Math 128C (De Loera) **Homework 3**

Due date: June 8th, 2018

INSTRUCTIONS

All homeworks will have many problems, both theoretical and practical. Programming exercises need to be submitted via CANVAS.

Theory part must be scanned or, even better, please use LateX! Be organized and use the notation appropriately. Show your work on every problem. Correct answers with no support work will not receive full credit.

- COMPUTER PROJECT 1:

A mathematical model that attempts to capture the Tacoma Narrows Bridge incident was proposed by McKenna and Tuama [2001]. The goal is to explain how torsional, or twisting, oscillations can be magnified by forcing that is strictly vertical.

Consider a roadway of width $2l$ hanging between two suspended cables, as in Figure 6.18(a). We will consider a two-dimensional slice of the bridge, ignoring the dimension of the bridge's length for this model, since we are only interested in the side-to-side motion. At rest, the roadway hangs at a certain equilibrium height due to gravity; let y denote the current distance the center of the roadway hangs below this equilibrium.

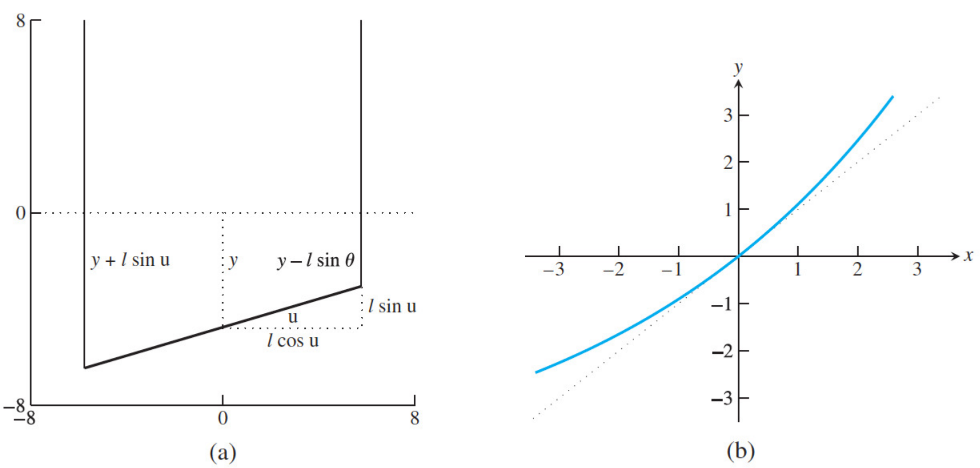

Figure 6.18 Schematics for the McKenna–Tuama model of the Tacoma Narrows Bridge. (a) Denote the distance from the roadway center of mass to its equilibrium position by y, and the angle of the roadway with the horizontal by θ. (b) Exponential Hooke's law curve $f(y) = (K/a)(e^{ay} - 1)$.

Hooke's law postulates a linear response, meaning that the restoring force the cables apply will be proportional to the deviation. Let θ be the angle the roadway makes with the horizontal. There are two suspension cables, stretched $y - l\sin\theta$ and $y + l\sin\theta$ from equilibrium, respectively. Assume that a viscous damping term is given that is proportional to the velocity. Using Newton's law $F = ma$ and denoting Hooke's constant by K, the equations of motion for y and θ are as follows:

$$
\begin{aligned}
y'' &= -dy' - \left[\frac{K}{m}(y - l\sin\theta) + \frac{K}{m}(y + l\sin\theta)\right] \\
\theta'' &= -d\theta' + \frac{3\cos\theta}{l}\left[\frac{K}{m}(y - l\sin\theta) - \frac{K}{m}(y + l\sin\theta)\right].
\end{aligned}
$$

However, Hooke's law is designed for springs, where the restoring force is more or less equal whether the springs are compressed or stretched. McKenna and Tuama hypothesize that cables pull back with more force when stretched than they push back when compressed. (Think of a string as an extreme example.) They replace the linear Hooke's law restoring force $f(y) = Ky$ with a nonlinear force $f(y) = (K/a)(e^{ay} - 1)$, as shown in Figure 6.18(b). Both functions have the same slope K at $y = 0$; but for the nonlinear force, a positive y (stretched cable) causes a stronger restoring force than the corresponding negative y (slackened cable). Making this replacement in the preceding equations yields

$$
\begin{aligned}
y'' &= -dy' - \frac{K}{ma}\left[e^{a(y-l\sin\theta)} - 1 + e^{a(y+l\sin\theta)} - 1\right] \\
\theta'' &= -d\theta' + \frac{3\cos\theta}{l}\frac{K}{ma}\left[e^{a(y-l\sin\theta)} - e^{a(y+l\sin\theta)}\right].
\end{aligned}
\tag{6.54}
$$

As the equations stand, the state $y = y' = \theta = \theta' = 0$ is an equilibrium. Now turn on the wind. Add the forcing term $0.2W\sin\omega t$ to the right-hand side of the y equation, where W is the wind speed in km/hr. This adds a strictly vertical oscillation to the bridge.

