We study a wide range of techniques from ACGE to answer questions about objects that are COMBINATORIAL, DISCRETE...

Some examples:
1) How many distinct necklaces can be made from 3 black beads and 13 white beads?

2) How many ways can the faces of a cube be colored so that there are two red faces, one white face, three blue?

3) How many "symmetries" does the dodecahedron have?

4) How many magic squares, Latin squares of size \( n \times n \) are there?

5) 15 - Puzzle

So we look to COUNT, show EXISTENCE, and sometimes find OPTIMAL configuration among a (huge) finite set of possibilities.
PERMUTATIONS. Let \([n] = \{1, 2, \ldots, n\}\) then a permutation is a function that is one-to-one and surjective. The set of all permutations is \(S_n\). \[\sigma: [n] \rightarrow [n] \text{ can be represented as a table function } \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 2 & 3 & 4 & 6 \\ 3 & 6 & 5 & 2 & 1 & 4 \end{pmatrix} \text{ or as a list or rearrangement } [3, 6, 5, 2, 1, 4] = \sigma \]

Here position of \(i\) indicates \(\sigma(i) = i\)

\[\text{Ex: } \sigma(2) = 6.\]

Functions can be composed, then we define the "multiplication of permutations" \(\alpha, \beta \rightarrow \beta \circ \alpha = \beta(\alpha(i))\)

\[\text{Ex: } \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix} \]

\[\beta \circ \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \\ 5 & 1 & 1 & 3 & 1 \end{pmatrix} \]

Lemma: \(|S_n| = n!\), in fact \(S_n\) is a GROUP:

i) if \(\pi, \sigma \in S_n \Rightarrow \pi \circ \sigma \in S_n\)

ii) For any \(\pi, \sigma, \tau \in S_n \Rightarrow (\pi \circ \tau) \circ (\sigma \circ \tau) = \pi \circ \sigma \circ \tau\)

iii) The identity permutation is a permutation and \(\text{id} \circ \sigma = \sigma = \sigma \circ \text{id}\)

iv) Each \(\pi \in S_n\) has an \underline{inverse} \(\pi^{-1} \in S_n\) such that \(\pi \circ (\pi^{-1}) = \pi^{-1} \circ \pi = \text{id}\).

\(S_n\) is called the \underline{symmetric group}.
There is a third representation of permutations. Let us motivate with a puzzle.

**Example:** A card game: Twelve cards are laid out on a table

<table>
<thead>
<tr>
<th>1 2 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 5 6</td>
</tr>
<tr>
<td>7 8 9</td>
</tr>
<tr>
<td>10 11 12</td>
</tr>
</tbody>
</table>

Then they are picked by row and reordered and then redistributed in the same 3x4 array but by columns

<table>
<thead>
<tr>
<th>1 5 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 6 10</td>
</tr>
<tr>
<td>3 7 11</td>
</tr>
<tr>
<td>4 8 12</td>
</tr>
</tbody>
</table>

**Question:** How many times must this shuffle be repeated before the cards reappear in the original position?

**Answer:** Keep track of where cards go!

1, 12 remain fixed

2 → 5 → 6 → 10 → 4 → 2 \(\xrightarrow{\text{creates a cycle}}\) \(\xrightarrow{\text{that repeats itself after 5 times}}\)

Shuffle has permutation = \((1 2 3 4 5 6 7 8 9 10 11 12)\)

\((1 5 9 2 6 10 3 7 11 4 8 12)\)

Its cycle decomposes as

\((1) (2, 5, 6, 10, 4) (3, 9, 11, 8, 7) (12)\)

So repeating five times is enough to be back to original position.
Lemma: Every permutation decomposes as the product of cycles that have disjoint elements.

Proof: Every permutation has smallest power $K$ such that $\pi^K = \text{id}$ (by finiteness of $S_n$) to do the cycle decomposition start with a symbol (say 2) trace effect of $\pi$ on its successors until we reach symbol 2 again. These symbols form a cycle.
- Choose a new symbol not present in a cycle so far.
- Repeat procedure until symbols are all touched.

Clearly $\text{LCM} (\text{cycle lengths}) = \text{smallest power } K$ such that $\pi^K = \text{id}$.

This is the ORDER of $\pi$.

We want to count permutations of different kinds!!

We will do this during the whole course.

Notation: $\binom{n}{j} = \# \text{ of subsets of cardinality } j \text{ in an } n \text{ element set}$.

$$\binom{n}{j} = \frac{n!}{(n-j)! \cdot j!}$$

Why? (Easy to prove using induction and $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$)
PROBLEM (ANGRY SECRETARY): An secretary is angry with her boss. To show her anger she inserts n letters in envelopes with addresses and postage without paying attention possibly inserting the wrong letter in a envelope.

What is the probability that she put every letter in an incorrect envelope?

