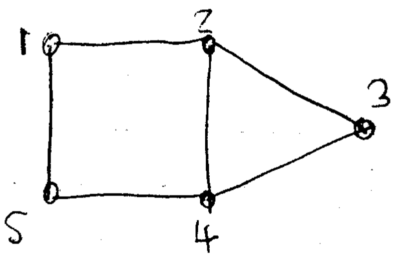
Question, which of the following subsets of S_5 are actually groups of permutations?

(i) $\{(1,2,3,4,5),\ (1,24)(35)\}$

(ii) $\{id,\ (12)(34),\ (13)(24),\ (14)(23)\}$

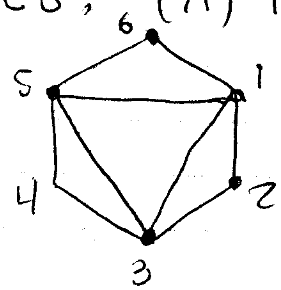
(iii) $\{id,\ (1,2,3,4,5),\ (1,3,5,2,4),\ (1,4253),\ (15432)\}$

Recall, an AUTOMORPHISM of a graph = is a permutation of labels which transforms edges into edges

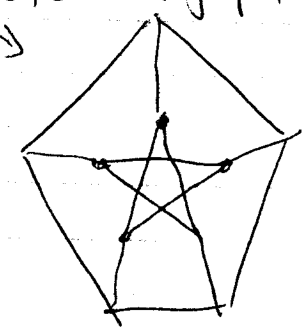


(15)(24) is an automorphism but not (12345)

Challenges; (A) Find the automorphism group of the graph



Petersen graph



(B) Show that the group of automorphis of has at least 120 elements

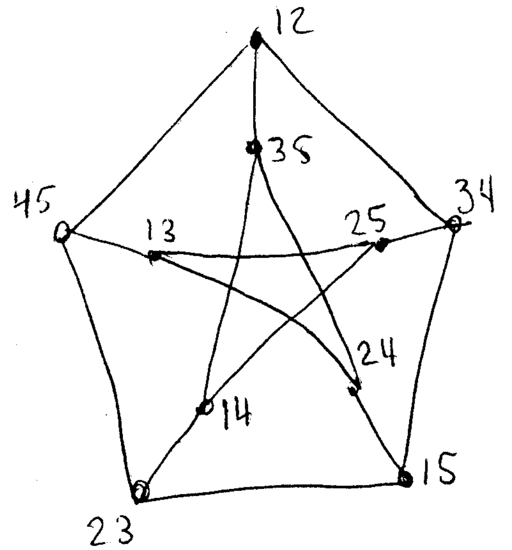
Solution: (A) $\{1,3,5\}$ has 4 edges each
$\{2,6,4\}$ have 2 edges each. Map into themselves

Now each permutation of $\{1,3,5\}$ forces a position for the other 3 vertices because of adjacencies.

$\Rightarrow$ id, (135)(246), (13)(46), (35)(26)
(153)(264), (15)(24)

(B) This is more difficult

We label each vertex with a pair of numbers out of $\{1, 2, \dots 5\}$ in such a way two vertices are adjacent iff their labels are disjoint



Observation: Each permutation of $\{1,2,3,4,5\}$ provides an automorphism of the labeling

$\Rightarrow$ S_5 is a subgroup of Automorphism(Petersen)

WE need more tools to prove these are <u>all</u>

Definition; Let G let be a group of permutations for a set X of labels (typically $1, 2, \dots n$ are used as labels but we saw others work nicely too) we define an equivalence relation: $x \sim y \Leftrightarrow g(x) = y$ for some permutation $g \in G$

$\Rightarrow$ We get a partition of X (the labels)

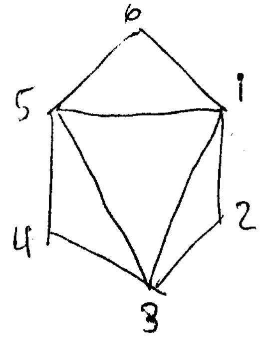
Each part is called an ORBIT

The ORBIT of $x \in X$

$$G_x = \{ g(x) \mid g \in G \}$$

Example: $\longrightarrow$

Example



There are two ORBITS $\{1,3,5\}$ and $\{2,4,6\}$

QUESTIONS How to find size of each orbit?
How many different orbits are there?

