_{43} = 24$, **third column**, and of course $x_{ij} \geq 0$



12	0	5	7
0	12	7	5
7	5	0	12
5	7	12	0

5

Figure 1.17: 4×4 magic squares with magic sum 24

One beautiful puzzle is to determine the number of magic squares of a given size. This problem is really asking about the number of lattice points in a particular polytope. We can ask

OPEN PROBLEM 5: Find a formula in terms of k for the number of 30×30 magic squares with magic sum k.

It is already non-trivial to figure out *one* concrete value, say $k = 10$. This is a challenge of computation which is well beyond humanity's reach today.

1.2 The Rest of this Book

- Basic polyhedral geometry. (week 2) Representation of polyhedra, dimension.

- Rules of computation and measuring efficiency introduction to POLYMAKE. (week 1)
- Computer representation of polytopes: facets vs. vertices, chirotopes. (week 3-4)
- Visualization of Polytopes: Schlegel Diagrams, Nets, Gale Transforms, Slices and Projections. (week 5)

- Polytope graphs (project).
- Finding decompositions and triangulations. (project)
- Volumes, integrals, and discrete Sums over polytopes. (project)
- Containment, Distance, Width, and Approximation problems. (project)
- Symmetry of polytopes and polyhedra. (project)

Chapter 2

PART II: Basics of Polyhedral Geometry

We will get more technical during this session (Let's make very precise the meanings of our words). We assume the reader is comfortable with linear algebra and the basics of analysis. We also assume the reader has studied the first thirteen sections of the lecture notes *Theory of Convex Sets* by G.D. Chakerian and J.R. Sangwine-Yager

2.1 Polyhedral Notions

Everything we do takes place inside Euclidean d-dimensional space $\mathbb{R}^d$. We have the traditional Euclidean distance between two points x, y defined by $\sqrt{(x_1 - y_1)^2 + \ldots (x_2 - y_2)^2}$. Given two points x, y. We will use the common

fact that $\mathbb{R}^d$ is a real vector space and thus we know how to add or scale its points.

Definition 2.1.1 *A subset S of $\mathbb{R}^n$ is called* convex *if for any two distinct points x_1, x_2 in S the line segment joining x_1, x_2, lies completely in S. This is equivalent to saying $x = \lambda x_1 + (1-\lambda)x_2$ belongs to S for all choices of λ between 0 and 1.*

In general, given a finite set of points $A = \{x_1, \ldots, x_n\}$, we say that a linear combination $\sum \gamma_i x_i$ is

- an *affine combination* if $\sum \gamma_i = 1$
- a *convex combination* if it is affine and $\gamma_i \geq 0$ for all i.

We will assume that the empty set is also convex. Observe that the intersection of convex sets is convex too. Let $A \subset \mathbb{R}^d$, the *convex hull* of A, denoted by $conv(A)$, is the intersection of all the convex sets containing A. In other words, A is the smallest convex set containing A. The reader can check that the image of a convex set under a linear transformation is again a convex set.

Recall from linear algebra that a linear function $f : \mathbb{R}^d \to \mathbb{R}$ is given by a vector $c \in \mathbb{R}^d, c \neq 0$. For a number $\alpha \in \mathbb{R}$ we say that $H_\alpha = \{x \in \mathbb{R}^d : f(x) = \alpha\}$ is an *affine hyperplane* or *hyperplane* for short. Note that a hyperplane divides $\mathbb{R}^d$ into two halfspaces $H_\alpha^+ = \{x \in \mathbb{R}^d : f(x) \geq \alpha\}$ and $H_\alpha^- = \{x \in \mathbb{R}^d : f(x) \leq \alpha\}$. Halfspaces are convex sets.

We begin with the key definition of this notes:

Definition 2.1.2 *The set of solutions of a system of linear inequalities is called a* polyhedron*. In its general form a polyhedron is then a set of the type*

$$P = \{x \in \mathbb{R}^d :< c_i, x >\leq \beta_i\}$$

for some non-zero vectors c_i in $\mathbb{R}^d$ and some real numbers β_i.

In other words A *polyhedron* in $\mathbb{R}^d$ is the intersection of finitely many halfspaces.

By the way, the plural of the word *polyhedron* is *polyhedra*.

Lemma 2.1.1 *Let $Ax \leq b$, $Cx \geq d$, be a system of inequalities. The set of solutions is a convex set.*

Write a proof!

Although everybody has seen pictures or models of two and three dimensional polyhedra such as cubes and triangles and most people may have a mental picture of what edges, ridges, or facets for these objects are, we will formally introduce them later on. Now another important definition:

Definition 2.1.3 *A* polytope *is the convex hull of a finite set of points in $\mathbb{R}^d$.*

Lemma 2.1.2 *For a set $A \subset \mathbb{R}^d$ we have that $conv(A)$ equals the set of all possible convex combinations. In particular, for a finite set of points in $\mathbb{R}^d$ $A := \{a_1, a_2, \ldots, a_n\}$ we have that $conv(A)$ equals*

$$\{\sum_{i=1}^{n} \gamma_i a_i : \gamma_i \geq 0 \text{ and } \gamma_1 + \ldots \gamma_n = 1\}$$

Exercise: Write a proof!

Related to minimal size representations of a vector as a convex combinations we have

Theorem 2.1.3 (Carathéodory's Theorem) *If $x \in \operatorname{conv}(S)$ in $\mathbb{R}^d$, then x is the convex combination of $d+1$ points in S.*

Write a proof

You can actually find $d+1$ points that actually suffice to write a point in the set. If you have, for instance

$$x \in \sum_{i=1}^{N} \gamma_i y_i$$

for some huge number N, the theorem here guarantees that you can do this using a sum of not so many summands. This is not necessarily about polytopes. This is a true fact about convexity in general. In polytopes, you have the advantage that you know the set S is already finite to start with. As a corollary, all triangles of a polygon (using the vertices of the polygon) must cover the polygon.

Now it is easier to speak about examples of polytopes. We invite you to find more on your own! Here are some basic examples.

