Math 146 (De Loera) **Name:**
Mid-term exam **Student ID#**
Spring 2009

INSTRUCTIONS

(1) READ INSTRUCTIONS CAREFULLY!!!

(2) DO NOT TURN THIS PAGE UNTIL INSTRUCTED TO DO SO

(3) FILL IN THE INFORMATION ON THIS PAGE (your name, etc) NOW!!

(4) SHOW YOUR WORK on every problem. Correct answers with no support work will not receive full credit.

(5) PLEASE WRITE LEGIBLY. Be organized and use the notation

(6) NO EXTRA ASSISTANCE ALLOWED. Assistance from classmates. Your notes and book is OK.

#	Student's Score	Maximum possible Sco
0		**10**
1		**5**
2		**5**
3		**5**
4		**10**
5		**5**
Total points		**40**

Final Exam

This is the final exam for Math 146, Spring 2011. The exam has 40 points, and you have until noon Monday June 6 to complete this exam. You may use any notes or books, but no outside help from others!! Please give *as much justification as you can* for all of your solutions.

0. (10 points) The following are all problems about balls distributed to boxes. Note that the difference lies on what assumption we make about the boxes and balls (are they distinguishable or indistinguishable?). You will need to use several of the counting sequences we learned this quarter!

- How many ways are there to distribute n identical balls in k identical boxes? What is the answer if each box has to get at least one ball? What about if each box gets at most one ball?
- How many ways are there to distribute n distinguishable balls in k identical boxes? What is the answer if each box has to get at least one ball? What about if each box gets at most one ball?
- How many ways are there to distribute n identical balls in k distinguishable boxes? What is the answer if each box has to get at least one ball? What about if each box gets at most one ball?
- How many ways are there to distribute n distinguishable balls in k distinguishable boxes? What is the answer if each box has to get at least one ball? What about if each box gets at most one ball?
- A shop sell k kinds of postcards. You wish to send two different postcards to each of n friends (but different persons may get the same card). How many ways are there to do this?

1. (5 points) Let a_n $(n \geq 0)$ be a fixed sequence with ordinary generating function $A(x) = \sum_{n\geq 0} a_n x^n$ and let b_n be another sequence defined by

$$b_n = (2^n+1)a_0 + (2^{n-1}+1)a_1 + (2^{n-2}+1)a_2 + \cdots + (2^1+1)a_{n-1} + (2)a_n$$

for all $n \geq 0$. Find the ordinary generating function $B(x) = \sum_{n\geq 0} b_n x^n$ in terms of $A(x)$.

2. (5 points) Let a_n be the number of nonnegative integers strictly less than 10,000 whose digits sum to n. For example, $a_2 = 10$ corresponding to the numbers

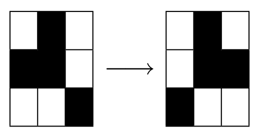
$$2000, 200, 20, 2, 1100, 1010, 1001, 110, 101, 11.$$

Find the ordinary generating function for $\{a_n\}_{n\geq 0}$.

(Hint: Every nonnegative integer less than 10,000 can be encoded as a 4-digit number if we allow leading zeros. Use this encoding together with one of the ogf rules to solve the problem.)

3. (5 points) Let $a_n = \sum_{i=1}^{n} i$ be the sum of the first n positive integers. For example, $a_0 = 0$, $a_1 = 1$, $a_2 = 1 + 2 = 3$ and $a_3 = 1+2+3 = 6$. Find a closed form expression for the ordinary generating function of a_n.

4. (10 points) We say that a *formal painting* is a coloring of the squares in a 3×3 grid using at most k colors. Two formal paintings are *essentially the same* if one can be obtained from the other by a rotation of the entire painting. For example, the two formal paintings with $k = 2$ colors below are essentially the same because they are related by a 90-degree rotation.



Find the number of essentially different formal paintings with at most k colors.

(a) Begin by writing down the automorphism group G of all rotations acting on the square 3×3 grid

1	2	3
8	9	4
7	6	5

(b) Next, compute the cycle index Z_G of G.

(c) Apply Pólya's theorem to get a formula for the number of essentially different formal paintings that use at most k colors.

5. (5 points) What is the probability that in a permutation of $\{1, 2, \ldots, n\}$ chosen at random, 1 and 2 belong to the same cycle?