Here is how to find size of an orbit:

Definition: Let G be a group of permutations of a set X of labels, let

$$G(x\to y) = \{ g\in G \mid g(x)=y \}$$

$$G(x\to x) = \{ g\in G \mid g(x)=x \} = \text{STABILIZER of } x.$$

Lemma: $G(x\to x)$ is a subgroup of G

proof Exercise

Lemma: Let G be a group of permutations of X and suppose $h\in G(x\to y)$ then

$$G(x\to y) = h\,G(x\to x)$$, i.e., $G(x\to y)$ is a COSET for $G(x\to x)$

proof: We show two containments

$$\alpha\in hG(x\to x) \Rightarrow \alpha = h\beta \text{ with } \beta(x)=x$$

$$\Rightarrow \alpha(x) = h\beta(x) = h(x) = y \Rightarrow \alpha\in G(x\to y)$$

$$\Rightarrow hG(x\to x) \subseteq G(x\to y)$$

Now if $\gamma \in G(x \to y)$ then

$$h^{-1}\gamma(x) = h^{-1}(y) = x \Rightarrow h^{-1}\gamma \in G(x \to x)$$

$$\Rightarrow \gamma = h\delta \text{ for some } \delta \in G(x \to x)$$

$$\Rightarrow hG(x \to x) = G(x \to y)$$

<u>MORAL OF STORY:</u>

$|G(x \to y)| = |G(x \to x)|$ when $y \in \text{orbit}(x)$

(otherwise $G(x \to y) = \phi$)

Why do we care?

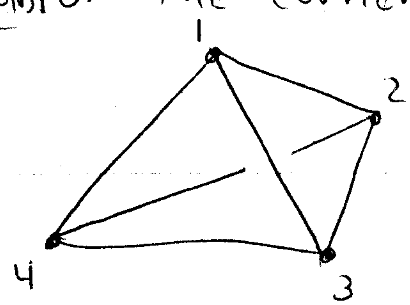
<u>Theorem:</u> G is a group of permutations of X

For any $x \in X$ $\quad |G(x \to x)| \cdot |G_x| = |G|$

We can use this to find out how big a group is.

<u>Example 1</u> What is the group of permutations for the corners of a regular tetrahedron?

<u>ROTATIONS</u>



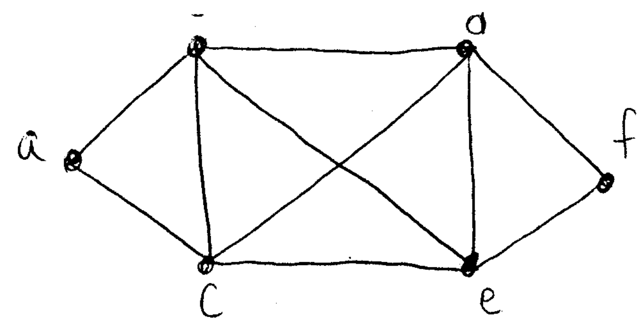
$G_1 = \{1, 2, 3, 4\}$

Now what is $G(1 \to 1)$?

3 rotations $\Rightarrow$

$|G| = 4 \cdot 3 = 12$

Now another example:



$G_a = \{a, f\}$

$G_b = \{b, c, d, e\}$

$|G(a \to a)| = 4$

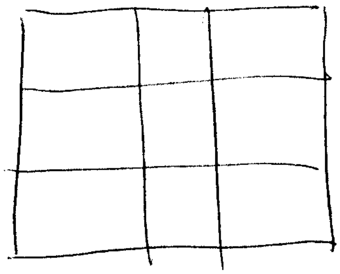
$\Rightarrow \quad |G| = 2 \cdot 4 = 8.$

Next time we will learn to use MAPLE to answers questions about graps of PERMUTATIONS!!!

See hand-out.