1. **Standard Simplex** Let $e_1, e_2, \ldots, e_{d+1}$ be the standard unit vectors in

   $\mathbb{R}^{d+1}$. The *standard* d-dimensional simplex Δ_d is $conv(\{e_1, \ldots, e_{d+1}\})$. From the above lemma we see that the set is precisely

   $\Delta_d = \{x = (x_1, \ldots, x_{d+1}) : x_i \geq 0 \text{ and } x_1 + x_2 + \cdots + x_{d+1} = 1\}$.

   Note that for a polytope $P = conv(\{a_1, \ldots, a_m\})$ we can define a linear map $f : \Delta_{m-1} \to P$ by the formula $f(\lambda_1, \ldots, \lambda_m) = \lambda_1 a_1 + \cdots + \lambda_m a_m$. Lemma 2.1.2 implies that $f(\Delta_{m-1}) = P$. Hence, every polytope is the image of the standard simplex under a linear transformation. A lot of the properties of the standard simplex are then shared by all polytopes.

2. **Standard Cube** Let $\{u_i : i \in I\}$ be the set of all 2^d vectors in $\mathbb{R}^d$ whose coordinates are either 1 or -1. The polytope $I_d = conv(\{u_i : i \in I\}$ is called the *standard* d-dimensional cube. The images of a cube under linear transformations receive the name of *zonotopes*.

   Clearly $I^d = \{x = (x_1, \ldots, x_d) : -1 \leq x_i \leq 1\}$

3. **Standard Crosspolytope** This is the convex hull of the $2d$ vectors $e_1, -e_1, e_2, -e_2, \ldots, e_d, -e_d$. The 3-dimensional crosspolytope is simply an octahedron.

4. A d**-pyramid** is the convex hull of a $(d-1)$-polytope Q and a point not in the affine hull of Q.

5. A d**-prism** is the convex hull of two $(d-1)$-polytopes P, P' where P' is a translate of P that does not lie in the affine hull of P.

6. A d**-bipyramid** is the convex hul of a $(d-1)$-polytope and a segment that intersects the interior of P, with one point on one side of the affine hull of P and the other end point on the other side.

Let P be a polytope in $\mathbb{R}^d$. A linear inequality $f(x) \leq \alpha$ is said to be *valid* on P if every point in P satisfies it. A set $F \subset P$ is a *face* of P if and only there exists a linear inequality $f(x) \leq \alpha$ which is valid on P and such that $F = \{x \in P : f(x) = \alpha\}$. In this case f is called a *supporting function* of F and the hyperplane defined by f is a *supporting hyperplane* of F.

For a face F consider the smallest affine subspace $aff(F)$ in $\mathbb{R}^d$ generated by F. Its dimension is called the dimension of F. Similarly we define the dimension of the polytope P. We like to think of the polytope and the empty set as faces. They are honorary faces of the polytope!

Another, equivalent definition of dimension is in terms of affine independence. This generalizes the notion of linear independence. We say that a set of points is *affinely dependent* if there is a linear combination of the x_i equal to 0, where $\sum \gamma_i = 0$ but not all γ_i are zero. One can prove that A set of $d+2$ points in $\mathbb{R}^d$ is always affinely dependent. A set of points that is not affinely dependent is affinely independent. The dimension of subset X of $\mathbb{R}^n$ can be defined as the size of any maximal affinely independent set of points in X.

Definition 2.1.4 *A point x is a convex set S is an* extreme point *of S if it is not an interior point of any line segment in S. This is equivalent to saying that when $x = \lambda x_1 + (1-\lambda)x_2$, then either $\lambda = 1$ or $\lambda = 0$.*

Lemma 2.1.4 *Every vertex of a polyhedron is an extreme point.*

Exercise: Write a proof!!

2.2 The characteristic features of Polyhedra

We want to be able to compute the essential features of a polyhedron or a polytope. Among them we wish to find out what are their faces or extreme points, we wish to know what is their dimension, whether the polyhedron in question is empty or not. We want to answer such questions with concrete practical algorithms. So far we have consider polyhedra as given by a system of inequalities of the form $Ax \leq b$. Now we disclose other equivalent representations:

2.2.1 Equivalent representations of polyhedra, dimension and extreme points

Lemma 2.2.1 *Given a bounded polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ then*

- *There is a translation of P that can be represented with in the form $\{x \in R^n : A'x \leq b'$ and $x \geq 0\}$.*
- *There is a polyhedron of the form $Q = \{x \in R^q : Bx = c, x \geq 0\}$ such that*
  - *the coordinate-erasing linear projection*

    $$\pi : \mathbb{R}^q \longrightarrow \mathbb{R}^n \ : \ x = (x_1, \dots, x_n, \dots x_q) \mapsto \pi(x) = (x_1, \dots, x_n)$$

    provides a bijection between Q and P.
  - *The bijection implies that $\pi(Q \cap \mathbb{Z}^q) = P \cap \mathbb{Z}^n$.*
  - *$y \in Q$ is an extreme point if and only if $\pi(x)$ is an extreme point of P*
  - *$dim(Q) = dim(P)$.*

Write a proof

Lemma 2.2.2 *The set of all $n \times n$ matrices whose sums along rows, columns, or diagonals are equal to the same constant (*magic squares*) k is a bounded polyhedron*

Exercise: Write a proof

We will state a lemma that will allow us to compute the dimension from a polyhedron. From Theorem **??** it is enough to explain it for polyhedra given in the form $\{x : Ax = b\}$. A face of dimension 0 is called a *vertex*. A face of dimension 1 is called an *edge*, and a face of dimension $dim(P) - 1$ is called a *facet*. The empty set is defined to be a face of P of dimension -1. Faces that are not the empty set or P itself are called proper. Let us look at a simple recipe for the dimension.

