Math 146
Homework 2

1. (A) Let G be a subgroup of S_n. Prove that either all of the permutations in G are even or exactly half of them are even.

   (B) An **inversion** in a permutation π in S_n is a pair (i, j) such that $i < j$ but $\pi(i) > \pi(j)$. Let $q(\pi)$ denote the number of inversions in π. Show that $sgn(\pi) = (-1)^{q(\pi)}$.

2. (A) The composition of permutations is not commutative in general. Find out how many permutations in S_8 commute with $(135)(24)(67)(8)$.

   (B) If π and γ are permutations prove that $\pi\gamma$ and $\gamma\pi$ have the same order.

3. (A) Find a list of all subgroups of the group of rigid motion symmetries of a square.

   (B) Let X be the set of corners of a regular cube and let G denote the subgroup of permutations of X which corresponds to rotations of the cube. How many G-orbits of elements in X are there? What is the cardinality of the stabilizer of a corner? What is the order of G?

   (C) What is the order of the group of rotational symmetries of a regular octahedron? Any relation to (B)?

4. (A) How many permutations belong to the group of automorphisms of the graph whose adjacency list is given by: (e, w), (f, w), (a, w), (a, b),(a, v), (v, w), (b, w),(b, v), (c, v),(d, v)?

   (B) Construct a graph with 6 vertices that has no automorphism other than the identity permutation. Can you do that with less vertices?

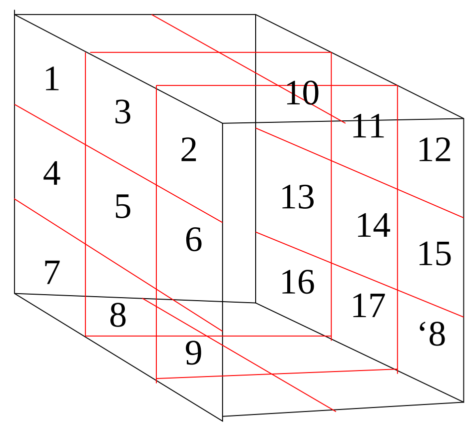
   (C) Let G be the subgroup of automorphisms of the graph $K_{3,3}$ and let v be any vertex of it. Calculate the order of G and the order of the stabilizer of v. Does it matter which vertex you choose?

5. (A) How many necklaces can be built using five white beads and three black beads?

   (B) You can paint a face of the cube red, white or blue. How many distinct cubes are there?

   (C) identity cards in planet $J(4)$ are manufactured using a square card with a 4×4 grid where two holes are punched. If all inhabitants of that planet carry a card, how many aliens live in that planet? The other planets in that solar system are $J(6), J(8), J(10)$. Where $J(n)$ uses an $n \times n$ grid with two holes are punched. What the populations of these planets? Can you come up with a formula in terms of n?

   (D) A circular disk in the state fair is divided into eight equal sectors, five of them are painted blue and three are painted red. How many different discs can be made this way?



6. The cheap Rubik box is a toy like the famous Rubik's cube (You can buy it at geektoysrus). Instead of a $3 \times 3 \times 3$ cube is a $3x3x2$ box. The moves are similar: Square faces rotate 90,180, or 270 degrees. The rectangular faces rotate only 180 degrees. You may want to use MAPLE to do calculations with groups and follow the information we saw in the computer lab.

   (A) What are the orbits for the cheap Rubik group on its action of the set of facets?

   (B) What is the order of the subgroup of moves for the cheap Rubik box generated by moves of orders two?

   (C) Decide whether the following configuration (see figure) is reachable from the initial situation of the cheap Rubik box by a finite set of moves:

7. Six distinguishable dice are thrown. Construct a generating function for number of ways of obtaining a given result when the first four dice show the same number and the last two dice show the same number. Find out the number of ways in which a total of 18 can be obtained in this manner.

8. (A) Write down the cycle index for A_5 and S_5.

   (B) Show that every odd permutation of the corners of a regular tetrahedron corresponds to either a reflection of the tetrahedron or the composite of a reflection and a rotation.

   (C) Write down the cycle index of the rotational symmetry group of the tetrahedron, regarded as a group of permutations of the edges.

9. Calculate the number of ways of coloring the faces of a regular dodecahedron in such a way that there are six red faces, four yellow faces, and two blue faces.

10. (A) Find the cycle index for the group of symmetries of a uniform straight rod divided along its length into m equal sections. Hence write down the

number of ways of coloring the sections with r colors available. (HINT: treat m odd and even separately).

(B) Five families arrange a party for their children. Each child will get a prize, either a balloon or a candy-bar, and children from the same family must get the same prize. Two families have four kids and the other two have two each. Find the number of ways in which six balloons and eight candybars can be used.