PROBLEM:

Suppose you must manufacture ID cards from plastic cards that look like



Punch 2 holes

How many ways are there to do this? $\binom{9}{2} = 36$

BUT, by symmetry of figure there are repetitions

The group of symmetries of the Square permutes on the set of all configurations

How many Orbits do I have? = # of different IDs

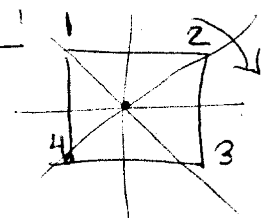
Theorem: The number of **orbits** of G acting on X is

$$\frac{1}{\text{order}(G)} \sum_{g \in G} |F(g)| \quad \left(\text{BURNSIDE'S LEMMA}\right)$$

$$F(g) = \{ x \in X \mid g(x) = x \}$$

HINT: USE TO COUNT PERMUTATIONS in S_8 that commute with $(12)(34)$

SOLUTION OF ID card problem:



for each

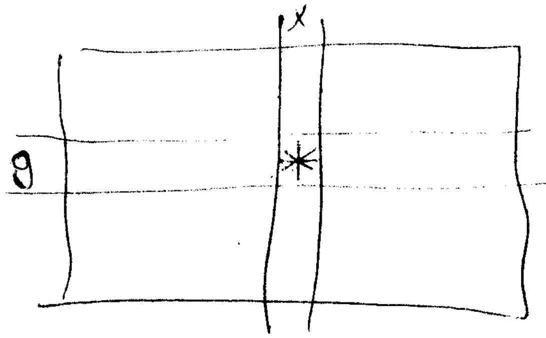
Square has 8 symmetries of them we need to find $F(g)$

g	$\|F(g)\|$
id	36
clockwise 90° rotation	0
clockwise 180° rotation	4
clockwise 270° rotation	0
Reflection in diagonal 13	6
Reflection over 24	6

Reflection over Bisector of 12	6
Reflection over Bisector of 14	6

Proof of Theorem: Let us consider the pairs

$$E = \{(g, x) \mid g \in G, x \in X \text{ and } g(x) = x\}$$



make a table an mark positions in E by an asterisk

Obs1: Sum of asterisks over row g
$= |\{x \mid g(x) = x\}|$

Obs2: Column sum of asterisks $= |\{g \mid g(x) = x\}|$
$= |G(x \to x)|$

$\Rightarrow$ Adding all rows

$$\sum_{g \in G} |F(g)| = \sum_{x \in X} |G(x \to x)| \qquad ☺$$

But we know $|G_x| \cdot |G(x \to x)| = |G|$

$\Rightarrow |G(x \to x)| = \dfrac{|G|}{|G_x|}$ thus ☺ becomes

$$\sum_{g \in G} |F(g)| = \sum_{x \in X} \frac{|G|}{|G_x|} = |G| \left(\sum_{O \text{ orbits}} \sum_{x \in O} \frac{1}{|G_x|} \right)$$

$$= \sum_{O \text{ orbits}} \frac{|G_x|}{|G_x|} = \# \text{ of orbits.}$$

Cycle Index and Polya's theory

Burnside's lemma is very useful as it is but there is a more efficient way to use it.

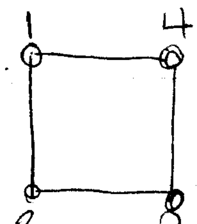
Suppose we are making necklaces with 4 blue or green beads

Note: Beads can be rotated and the necklace can be flipped over and worn.

SO HOW MANY DISTINCT NECKLACES CAN WE BUILD?

Note: 16 ways to choose colors but many are equivalent.

Suppose we use



this model of the necklace

then the symmetries are the symmetries of a square

$(1\,2\,3\,4) = \pi_1$
$(13)(24) = \pi_2$
$(1\,4\,3\,2) = \pi_3$
$(12)(34) = \pi_4$
$(14)(23) = \pi_5$
$(24)(1)(3) = \pi_6$
$(13)(2)(4) = \pi_7$
$(1)(2)(3)(4) = id$

Instead of Brute force counting which necklaces are fixed by π_7

let us Think!