Lemma 2.2.3 *Given $P = \{x \in \mathbb{R}^n \mid Ax = b \ x \geq 0\}$, the dimension of P is $n - \text{rank}(A)$.*

Write a proof

Given the example we have for the 4×4 magic squares, the matrix A is

$$\begin{bmatrix}
1 & 1 & 1 & 1 & & & & & & & & & & & & \\
& & & & 1 & 1 & 1 & 1 & & & & & & & & \\
& & & & & & & & 1 & 1 & 1 & 1 & & & & \\
& & & & & & & & & & & & 1 & 1 & 1 & 1 \\
1 & & & & 1 & & & & 1 & & & & 1 & & & \\
& 1 & & & & 1 & & & & 1 & & & & 1 & & \\
& & 1 & & & & 1 & & & & 1 & & & & 1 & \\
& & & 1 & & & & 1 & & & & 1 & & & & 1
\end{bmatrix} \tag{2.1}$$

This matrix is not full rank. The rank of A is 7. There are sixteen variables. So, by the lemma above, the dimension is $16-7$. The formula has been verified in this example.

Lemma 2.2.4 *The dimension of the polyhedron $M_{n\times n}(k)$ of all $n \times n$ magic squares with magic sum is equal to $(n-1)^2$.*

Write a proof

Theorem 2.2.1 *Consider the polyhedron given by $Ax = b, x \geq 0$. Suppose the m columns $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$ of the $m \times n$ matrix A are linearly independent and there exist non-negative numbers x_{i_j} such that*

$$x_{i_1}A_{i_1} + x_{i_2}A_{i_2} + \cdots + x_{i_m}A_{i_m} = b.$$

Then the points with entry x_{i_j} in position i_j and zero elsewhere is an extreme point of the polyhedron $P = \{x : Ax = b, x \geq 0\}$.

Proof: Suppose x is not a extreme point. Then x lies in the interior of a line segment in $P = \{y : Ay = b, y \geq 0\}$. Thus $x = \lambda u + (1-\lambda)v$ with λ between 0 and 1. But this implies, by looking at the entries of x that are zero, that u, v also have that property. Now, consider $y = x - u$. $A(x-u) = 0$ but since the columns $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$ are linearly independent, $x = u$, a contradiction.

Theorem 2.2.2 *Suppose $x = (x_1, \ldots, x_n)$ is an extreme point of a polyhedron $P = \{x : Ax = b,\ x \geq 0\}$ with A an $m \times n$ matrix. Then*

1) the columns of A which correspond to positive entries of x form a linearly independent set of vectors in $\mathbb{R}^m$

2) At most m of the entries of x can be positive, the rest are zero.

Proof: Suppose the columns are linearly dependent. Thus there are coefficients, not all zero, such that $c_{i_1}A_{i_1} + c_{i_2}A_{i_2} + \cdots + c_{i_m}A_{i_m} = 0$

Thus we can form points
$(x_{i_1} - dc_{i_1})A_{i_1} + (x_{i_2} - dc_{i_2})A_{i_2} + \cdots + (x_{i_m} - dc_{i_m})A_{i_m} = b$
$(x_{i_1} + dc_{i_1})A_{i_1} + (x_{i_2} + dc_{i_2})A_{i_2} + \cdots + (x_{i_m} + dc_{i_m})A_{i_m} = b$

Since d is any scalar, we may choose d less than the minimum of $x_j/|c_j|$ for those $c_j \neq 0$.

We have reached a contradiction! Since $x = 1/2(u) + 1/2(v)$ and both u, v are inside the polyhedron. For part (2) simply observe that there cannot be more than m linearly independent vectors inside $\mathbb{R}^m$.

Exercise 2.2.5 *Find all the extreme points of the polyhedron $M_{3\times 3}(1)$.*

Write a solution!

Definition 2.2.3 *A* basic solution *is a a solution of the system $Ax = b$ where $n - m$ variables are set to zero. If in addition the solutions happens to have $x \geq 0$ then we say is* basic feasible solution.

In any basic solution, the $n - m$ variables which are set equal to zero are called nonbasic *variables and the m variables we solved for are called the* basic *variables.*

We can rephrase the above results by saying that the extreme points of the polyhedron $P = \{x : Ax = b, x \geq 0\}$ are precisely the basic feasible solutions. By Krein-Milman's theorem we know now that if P is bounded then it is the convex hull of its basic feasible points.

Lemma 2.2.6 *Every basic feasible of a polyhedron $\{x : Ax = b, x \geq 0\}$ is a vertex.*

Corollary 2.2.7 *For a polyhedron of the form $P = \{x : Ax = b, x \geq 0\}$, the sets of basic feasible solutions, vertices, and extreme points are identical.*

Write a proof!

2.3 Weyl-Minkowski and Polarity

It makes sense to study the relation between polytopes and polyhedra. Clearly standard cubes,simplices and crosspolytopes are also polyhedra, but is this the case in general? What one expects is really true. Polytopes are special kind of polyhedra, but not all polyhedra are polytopes.

Theorem 2.3.1 (Weyl-Minkowski theorem) *Every polytope is a polyhedron. Every bounded polyhedron is a polytope.*

Figure 2.1: H. Weyl and H. Minkowski

This theorem is very important. Having this double way of representing a polytope allows you to work, using either the vertex representation or the inequality representation representation, what would be hard to prove using a single representation becomes easy with the other. For example, every intersection of a polytope with an affine subspace is a polytope. Similarly, the intersection of finitely many polytopes is a polytope. Both statements are rather easy to prove if one knows that polytopes are just given by systems of linear inequalities, since then the intersection of polytopes is just adding new equations. On the other hand, It is known that every linear projection of a bounded polyhedron is a bounded polyhedron. To prove this from the inequality representation is difficult, but it is easy when one observes that the projection of convex hull is the convex hull of the projection of the vertices of the polytope. In addition, the Weyl-Minkowski theorem is very useful in applications! its existence is key in the field of combinatorial and linear optimization. We can give a non-constructive proof here and later an algorithmic proof.

Definition 2.3.1 *A representation of a polytope by a set of inequalities, is an* H-representation. *On the other hand, when a polytope is given a convex hull of a set of points we have a* V-representation.