Useful estimates for the physical constants can be made. The mass of a one-foot length of roadway was about 2500 kg, and the spring constant K has been estimated at 1000 Newtons. The roadway was about 12 meters wide. For this simulation, the damping coefficient was set at $d = 0.01$, and the Hooke's nonlinearity coefficient $a = 0.2$. An observer counted 38 vertical oscillations of the bridge in one minute shortly before the collapse—set $\omega = 2\pi(38/60)$. These coefficients are only guesses, but they suffice to show ranges of motion that tend to match photographic evidence of the bridge's final oscillations. MATLAB code that runs the model (6.54) is as follows:

It is provided in the file *tacoma.m* available at `https://www.math.ucdavis.edu/~deloera/TEACHING/MATH160/tacoma.m` In the next problems you will do what engineers would need to do explore the behavior of the system.

This project is an example of how experimental mathematics can be used in real life. The equations are way too difficult to derive a closed-form solution (a la 22B), and even too difficult to prove qualitative result about them (a la 119A). BUT using reliable ODE solvers we can generate numerical trajectories for various parameter settings to illustrate the types of phenomena available to this model. ODE models can predict behavior and shed light in scientific and engineering problems.

Run `tacoma.m` with the default parameter values to see the phenomenon postulated earlier. If the angle θ of the roadway is set to any small nonzero value, vertical forcing causes θ to eventually grow to a macroscopic value, leading to significant torsion of the roadway. The interesting point is that there is no torsional forcing applied to the equation; the unstable "torsional mode" is excited completely by vertical forcing.

Run `tacoma.m` with wind speed $W = 80$ km/hr and initial conditions $y = y' = \theta' = 0$, $\theta = 0.001$. The bridge is stable in the torsional dimension if small disturbances in θ die out; unstable if they grow far beyond original size. Which occurs for this value of W?

Replace the Trapezoid Method by fourth-order Runge–Kutta to improve accuracy. Also, add new figure windows to plot $y(t)$ and $\theta(t)$.|

Find the minimum wind speed W for which a small disturbance $\theta(0) = 10^{-3}$ has a magnification factor of 100 or more. Can a consistent magnification factor be defined for this W?

Try some larger values of W. Do all extremely small initial angles eventually grow to catastrophic size?

What is the effect of increasing the damping coefficient? Double the current value and find the change in the critical wind speed W. Can you suggest possible changes in design that might have made the bridge less susceptible to torsion?

- This problem will force you to think of the principles of finite differences

Given the relation

$$y(x+h) = 2y(x) - y(x-h) + \frac{h^2}{12}[y''(x+h) + 10y''(x) + y''(x-h)] + O(h^6)$$

Show how it can be used for a finite difference method to obtain an approximate numerical solution of the following boundary-value problem

$$y'' = f(x)y + g(x), \quad x \in [a,b] \quad y(a) = \alpha, \quad y(b) = \beta.$$

- Consider the multistep method of the form

$$w_{n+1} = 4w_n - 3w_{n-1} - 2hf(t_{n-1}, w_{n-1}), \quad n \geq 1$$

Give an argument whether this method is stable or not? Can you say something about the region of absolute convergence?

- Consider the boundary value problems

$$y'' = xy' + y + 2cos(x), \quad y(0) = 1, \quad y(1) = 9$$

$$y'' = y^2 - x + yx, \quad y(0) = 1, \quad y(1) = 3$$

write down the initial value problem(s) that need(s) to be solved when using the shooting method. Do not solve them (it). Explain your reasoning.

- COMPUTER PROBLEM 2: This last problem explores the very last topic we saw in the course: *Non-Linear Partial Differential Equations*. We discussed Burger's equation and its solution in class. Now you apply this to specific instances.

  - Solve the Burger equation of the form

$$\begin{cases} u_t + uu_x = Du_{xx} \\ u(x,0) = f(x) \text{ for } x_l \leq x \leq x_r \\ u(x_l, t) = l(t) \text{ for all } t \geq 0 \\ u(x_r, t) = r(t) \text{ for all } t \geq 0, \end{cases}$$

    over the interval $[0, 1]$ with initial condition $f(x) = sin(2\pi x)$ and boundary conditions $l(t) = r(t) = 0$, step sizes (a) $h = k = 0.1$ and (b) $h = k = 0.02$. Plot the approximate solutions for $0 \leq x \leq 1$. Which equilibrium solution does the solution approach as time increases.

  - Verify that the function

$$u(x,t) = \frac{2D\beta\pi e^{-D\pi^2 t} \sin \pi x}{\alpha + \beta e^{-D\pi^2 t} \cos \pi x}.$$

    is indeed a solution of the Burger equation with homogeneous Dirichlet boundary conditions and initial conditions given by the equation

$$\begin{cases} u_t + uu_x = Du_{xx} \\ u(x,0) = \dfrac{2D\beta\pi \sin \pi x}{\alpha + \beta \cos \pi x} \text{ for } 0 \leq x \leq 1 \\ u(0,t) = 0 \text{ for all } t \geq 0 \\ u(1,t) = 0 \text{ for all } t \geq 0. \end{cases}$$

    Solve the problem numerically over the interval $[0, 1]$ with specific parameters $\alpha = 4, \beta = 3$, and $D = 0.2$. Plot the approximate solution using step sizes $h = 0.01$ and $k = 1/16$, and make a log-log plot of the approximation error at $x = 1/2, t = 1$, as a function of k for $k = 2^{-p}$, $p = 4, 5, 6, 7, 8$