Math 146
Homework 4

1. Let q_n be the number of words of length n in the alphabet $\{a, b, c, d\}$ which contain an odd number of b's. Show that $q_{n+1} = 4^n + 2q_n$. Find the generating function of q_n and extract a formula for it.

2. Solve problems from Wilf's book Section 2.7:
   (a) 1.ab,
   (b) 2.ade,
   (c) 4.abcef,
   (d) 6, 7, 20,
   (e) 21.ab, 27, 32

3. Recall the permutations that are derangements are those without fixed points. Denote by D_n the number of derangements of S_n. Prove that $D_n = nD_{n-1} + (-1)^n$. Use this recurrence relation to prove that the exponential generating function for D_n is $\frac{e^{-x}}{1-x}$. HINT: Finding a purely bijective proof of the recurrence above is kind of hard. Instead you can either do a purely algebraic proof of the recurrence OR a hybrid proof as follows: Give a purely bijective proof of the fact that $D_n = (n-1)(D_{n-2} + D_{n-1})$, then use this new recurrence to do a simple derivation of the original recurrence relation.

4. For the following problems find the ordinary generating function of the sequence $\{a_n\}_{n\geq 0}$ (where $a_0 = 1$ for each problem).
   - a_n is the number of partitions of n with all parts ≤ 4.
   - a_n is the number of partions of n with largest part $= 4$.
   - a_n is the number of partions of n with no part appearing more than 2 times.
   - a_n is the number of partions of n with no part divisible by 3.
   - a_n is the number of partions of n in which each odd part appears at most twice and each even part appears an even number of times.