Before we discuss a proof of Weyl-Minkowski theorem we need to introduce a useful operation. To every subset of Euclidean space we wish to associate a

convex set. Given a subset A of $\mathbb{R}^d$ the *polar* of A is the set A^o in $\mathbb{R}^d$ defined as the linear functionals whose value on A is not greater than 1, in other words:

$$A^o = \{x \in \mathbb{R}^d :< x, a > \leq 1 \; for \; every \; a \in A\}$$

Another way of thinking of the polar is as the intersection of the halfspaces, one for each element $a \in A$, of the form

$$\{x \in \mathbb{R}^d :< x, a > \leq 1\}$$

Proposition 2.3.2 *For any point in $\mathbb{R}^n$, x^o is a closed halfspace whose bounding hyperplane is perpendicular to the vector x and which intersects the segment from the origin O to x at a point p such that $d(O,p)d(O,x) = 1$*

write a proof!

Lemma 2.3.3 *For any sets $A, B \subset \mathbb{R}^n$, we have*

1. *the polar A^o is closed, convex, and contains the origin O*
2. *If $A \subset B$ then $B^o \subset A^o$.*
3. *If $A = conv(S)$, then $A^o = S^o$.*
4. *If A is a convex body, then A^o is the intersection of the duals of the extreme points of A.*

Proof: For part (1) A^o is the intersection of closed convex sets, thus it is closed and convex. It is immediate that the origin is always in the polar. Part (2) is easy. Now for a proof of part (3), part (2) implies that $A^o \subset S^o$. Pick $x \in S^o$. We need to show that $< x, z > \leq 1$ for all $z \in A$. We have that $z = \sum \lambda_i x_i$, with $x_i \in S$ and $\sum \lambda_i = 1$, $\lambda_i \geq 0$. By linearity of the inner product $< z, x >= \sum \lambda_i < x_i, x > \leq 1$. Thus $x \in A^o$, then $A^o = S^o$. Part (4) is direct from part (3) because Krein-Milman says a convex body is the convex hull of its extreme points.

Here are two more useful examples of polarity: Take L a line in $\mathbb{R}^2$ passing through the origin, what is L^0? Well the answer is the perpendicular line that passes through the origin. If the line does not pass through the origin the answer is different. What is it? Answer: it is a clipped line orthogonal to the given line that passes through the origin. To see without loss of generality rotate the line until it is of the form $x = c$ (because the calculation of the polar boils down to checking angles and lengths between vectors we must get the same answer up to rotation).

What happens with a circle of radius one with center at the origin? Its polar set is the disk of radius one with center at the origin. Next take $B(0, r)$. What is $B(0, r)^o$? The concept of polar is rather useful. We use the following lemma:

Lemma 2.3.4 *1. If P is a polytope and $0 \in P$, then $(P^o)^o = P$.*

2. Let $P \subset \mathbb{R}^d$ be a polytope. Then P^o is a polyhedron.

Write a proof!

Now, using the above lemma, we are ready to prove the Weyl-Minkowski theorem:

Proof: **(of Weyl-Minkowski)** First we verify that a bounded polyhedron is a polytope: Let P be $\{x \in \mathbb{R}^d :< x, c_i >\leq b_i\}$.

Consider the set of points E in P that are the unique intersection of d or more of the defining hyperplanes. The cardinality of E is at most $\binom{m}{d}$ so it is clearly a finite set and all its element are on the boundary of P. Denote by Q the convex hull of all elements of E. Clearly Q is a polytope and $Q \subset P$. We claim that $Q = P$. Suppose there is a $y \in P - Q$. Since Q is closed and bounded (bounded) we can find a linear functional f with the property that $f(y) > f(x)$ for all $x \in Q$. Now P is compact too, hence f attains its maximum on the boundary moreover we claim it must reach it in a point of E. The reason is that a boundary point that is not in E is in the solution set

We verify next that a polytope is indeed a polyhedron: We can assume that the polytope contains the origin in its interior (otherwise translate). So for a sufficiently small ball centered at the origin we have $B(0, r) \subset P$. Hence $P^o \subset B(0, r)^o = B(0, 1/r)$. This implies that P^o is a bounded polyhedron. But we saw in the first part that bounded polyhedra are polytopes. Then P^o is a polytope. We are done because we know from the above lemma that $(P^o)^o = P$ and polar of polytopes are polyhedra.

From this fundamental theorem several nice consequences follow:

Corollary 2.3.5 *Let P be a d-dimensional polytope in $\mathbb{R}^d$. Then*

1. *The intersection of P with a hyperplane is a polytope. If the hyperplane passes through a point in the relative interior of P then the intersection is a $(d-1)$-polytope.*

2. *Every projection of P is a polytope. More generally, the image of P under a linear map is another polytope.*

Write a proof!

Proof: Part (1) follows because a polytope is a bounded polyhedron but the intersection of a polyhedron with a hyperplane gives a polyhedron of lower dimension which is still bounded, thus the result is a polytope. The points of P are convex combinations of vertices $v_1, \dots, v_m$ then applying a linear transformation π, we see linearity implies that any point of $\pi(P)$ is a convex combination of $\pi(v_1), \dots, \pi(v_m)$.

Chapter 3

PART III: Fourier-Motzkin Elimination and its Applications

3.1 Feasibility of Polyhedra and Facet-Vertex representability

In this section we take care of various important issues. Our very first algorithm will be about deciding when a polyhedron (system of inequalities) is empty or not. We are also interested in practical ways of going from H-representation to V-representation or vice versa. Our new proof of Weyl-Minkowski was nonconstructive.

3.1.1 Solving Systems of Linear Inequalities in Practice

When is a polytope empty? Can this be decided algorithmically? How can one solve a system of linear inequalities $Ax \leq b$? We start this topic looking back on the already familiar problem of how to solve systems of linear equations. It is a crucial algorithmic step in many areas of mathematics and also would help us better understand the new problem of solving systems of linear inequalities. Recall the fundamental problem of linear algebra is

Problem: Given an $m \times n$ matrix A with rational coefficients, and a rational vector $b \in \mathbb{Q}^m$, is there a solution of $Ax = b$? If there is solution we want to find one, else, can one produce a proof that no solution exist?