Definition: A permutation with the property \( P(i) \neq i \) for all \( i \in \{1, 2, \ldots, n\} \) is called a DERANGEMENT.

How many derangements are there in \( S_n \)?

Theorem (INCLUSION-EXCLUSION PRINCIPLE): For any sets \( B_1, B_2, \ldots, B_s \) finite sets (not necessarily disjoint),

\[
|B_1 \cup B_2 \cup \ldots \cup B_s| = \sum_{i=1}^{s} |B_i| - \sum_{i \neq j} |B_i \cap B_j| + \sum_{i \neq j \neq k} |B_i \cap B_j \cap B_k| - \cdots + (-1)^{s+1} |B_1 \cap B_2 \cap \ldots \cap B_s|
\]

\[\cdots\]

\[\cdots + (-1)^s \sum_{ \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, s\} } \left| B_{i_1} \cap \ldots \cap B_{i_k} \right| + \cdots + (-1)^s \left| B_1 \cap \ldots \cap B_s \right|\]

\[\text{proof: Take } x \in B_1 \cup \ldots \cup B_s.\]

If \( x \) belongs to \( B_{i_1}, \ldots, B_{i_r} \), how many times was \( x \) counted in \( \smiley \)?

\( x \) is counted ONCE in the size \( r \) intersection.

\( x \) is counted \( r \) times in size \( (r-1) \) intersections.

Thus \( x \) is counted \( \binom{r}{2} \) times in size \( j \) intersections.

Therefore \( x \) is counted a TOTAL OF...
\[(1) - (2) + (3) - ... + (-1)^n (n)\]

Contribution of \(x\) to right hand side equation.

Now using \(\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}\)

\[\star \text{ becomes} \]

\[\left\{ \binom{r-1}{0} + \binom{r-1}{1} \right\} - \left\{ \binom{r-1}{1} + \binom{r-1}{2} \right\} + \left\{ \binom{r-1}{2} + \binom{r-1}{3} \right\} \]

\[= \ldots \cdot (-1)^r \cdot \binom{r}{r} = \binom{r-1}{0} + \binom{r-1}{1} + (-1)^r \binom{r}{r} \]

\[= 1 + (-1)^{r-1} + (-1)^r = 1\]

\[\Rightarrow \text{ Each } x \in B_1 V \ldots UB_s \text{ is counted ONCE on the } \text{ RHS equation.}\]

Now we apply this to counting derangements

Let \(A_j = \{ \pi \in S_n \mid \pi(j) = j \} \) consider

\[\alpha_r = \sum_{i_1, \ldots, i_r} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}|\]

\(i_1, \ldots, i_r\) runs over all \(r\)-subsets

What is the value of \(\alpha_r\)? Easy!

- Choose set of "fixed" guys \(i_1, \ldots, i_r\) \((r)\) ways to do it.
- Remaining \((n-r)\) can be permuted in any way! \((n-r)!\)
The two decisions are done independently

\[ \alpha_r = \binom{n}{r} (n-r)! = \frac{n!}{r!} \]

Corollary: Using Inclusion-Exclusion Principle we know

\[ |A_1 \cup \ldots \cup A_n| = \alpha_1 - \alpha_2 + \alpha_3 - \ldots - (-1)^n \alpha_n \]

\[ = n! - n! \cdot \frac{1}{2!} + \frac{n!}{3!} - \frac{n!}{4!} + \ldots + \frac{n!}{n!} \]

\( \Rightarrow \) \# Derangements \( S = n! - \left( n! - \frac{n!}{2} + \frac{n!}{3} - \ldots + (-1)^n \right) \)

\[ = \frac{n!}{2} - \frac{n!}{3} + \frac{n!}{2} - \ldots + (-1)^{n+1} \]

Notice that that given a permutation \( \pi \), its cycle decomposition partitions the set \( \{1, 2, \ldots, n\} \).

Definition: A partition of a set \( X \) is a family of non-empty subsets \( \{Y_i : i \in I\} \) such that

1. \( X = \bigcup_{i \in I} Y_i \)
2. \( Y_i \cap Y_j = \emptyset \)

Definition: The type of a permutation \( \pi \) is the expression \( [x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}] \) where \( x_i \) is the number of cycles of length \( i \) in \( \pi \).

Example: \( \pi = (1, 2, 3) (7, 8) (4)(5)(6) \)

has type \( 1^3 2^1 3^1 \)

How many permutations are there of type \( [x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}] \) ??
Theorem: \# of \( S_n \) permutations of type \([a_1^x, a_2^z, \ldots, a_n^x] \) is \( \frac{n!}{1^{a_1} 2^{a_2} \cdots n^{a_n}} \).

