Math and Computers, Math 165

1. Applying the multivariate division algorithm we learned in class compute the remainded of $f = xy^2z^2 + xy - yz$ with respect to the polynomials $\{x - y^2, y - z^3, z^2 - 1\}$ given in that order. Repeat the exercise with the order of the set permuted cyclically. Check your hand-made answer with MAPLE.

2. Applying the multivariate division algorithm by hand, and using graded lexicographic order compute

   (a) the remainder of $f = xy^2z^2 + xy - yz$ of division by the polynomials $\{f_1 = x - y^2, f_2 = y - z^3, f_3 = z^2 - 1\}$ given in the order (f_1, f_2, f_3)

   (b) Repeat part (a) with the order of the dividing polynomials now (f_2, f_3, f_1).

   (c) the remainder of $f = x^3 - x^2y - x^2z$ by the polynomials $f_1 = x^2y - z$ and $f_2 = xy - 1$. First find the remainder of division by (f_1, f_2), then find the remainder of division by (f_2, f_1). What do you notice?

3. Let $I = \langle x^6, x^2y^3, xy^7 \rangle$. In the plane, plot the set of all exponent vectors (m, n) of monomials $x^m y^n$ appearing in I. If we apply the division algorithm to a polynomial f using the generators of I, what terms can appear in the remainder? Show it in a nice picture!

4. A basis for a monomial ideal $\{x^{a_1}, x^{a_2}, \dots, x^{a_s}\}$ is said to be minimal if no x^{a_i} in the basis divides another x^{a_j}. Prove that a monomial ideal has a unique minimal basis

5. Show that for each n there exist a monomial ideal $I \subset K[x, y]$ such that every basis of I has at least n elements.

6. **Programming Project is DUE MAY 9 2011**

   (a) The following project investigates the geometry of the complex roots of a polynomial $f(x)$ and of its derivative $f'(x)$. You will need to learn to use the geometric and graphical capabilities of MAPLE. You will need to write code to perform the following steps (you will only need to hand in the last code):

   (b) Write MAPLE code that plots the complex roots of a univariate polynomial $f(x)$. The roots must be labeled differently (by symbol or by colors) to distinguish them from other points.

   (c) Write a MAPLE subroutine that for an arbitrary, not identically constant polynomial $f(x)$, computes and draws the smallest convex polygon containing the zeros of the original polynomial. Such polygon is called the convex hull. We suggest that to find this polygon you have convert complex numbers (the roots of $f(x)$) into vectors, then use the command simplex[convexhull] to find the desired polygon. Let us call the resulting polygon the **Root Polygon** of $f(x)$. To use graphics you may want to run this as a MAPLE worksheet.

(d) Next consider many polynomials of degree at least 5, whose roots are not all real. Draw in the same picture the root polygons of $f(x)$ and $f'(x)$, $f''(x)$, $f'''(x)$, etc. Do this for at least 20 different polynomials. Observe what happens. What do you notice?

(e) Write MAPLE code to automatically generate the pictures above from any given polynomial. Use colors to make the picture clear. You will have to hand in the MAPLE code that automatically generated your pictures AND to write a conjecture based on your empirical observation.