I am sure you are well-aware of the Gaussian elimination algorithm to solve such systems. Thanks to this and other algorithms we can answer the first question. Something that is usually not stressed in linear algebra courses is that when the system is infeasible (this is a fancy word to mean no solution exists) Gaussian elimination can provide a *proof* that the system is indeed infeasible!

This is summarized in the following theorem:

Theorem 3.1.1 (Fredholm's theorem of the Alternative) *The system of linear equations Ax=b has a solution if and only for each y with the property that $yA = 0$, then $yb = 0$ as well.*

In other words, one and only one of the following things can occur: Either $Ax = b$ has solution or there exist a vector y with the property that $yA = 0$ but $yb \neq 0$. Similarly $\{x \mid Ax = b\}$ is non-empty if and only if $\{y \mid y^T A = 0, y^T b = -1\}$ is empty.

So think of this as a mathematical proof that some system has no solution! The vector y above is a **mathematical proof** that $Ax = b$ has no solution. To solve $Ax = b$? We teach them Gaussian elimination. We teach row operations that put the matrix in a certain reduced form. We know then that the entries of b change. What we teach undergraduates is there is no solution when there is a row zeroes in the reduced A but a non-zero entry in the corresponding b. If the string of matrices

$$M_k \cdots M_2 M_1$$

are the elementary matrices that reduce A, then

$$y = e_d M_k \cdots M_2 M_1$$

This shows that $y^T A = 0$ and $y^T b = -1$.

Thus when $Ax = b$ has no solution we get a certificate, a proof that the system is infeasible. But, how does one compute this special certificate vector y? With care, it can be carefully extracted from the Gaussian elimination. Here is how: The system $Ax = b$ can be written as an extended matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

We perform *row operations* to eliminate the first variable from the second, third rows. Say a_{11} is non-zero (otherwise reorder the equations). Substract multiples of the first row from the second row, third row, etc. Note that this is the same as multiplying the extended matrix, on the left, by elementary lower triangular matrices. After no more than m steps the new extended matrix looks like.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ 0 & a'_{22} & \dots & a'_{2n} & b'_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a'_{m2} & \dots & a'_{mn} & b'_m \end{bmatrix}$$

Now the last $m-1$ rows have one less column. Recursively solve the system of $m-1$ equations. What happens is the variables have to be eliminated in all but one of the equations creating eventually a row-echelon shaped matrix B. Again all these row operations are the same as multiplying A on the left by a certain matrix U. If there is a solution of this smaller system, then to obtain the solution value for the variable x_1 can be done by substituing the values in the first equation. When there is no solution we detect this because one of the rows, say the i-th row, in the row-echelon shaped matrix B has zeros until the last column where it is non-zero. The certificate vector y is given then by the i-th row of the matrix U which is the one producing a contradition $0 = c \neq 0$.

If you are familiar with the concerns of numerical analysis, you may be concerned about believing the vector y is an exact proof of infeasibility. "What if there are round of errors? Can one trust the existence of y?" you will say. Well, you are right! It is good time to stress a fundamental difference in this lecture from what you learned in a numerical analysis course: Operations are performed using *exact arithmetic* not *floating point arithmetic*. We can trust the identities discovered as exact.

Unfortunately, in many situations finding just *any* solution might not be enough. Consider the following situations:

Suppose a friend of yours claims to have a $3 \times 3 \times 3$ array of numbers, with the property that when adding 3 of the numbers along vertical lines or any horizontal row or column you get the numbers shown in Figure 3.1:

The challenge is to figure out whether your friend is telling the truth or not? Clearly because the numbers in the figure are in fact integer numbers one can hope for an *integral* solution, or even for a *nonnegative integral* solution because the numbers are non-negative integers. This suggests three interesting variations of linear algebra. We present them more or less in order of difficulty here below. We begin now studying an algorithm to solve problem A. We will encounter problems B and C later on. Can you guess which of the three problems is harder in practice?

Problem A: Given a rational matrix $A \in Q^{m \times n}$ and a rational vector $b \in Q^m$. Is there a solution for the system $Ax = b$, $x \geq 0$, i.e. a solution with all non-negative entries? If yes, find one, otherwise give a proof of infeasibility.

Problem B: Given an integral matrix $A \in \mathbb{Z}^{m \times n}$ and an integral vector $b \in \mathbb{Z}^m$. Is there a solution for the system $Ax = b$, with x an integral vector? If yes, find a solution, otherwise, find a proof of infeasibility.

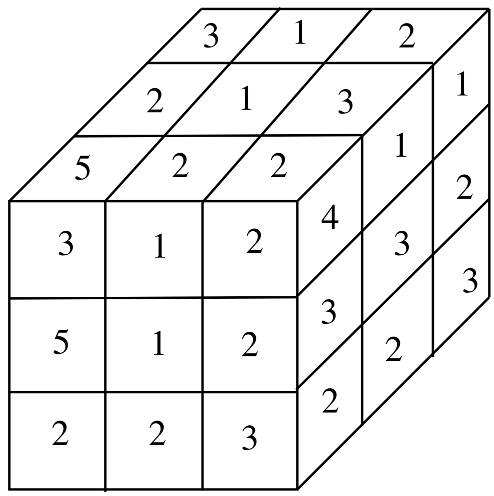


Figure 3.1: A cubical array of 27 seven numbers and the 27 line sums

Problem C: Given an integral matrix $A \in \mathbb{Z}^{m \times n}$ and an integral vector $b \in \mathbb{Z}^m$. Is there a solution for the system $Ax = b,\ x \geq 0$? i.e. a solution x using only non-negative integer numbers? If yes, find a solution, otherwise, find a proof of infeasibility.

I want this similar result for polyhedra. It's called Farkas lemma:

Theorem 3.1.1 (Farkas Lemma) *A polyhedron* $\{x \mid Ax \leq b\}$ *is non-empty if and only if there is no solution to* $\{y \mid y^T A = 0, y^T b < 0, y \geq 0\}$.