**Proof:**

\[ S_n \xrightarrow{\psi} \text{set of all permutations of type } [1^{a_1} 2^{a_2} \cdots n^{a_n}] \]

\( \pi(1 \pi(2)) \cdots \pi(n) \xrightarrow{\psi} (\pi(1) \pi(2) \cdots \pi(n)) \cdots (\pi(n) \pi(1) \cdots \pi(n-1)) \)

We put the symbols into brackets.

Clearly this map is surjective.

Now this map is not injective! How many \( \pi \in S_n \) has to same cycle decomposition?

1) (**) inside any i-cycle we can shift on make any of the i symbols to be first (i ways to choose who is first)

2) The order in which we put the j-cycles is irrelevant for the purpose of map \( \psi \) (j! ways to order them)

Cardinality of \( \psi^{-1}(\pi) \)

\( \pi \) permutation of type \([1^{a_1} 2^{a_2} \cdots n^{a_n}] \)
Definition: We say two permutations $\alpha, \beta$ are conjugate if there exist another permutation $\sigma$ such that $\sigma \alpha \sigma^{-1} = \beta$ or equivalently $\sigma \alpha = \beta \sigma$.

Question: Can two conjugate permutations have different type? 

NO!

Theorem: Two permutations have the same type if and only if they are conjugate.

Proof: for each $k$-cycle of $\alpha$ we can use $\sigma \alpha = \beta \sigma$ to recover a $k$-cycle of $\beta$:

\[ x_1 \xrightarrow{\alpha} x_2 \xrightarrow{\alpha} x_3 \xrightarrow{} \cdots \xrightarrow{} x_r \]

\[ \sigma \]

\[ y_1 \xrightarrow{\sigma} y_2 \xrightarrow{\sigma} y_3 \xrightarrow{} \cdots \]

$\Rightarrow$ if $\sigma \alpha = \beta \sigma \Rightarrow$ they have the same type.

Now conversely suppose $\alpha, \beta$ have the same type. How to find $\sigma$?

Example: \( (1 3 6 2 4) \) \( (5 8 7) \) (9) = $\alpha$

\( (1 5 8 6 2) \) \( (3 9 4) \) (7) = $\beta$

WHAT is the natural thing to do?
set-up bijection by aligning cycles of similar sizes.

\[ 13624, 5879 \]
\[ 15862, 3947 \]

We get (in example) a permutation
\[ \sigma = (1) (268974) (35) \] which does the job!

**NOTE:** \( \sigma \) is not unique.

But this is the general way of constructing it.

We saw permutations are always products of cycles.

**Definition:** A permutation that interchanges two symbols only and leaves the rest unaltered is a **transposition** (it has type \( [1^{n-2} 2] \)).

**Lemma:** Every cycle \( (x_1, x_2, \ldots, x_{r-1}, x_r) \) can be written as a product of transpositions
\( (x_1 x_r) (x_1 x_{r-1}) (x_1 x_{r-2}) \ldots (x_1 x_3) (x_1 x_2) \).

**proof:** Checking this by element

The only difficult one is \( x_r \rightarrow x_1 \), but this can be seen as consecutive exchanges
\[ x_1 x_2 x_3 \ldots x_{r-1} x_r \]
\[ x_2 x_1 x_3 \ldots \]
\[ x_2 x_3 x_1 \ldots x_{r-1} x_r \]
\[ x_2 x_3 x_4 \ldots x_1 x_r \]
Corollary: Every permutation can be written as the product of transpositions.

**WARNING:** Decomposition is not always unique!


**Definition:** For a permutation of type \([1^{a_1} 2^{a_2} \ldots n^{a_n}]\)

\[l(e + C(T)) = a_1 + a_2 + \ldots + a_n = \text{# of disjoint cycles}\]

**Question:** What is the relation between \(C(T \Pi)\) and \(C(\Pi)\) for \(T\) a transposition?

**Answer:** It changes the \# of cycles by one (it either adds a cycle or removes a cycle)

**Proof:** Suppose \(T(a) = b, T(b) = a\)

**Case 1:** Both \(a, b\) are in the same cycle of \(\Pi\)

\[
(x_1 x_2 \ldots a \ldots b \ldots x_n)
\]

\[
\begin{align*}
T \Pi(x_i) &= \Pi(x_i) \text{ for } x_i \neq y, z \\vspace{0.1cm} \\
T \Pi(y) &= T(a) = b, \quad T \Pi(z) = T(b) = a
\end{align*}
\]
Case 2: suppose \( a, b \) are indiffrent cyclus \\
\[ Theorem: \text{Every two decompositions of a permutation into transpositions have the same parity, namely if} \]
\[ \Pi = T_r T_{r-1} \ldots T_z T_i = \sigma_s \sigma_{s-1} \ldots \sigma_2 \sigma_1 \]
\[ \text{where} \ T_i, \ \sigma_j \ \text{are permutations} \]
\[ \Rightarrow r, s \ \text{are both even or both odd.} \]