Math and Computers, Math 165
Homework four

1. Compute the S-pairs of the following polynomials using the lexicographic monomial order.

   - $f = 4x^2z - 7y^2$, $g = xyz^2 + 3xz^4$.
   - $f = x^4y - z^2$, $g = 3xz^2 - y$
   - $f = xy + z^3$, $g = z^2 - 3z$.
   - If we use graded lexicographic order with $x > y > z$ is the set of polynomials $x^4y^2 - z^5, x^3y^3 - 1, x^2y^4 - 2z$ a Gröbner basis for the ideal generated by these polynomials? Why or why not?
   - Show that if we believe that every ascending chain of ideals in $C[x_1, \ldots, x_n]$ stabilizes, then the conclusion of Hilbert's basis theorem is a consequence (Hint: try a proof by contradiction).
   - Show that if G is a basis of the ideal I with the property that the remainder of f modulo G is zero for all $f \in I$, then G is a Gröbner basis for I.
   - Does the s-pair $S(f, g)$ depend on which monomial order is used? Illustrate your answer with an example.
   - Show that $\{y - x^2, z - x^3\}$ is not a Gröbner basis for lexicographic order with $x > y > z$.
   - Is $G = \{x^4y^2 - z^5, x^3y^3 - 1, x^2y^4 - 2z\}$ a Gröbner basis for the ideal it generates in gradedlex order?
   - Use Buchberger's algorithm to compute a Gröbner basis for the following ideal I in lex order: $I =< x^2 + y, x^4 + 2x^2y + y^2 + 3 >$
   - Compute a Gröbner basis for the ideal $I =< x^2 + y - 1, xy - x >\subset Q[x, y]$ with respect to lexicographic order using Buchberger's algorithm. If your basis is not minimal compute a minimal one.
     Which of the following polynomials belongs to the ideal I? $f_1 = x^2 + y^2 - y$, $f_2 = 3xy^2 - 4xy + x + 1$.
   - Now that you know Gröbner bases, you can solve difficult problems like this arising in calculus (you could not do this in 21D!!): Find the critical points of the functions

     $$f(x, y) = (x^2 + y^2 - 4)(x^2 + y^2 - 1) + (x - 3/2)^2 + (y - 3/2)^2.$$

     $$f(x, y, z, w) = x^3 + y^3 + z^3 + w^3.$$

   - For the following systems of polynomial equations answer the following questions: Is the system solvable? Does it have a finite number of solutions? If so count the solutions, if not determine the dimension of the solution space.
     a) $x^2 - 2x + 5, xy^2 + yz^3, 3y^2 - 8z^3$.
     b) $x^2z^2 + x^3, xz^4 + 2x^2z^2 + x^3, y^2z - 2yz^2 + z^3$.

- Find the common zeros of the polynomials $xyz - w, yzw - x, zwx - y, xyw - z$.
- Find the maximum of $f = x^2 + y^2 + xy$ subject to $x^2 + 2y^2 = 1$.
- Solve the following system of equations in the variables x, y, z over the reals (here a is a real constant). $F := \{z^2 - x^2 - y^2 + 2ax + 2az - a^2 = 0, yz - ay - ax + a^2 = 0, -2a + x + y = 0\}$.