To prove this theorem, I'm actually going to give you an algorithmic proof. If the polyhedron is non-empty, it will give you an explicit solution. Otherwise, it will tell you how to develop a certificate y. The algorithm is an inefficient (non-polynomial time algorithm) algorithm to decide this. The algorithm is called the *Fourier-Motzkin algorithm*. It was reinvented by Motzkin. Fourier discovered this centuries ago, and it was completely forgotten. Moreover, later we will derive the transformation from H-representation to V-representation and back.

3.1.2 Fourier-Motzkin Elimination and Farkas Lemma

The input of this algorithm is a polyhedron $\{x \mid Ax \leq b\}$. The output is a yes or a no.

Let's deal with the case of just one variable. For example,

Example 3.1.2

$$7x_1 \leq 3$$
$$3x_1 \leq 2$$
$$2x_1 \leq 0$$
$$-4x_1 \leq 4$$

We know this one is possible, because we write this as

$$x_1 \leq \frac{3}{7}$$

$$x_1 \leq \frac{2}{3}$$

$$x_1 \leq 0$$
$$x_1 \geq -1$$

If P is described by a single variable x, P is feasible if

$$\max(b_i/a_i \mid b_i/a_i < 0 \leq \min(b_j/a_j \mid b_j/a_j > 0.$$

So, we know what to do in one variable. We should do an induction. If there is more than one variable, then we eliminate the leading variable x_1. We rewrite the inequalities to be regrouped into three kinds:

$$x_1 + (a_i')^T x' \leq b_i', \quad \textbf{(if coefficient of } a_{i1} \textbf{ is positive) (TYPE I)}$$

$$-x_1 + (a_j')^T x' \leq b_j', \quad \textbf{(if coefficient of } a_{j1} \textbf{ is negative) (TYPE II)}$$

$$(a_k')^T x' \leq b_k', \quad \textbf{(if coefficient of } a_{k1} \textbf{ is zero) (TYPE III)}$$

Here $x' = (x_2, x_3, \ldots, x_n)$.

For the type IIs, these are obtained by dividing by $|a_{j1}|$ on both sides. Here comes the hocus pocus. The $+1$ and -1 coefficients on x_1 on the types I and types II cancel out. Now, the computational trouble is that if you have 1 million of type I and 1 million of type II, there's 1 million squared of these. We just keep the equations of type III. Original system of inequalities has a solution if and only if the system $(*)$ is feasible

$$(*) \textbf{ is equivalent to } (a_j')^T x - b_j \leq b_i - (a_i')^T x', \textbf{ and } (a_k')^T x' \leq b_k'$$

If we find $x_2, x_3, \ldots, x_n$ satisfying $(*)$, find

$$max((a_j')^T x - b_j) \leq x_1 \leq min(b_i - (a_i')^T x').$$

The process ends when we have a single variable.

An equation of the first type

$$x_1 + (a_i')^T x' \le b_i'$$

is now

$$x_1 \le b_i' - (a_i')^T x'$$

and the second type

$$-x_1 + (a_j')^T x' \le b_j'$$

is now

$$-x_1 \ge (a_j')^T x' - b_j'$$

Then we can look at the solution for x_1.

This is essentially the algorithm. It's a beautiful method, but I wouldn't suggest it for my family to do. There is a polynomial time method, called the ellipsoid method. But it's not a *strongly* polynomial time method. Now, I wouldn't recommend my family to do this method either! It is a polynomial time feasibility/decision problem. However, it's not really practical quite yet. The state of the art is really the simplex method. It can be used (and is used by engineers) to see if a polyhedron is empty or not.

Now, is the simplex method polynomial time? Actually the method depends on your choice of pivot, given by a pivot rule. There are actually, then, many simplex methods! My academic grandfather Victor Klee[1] crafted an example with Minty that takes an exponential number of pivots. In some sense, here's the irony of life: in practice, people use the simplex method, though theoretically exponential.

So, let's try to prove the Farkas lemma. We kill one variable at each iteration until we are in a single variable system. The new system will have no variables:

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \le \begin{bmatrix} b_1' \\ b_2' \\ \vdots \\ b_n' \end{bmatrix}$$

The polyhedron $\{x \mid Ax \le b\}$ is infeasible if and only if $b_i' < 0$ for some i.

So the rewriting and addition steps correspond to row operations on the original matrix A.

$$r = MAx \ge Mb = b'$$

with matrix M with non-negative entries. We set $y^T = (e_i)^T M$, with e_i standard i-th unit vector then

$$0 = y^T A,\ y^T b < 0,\ \text{and } y \ge 0.$$

Here is another form of Farkas lemma:

[1] A great supporter of the MAA, he passed away recently

Corollary 3.1.3 (Farkas Lemma I) *$\{x : Ax = b, x \geq 0\} = \emptyset$ if and only if $\{y : y^T A \geq 0, y^T b < 0\} \neq \emptyset$.*

This is perhaps the most famous version quoted as the Farkas lemma. I'll explain it in a picture. Many people call this the separation theorem. I will think of the columns of the matrix A as "fingers" coming out of the origin. When I consider Ax (if x has to be non-negative), I'm talking about the polyhedral cone. Then, this question is: is the vector b in the cone? If the vector b is **not** in the cone, it's because you can find a hyperplane (with normal vector y) that puts the cone on one side of the hyperplane and the vector y on the other side. This is unbelievably important in mathematics, because separation theorems tell you when you are done in optimization.

Now, we can prove this version as a corollary of the previous version:
Proof: We first to a rewriting of each equality as two inequalities: $\{x : Ax = b, x \geq 0\} \neq \emptyset \iff \{x : Ax \leq b,\ -Ax \leq -b,\ -Ix \leq 0\} \neq \emptyset$. Why did I do this change of representation from equality to inequality? It's so that I could use the previous version of the Farkas lemma. By previous version of Farkas, this happens if and only if no solution of the form $y^T = [y_1\ y_2\ y_3]^T$ exists[2] of with

$$[y_1\ y_2\ y_3]^T \begin{bmatrix} A \\ -A \\ -I \end{bmatrix} = 0,\ [y_1\ y_2\ y_3]^T \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix} < 0,\ y^T \geq 0$$

The vector $y_1 - y_2$ has the desired property.