Clearly (1, 3) must have same color in those positions to make it fixed

2 can be of either color

4 " " " " "

MORAL: If there are k cycles in the representation of a permutation π and a colors are available

$$\Rightarrow \left| \{ x \in X \mid \pi(x) = x \} \right| = a^k$$

$\Rightarrow$ We can simplify Burnside's formula

$$\frac{1}{|G|} \sum_{g \in G} \left| \{ x \in X \mid g(x) = x \} \right| = \frac{1}{|G|} \sum_{g \in G} a^{k_g}$$

where k_g = # of disjoint cycles of g as a permutation of the "positions" in the object.

Example: For the Necklace example

$$\frac{1}{|G|} \sum_{g \in G} \left| \{ x \in X \mid g(x) = x \} \right| = \frac{1}{8} \sum_{\substack{\text{square} \\ \text{symmetries}}} 2^{\#\text{ cycles}}$$

$$= \frac{1}{8}\left(2^1 + 2^2 + 2^1 + 2^2 + 2^2 + 2^3 + 2^3 + 2^4\right) = \frac{48}{8} = 6$$

But if we have 5 colors instead of 2

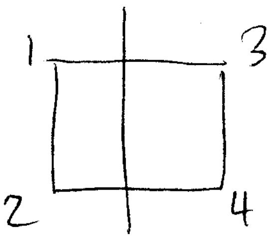
$$\frac{1}{8}\left(5^1 + 5^2 + 5^1 + 5^2 + 5^2 + 5^3 + 5^3 + 5^4\right) = \frac{960}{8} = 120$$

Now we can use a polynomial as a Book keeping device

<u>Definition</u> Given a permutation $g \in G \subseteq S_n$ of type $[1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}]$ let

$$\Psi_g(x_1, \ldots, x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

<u>Example:</u>



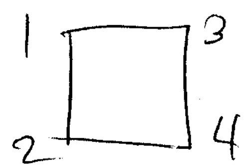
$g = (13)(24)$

$\Psi_g(x_1, x_2, x_3, x_4) = x_2^2$

<u>Definition:</u> The <u>cycle index</u> of a permutation group $G \subseteq S_n$ is the multivariate polynomial

$$\Psi_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{g \in G} \Psi_g(x_1, \ldots, x_n)$$

<u>Example:</u> For square



$$\Psi_G = \frac{1}{8}\left(x_1^4 + 2x_1^2 x_2 + 3x_2^2 + 2x_4\right)$$

<u>NOTE:</u> The cycle index can be written as

$$\Psi_G(x_1, \ldots x_n) = \frac{1}{|G|} \sum C_G(\alpha_1, \alpha_2, \ldots, \alpha_n)\, x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

where $C_G(\alpha_1, \ldots \alpha_n) = \#$ of permutations on G of type $[1^{\alpha_1} 2^{\alpha_2} \ldots n^{\alpha_n}]$

Sum over all types.

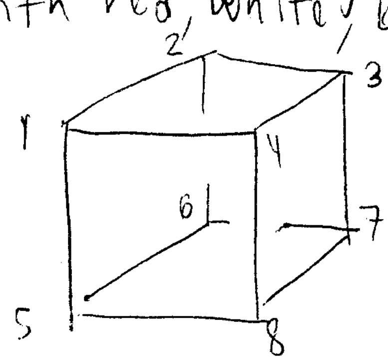
The great mathematician G. Polya created this theory for Chemists!

Final Example: The group of rotational symmetries of a cube has 24 elements (homework)
They are determined by the axis of rotation

- (1) Axis joining midpoints of opposite faces we have order 4 and order 2 rotations (corresponding to 90° and 180° angles)
  Example (1234)(5678)
- (2) Axis is line joining midpoints of opposite sides (order 2)
  example (15)(28)(37)(46)
- (3) Axis is line joining opposite corners (order 3)
  Example (245)(386)
- (4) identity

$$\text{Cycle index} = \frac{1}{24}\left(x_1^8 + 8x_1^2x_3^2 + 9x_2^4 + 6x_4^2\right)$$
Cube
Faces

How many ways can one color the VERTICES of a cube with red, white, blue?



EASY apply formula with $x_i = 3$