There is something really profound in Farkas'lemma and we will look at another nice new proof of it.

Theorem 3.1.2 *For a system of equations $Ax = b$, where A is a matrix and b is a vector. One and only one of the following choices holds:*

- *There is a non-negative vector x with $Ax = b$.*
- *There exists a non-trivial vector c such that $cA \geq 0$ but $c \cdot b < 0$.*

Proof: Clearly if the second option holds there cannot be positive solution for $Ax = b$ because it gives $0 \leq (cA)x = c(Ax) = cb < 0$.

Now suppose that $yb \geq 0$ for all y such that $yA \geq 0$. We want to prove that then b is an element of the cone K generated by the non-negative linear combinations of columns of A. For every b in $\mathbb{R}^n$ there exist in the cone $K = \{Ax | x \geq 0\}$ a point a that is closes to b and $Ax = a$ for $x \geq 0$. This observation is quite easy to prove and we leave it as an exercise (there are very easy arguments when the cone K is pointed). Now using this observation we have that

$$(A_j, b - a) \leq 0, \quad j = 1 \ldots k \tag{3.1}$$

[2]It's a bigger vector now, because of how we changed the matrix A, right?

and

$$(-a, b-a) \leq 0. \tag{3.2}$$

Why? the reason is a simple inequality on dot products. If we do not have the inequalities above we get for sufficiently small $t \in (0,1)$:

$$|b-(a+tA_j)|^2 = |(b-a)-tA_j|^2 =$$
$$|b-a|^2 - 2t(A_j, b-a) + t^2|A_j|^2 < |b-a|^2$$

or similarly we would get

$$|b-(a-ta)|^2 = |(b-a)+ta|^2 = |b-a|^2 - 2t(-a, b-a) + t^2|a|^2| < |b-a|^2$$

Both inequalities contradict the choice of b because $a + tA_j$ is in K and the same is true for $a - ta = (1-t)a \in K$. We have then that from the hypothesis and the equations in (ONE) that $(b, -(b-a)) \geq 0$, which is the same as $(b, b-a) \leq 0$ and this together with equation (TWO) $(-a, b-a) \leq 0$ gives $(b-a, b-a) = 0$, and in consequence $b = a$. □

The theorem above is equivalent to

Theorem 3.1.3 *For a system of inequalities $Ax \leq b$, where A is a matrix and b is a vector. One and only one of the following choices holds:*

- *There is a vector x with $Ax \leq b$.*
- *There exists a vector c such that $c \cdot b < 0$, $c \geq 0$, $\sum c_i > 0$, and $cA = 0$.*

The reason is simple, The system of inequalities $Ax \leq b$ has a solution if and only if for the matrix $A' = [I, A, -A]$ there is a non-negative solution to $A'x = b$. The rest is only a translation of the previous theorem in the second alternative. If you tried to solve the **strict** inequalities in the system $Bx < 0$, like the one we got for deciding convexity of pictures, you would run into troubles for most computer programs (e.g. MAPLE, MATHEMATICA,etc). One needs to observe that a system of strict inequalities $Bx < 0$ has a solution precisely when $Bx \leq -1$ has a solution. If the solution x gives $Bx < -1/q$ for instance px is a solution for the strict inequality and vice versa. Thus the above theorem implies the Farkas' lemma version we saw earlier.

There are millions of version of Farkas lemmas. Here is another one:

Corollary 3.1.4 (Farkas Lemma II) $\{x : Ax \leq b, x \geq 0\} \neq \emptyset \iff$ *When* $y^T A \geq 0$, *then* $y^T b \geq 0$

Pretty much you can prove that Farkas implies everything you do in life!

3.1.3 The Face Poset of a Polytope

Now is time to look carefully at the partially ordered set of faces of a polytope.

Proposition 3.1.4 *Let $P = conv(a_1, \dots, a_n)$. and $F \subset P$ a face. Then $F = conv(a_i, a_i \in F)$. Hence, every face of a polytope is a polytope.*

Proof: Let $f(x) = \alpha$ be the suporting hyperplane. That $Q = conv(a_i, a_i \in F)$ is contained in F is clear. For the converse take $x \in F - Q$. We can still write x as $\lambda_1 a_1 + \cdots + \lambda_n a_n$ with the lambdas as usual. Applying f we get that if $\lambda_j > 0$ for an index not in Q, then we get $f(x) < \alpha$ because $f(a_j) < \alpha$ thus $\lambda_1 f(a_1) + \cdots + \lambda_n f(a_n) < \lambda_1 \alpha + \dots \lambda_n \alpha = \alpha$. Thus we arrive to a contradiction

Corollary 3.1.5 *A Polytope has a finite number of faces, in particular a finite number of vertices and facets.*

We also have the following properties:

Theorem 3.1.5 *Let P be a d-polytope in $\mathbb{R}^n$.*

1. *Every point on the boundary of P lies in a facet of P. Thus the boundary of P is the union of its facets.*
2. *Each $(d-1)$-dimensional face of P, a ridge, lies in exactly two facets of P.*
3. *Each vertex of P lies in at least d-facets of P and at least d edges of P.*

For a polytope with vertex set $V = \{v_1, v_2, \dots, v_n\}$ the *graph of P* is the abstract graph with vertex set V and the set of edges $E = \{(v_i, v_j) : [v_i, v_j] \textit{is an edge of } P\}$. You can have a very entertaining day by drawing the graphs of polytopes. Later on we will prove a lot of cute properties about the graph of a polytope. Now there is a serious problem. We still don't have a formal verification that the graph of a polytope under our definition is non-empty! we must verify that there is always at least a vertex in a polytope. Such a seemingly obvious fact requires a proof. From looking at models of polyhedra one is certain that there is a containment relation among faces: a vertex of an edge that lies on the boundary of several facets, etc. Here is a first step to understand the

Lemma 3.1.6 *Let P be a d-polytope and $F \subset P$ be a face. Let $G \subset F$ be a face of F. Then G is a face of P as well.*

Write a proof!
Proof: Suppose $P = conv(a_1, \dots, a_n)$ and $f(x) = \alpha$ is a supporting linear functional to the face F. We saw $F = conv(a_i : i \in I_F)$, and by definition of being a face $f(a_i) = \alpha$ if $i \in I_F$ and $f(a_i) < \alpha$ otherwise. At the same time we have $g(x) = \beta$ such that for $G = conv(a_j : j \in I_G \subset I_F)$. We have again that $g(a_i) = \beta$ if $i \in I_G \subset I_F$ and $g(x) < \beta$ otherwise.

Construct the hyperplane $h = f + \epsilon g$. Note that for a_j with $j \in I_G$ we have $h(a_i) = \alpha + \epsilon\beta$. Now for a_j with $j \in I_F - I_G$ we have $h(a_i) < \alpha + \epsilon\beta$. For all the other indices we have, by choosing ϵ small enough, we have that $h(a_i) < \alpha + \epsilon\beta$ too.

That is, there is a transitivity of face containments! You can also see it from the earlier lemma. Then you can actually do a partial order of faces by inclusion. If you have a poset, you know you can represent it by its *Hasse diagram*. The Hasse diagram for the face lattice of the 3-dimensional simplex is the Boolean lattice B_4. The face poset is actually a graded lattice. This is related to Euler's formula. Let f_i represent the number of i-dimensional faces. So, Euler's formula is related to the number of elements in the poset. You have Möbius theory and so on.

What do the faces of polyhedra look like?

Theorem 3.1.7 *Let $Q = \{x : Ax \leq b\}$ a polyhedron. A non-empty subset F is a face of P if and only if F is the set of solutions of a system of inequalities and equalities obtained from the list $Ax \leq b$ by changing some of the inequalities to equalities.*

You just need to change some inequalities into equalities! Thus, similar to the story for polytopes, we have the same story for polyhedra:

Corollary 3.1.8 *The set of faces of a polyhedron forms also a poset by containment and it is finite.*

Corollary 3.1.6 *Every non-empty polytope has at least one vertex.*

Write a proof!

Theorem 3.1.7 *Every polytope is the convex hull of the set of its vertices.*

Write a proof!

3.1.4 Polar Polytopes and Duality of Face Posets

Now we know that a polytope has a canonical representation as the convex hull of its vertices. The results above establishes that the set of all faces of a polytope form a partially ordered set by the order given by containment. This poset receives the name of the *face poset* of a polytope. We say that two polytopes are *combinatorially equivalent* or *combinatorially isomorphic* if their face posets are the same. In particular, two polytopes P, Q are isomorphic if they have the same number of vertices and there is a one-to-one correspondence p_i to q_i between the vertices such that $conv(p_i : i \in I)$ is a face of P if and only if $conv(q_i : i \in I)$ is a face of Q. The bijection is called an isomorphism.

A property that can guess from looking at the Platonic solids is that there is a duality relation where two polytopes are matched to each other by paring the vertices of one with the facets of the other and vice versa. We want now to make this intuition precise. We will establish a bijection between the faces of P

and the faces of P^o. Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope containing the origin as its interior point. For a non-empty face F of P define

$$\hat{F} = \{x \in P^o :< x, y >= 1 for\ all\ y \in F\}$$

and for the empty face define $\hat{} = Q$.

Theorem 3.1.8 *The hat operation applied to faces of a d-polytope P satisfies*

1. *The set $\hat{F}$ is a face of P^o*
2. *$dim(F) + dim(\hat{F}) = d - 1$.*
3. *The hat operation is involutory: $(\hat{\hat{F}}) = F$.*
4. *If $F, G \subset P$ are faces and $F \subset G \subset P$, then $\hat{G}, \hat{F}$ are faces of P^o and $\hat{G} \subset \hat{F}$.*

Proof: To set up notation we take $P = conv(a_1, a_2, \ldots, a_m)$ and $F = conv(a_i : i \in I)$.
(1) Define $v := 1/|I| \sum_{i \in I} a_i$. We claim that in fact, $\hat{F} = \{x \in P^o :< x, v >= 1\}$. It is clear that $\hat{F} \subset \{x \in P^o :< x, v >= 1\}$ The reasons for the other containment are: we already know that $< x, a_i > \leq 1$ and $< x, v >= 1$ implies then that $< x, a_i >= 1$ for all $i \in I$. Since all other elements of F are convex linear combinations of a_i's we are done.

Now that the set $\hat{F}$ is a face of P^o is clear because the supporting hyperplane to the face is the linear functional $< x, v >= 1$. Warning! the $\hat{F}$ could be still empty face!!
(2) Now we convince ourselves that if F is a non-empty face, then $\hat{F}$ is non-empty and moreover the sum of their dimensions is equal to $d - 1$.

By definition of face $F = \{x :< x, c >= \alpha\}$ and for other points in P we have $< y, c >< \alpha$. Because the origin is in P we have that $\alpha > 0$, which means that $b = c/\alpha \in \hat{F}$ because 1) $< b, a_i >= 1$ for $i \in I$ and 2) $< b, a_i > \leq 1$ (this second observation is a reality check: b is in P^o). Hence $\hat{F}$ is not empty.

Suppose $dim(F) = k$ and let $h_1, \ldots, h_{d-k-1} \in \mathbb{R}^d$ be linear independent vectors orthogonal to the linear span of F. The orthogonality means that $< h_i, a_j >= 0$ for $j \in I$ and all h_i. We complete to a basis!

For all sufficiently small values $\epsilon_1, \ldots, \epsilon_{d-k-1}$ we have that $r := b + \epsilon_1 h_1 + \epsilon_2 h_2 + \cdots + \epsilon_{d-k-1} h_{d-k-1}$ satisfies $< r, a_i >= 1$ for $i \in I$ and $< r, a_j >< 1$ for other indices. Hence r is in $\hat{F}$ proving that $dim(\hat{F}) \geq d - 1 - dim(F)$.

On the other hand $\hat{F}$ is in the intersection of the hyperplanes $\{x \in \mathbb{R}^d :< x, a_i >= 1\}$ therefore $dim(\hat{F}) \leq d - 1 - dim(F)$. We are done.