Vector Spaces and Subspaces (Chapter 3)
A vector space $V$ is a set with two operations addition and scalar multiplication. The scalars are members of a field $K$, in which case is called a vector space over $K$. The following conditions must hold for all elements and any scalars:

1. $V$ must be closed under both operations (*no escaping to a different space*).
2. Commutativity of vector addition: $v + w = w + v$
3. Associativity of vector addition: $(v + w) + z = v + (w + z)$
4. Existence of additive identity $v + 0 = v$
5. Existence of additive inverse: $v + (-v) = 0$
6. Associativity of scalar multiplication: $\lambda(\mu v) = (\lambda \mu)v$, where $\lambda, \mu$ are numbers, $v$ vector.
7. Distributivity of scalar sums $(\lambda + \mu)v = \lambda v + \mu v$
8. Distributivity of vector sums $\lambda(v + w) = \lambda v + \lambda w$

**Key Example** $\mathbb{R}^d$: every element is an ordered list of $d$ real numbers, scalars are real numbers, addition is component-wise, and scalar multiplication is multiplication on each component separately. YOU KNOW THIS ONE!!!
Examples of vector spaces
1. The space $M_{n,m}(\mathbb{R})$ of $n \times m$ matrices with real coefficients.
2. Polynomials with real coefficients of degree less than 4.
3. Polynomials with real coefficients of any degree.
4. Space of all continuous functions defined on the interval $[0, 1]$.

Definition A subspace of a vector space is a non-empty subset that satisfies all the requirements above, linear combinations stay in the subspace.

Examples of subspaces:
1. The set of 2 matrices is a vector space. The subset of matrices with $a_{21} = 0$ and $a_{13} = 0$ is a subspace!
2. Example: Polynomials are a subspace of the continuous functions defined on the $[0, 1]$ interval.
3. Example: Let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbb{R}^n$. The set of ALL linear combinations $a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$ is a vector subspace of $\mathbb{R}^n$. Subspace generated or spanned by $v_1, \ldots, v_k$. = $\text{span}(v_1, v_2, \ldots, v_k)$
4. Example: If $v_1, v_2, \ldots, v_k$ are the COLUMNS of an $m \times k$ matrix $A$, their span is the COLUMN SPACE, a subspace of $\mathbb{R}^n$

Important: $Ax = b$ has a solution if $b$ belongs to the column space of $A$. 
Lemma Let $V$ be a vector space, $W$ a non-empty subset of $V$, then $W$ is a subspace if and only if two conditions hold:

1. $u, v \in W$ implies that $u + v$ is in $W$ too.
2. $u \in W$ implies that $cu \in W$ too.

QUESTIONS Which of the following are subspaces (operations used are normal operations in those spaces)?

1. Set of all solutions to the equation $z = x - y$,
2. Set of all solutions to $z = xy$.
3. Set of all vectors in $R^n$ whose first component is zero.
4. Set of differentiable functions $f(x) : R \rightarrow R$ defined in the interval $[0, 1]$
5. All vectors perpendicular to $u = (1, 2, 1)$
6. All symmetric $3 \times 3$ matrices
7. For an $m \times n$ matrix $A$, the **nullspace** of $A$.
8. The set of all vectors in $R^2$, $(x, y)$ with $x \geq 0$.
9. The set $W_1 \cap W_2$ with both $W_1, W_2$ known to be subspaces.
10. The set of singular $3 \times 3$ matrices.
Say we are given an $m \times n$ matrix $A$

- **Column Space:** Also known as the range of $A$, denoted as $C(A)$; this is subspace of $\mathbb{R}^m$.

- **Row Space:** Contains all combinations of the rows of $A$; same as the Column Space of $A^T$. It is a subspace of $\mathbb{R}^n$.

- **Nullspace** $N(A)$, vectors elements that (right) multiply to zero with $A$

- **Left Nullspace:** Same as the Nullspace of $A^T$.

**IMPORTANCE?:** Most algorithms and applications of linear algebra are understood by moving these 4 subspaces.

- $C(A), C(A^T)$ and $N(A), N(A^T)$ behave well under elementary row operations!
Ax = \begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix}\begin{bmatrix}x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix} = x_1(\text{column 1}) + x_2(\text{column 2}) + \ldots + x_n(\text{column n}) = b.

Therefore $b \in C(A)$, a linear combination of the columns of $A$. 
What are these 4 subspaces? (continued)

- **Nullspace**: nullspace $N(A)$, a subspace of $\mathbb{R}^n$.

- Solutions to the system $Ax = 0$ are not changed by elimination operations.

  \[
  Ax = \begin{bmatrix}
  \ldots \text{row} & 1 & \ldots \\
  \ldots \text{row} & 2 & \ldots \\
  \ldots \text{row} & m & \ldots 
  \end{bmatrix}
  \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_m 
  \end{bmatrix}
  = \begin{bmatrix}
  0 \\
  0 \\
  0 
  \end{bmatrix}
  \]

  where $x \in N(A)$.

- **Left Nullspace**: nullspace of $A^T$, a subspace of $\mathbb{R}^m$; all solutions to $A^Ty = 0$. 
More on the Nullspace

- The nullspace of a matrix $A$ is nothing else but the set of solutions of a homogeneous system of equations!!
- We can use the elimination procedure until the matrix $A$ is in row echelon form $U$ and then read the answer BUT we can also continue!!
- We can make zeros OVER the pivots (not just below) then we get the **reduced row echelon form** $RREF$
- When we get $RREF$ we get something important: There are special solutions set ONE free variable 1 the others to zero.
- **KEY POINT** Any other solution of $Ax = 0$ is a linear combination of the special solutions!!!
LAST EPISODE OF THIS ADVENTURE WE SAW....
Example Understanding Special solutions of Nullspace

\[
\begin{bmatrix}
0 & 1 & 2 & 1 & 1 \\
1 & 1 & 4 & 1 & 2 \\
0 & 0 & 0 & 1 & 2 \\
1 & -1 & 0 & 0 & 2 \\
2 & 1 & 6 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 4 & 1 & 2 \\
0 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & 2 \\
1 & -1 & 0 & 0 & 2 \\
2 & 1 & 6 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 4 & 1 & 2 \\
0 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & 2 \\
0 & -2 & -4 & -1 & 0 \\
0 & -1 & -2 & -2 & -3
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 4 & 1 & 2 \\
0 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & 2 \\
0 & -1 & -2 & -2 & -3 \\
0 & 0 & 0 & 1 & 2
\end{bmatrix}
\]
We have arrived to row echelon form!!! But now let us keep going!
Find specially solutions!!!
We have arrived to the row REDUCED echelon form. Here is is cleaner to see the SPECIAL SOLUTIONS.
We can write the (basic) pivot variables in terms of the free variables so the solutions are

\[ x_1 = -2x_3 - x_5 \]
\[ x_2 = -2x_3 + x_5 \]
\[ x_3 \text{ free} \]
\[ x_4 = -2x_5 \]
\[ x_5 \text{ free} \]

Now let us write this as a vector!!

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5
\end{pmatrix} =
\begin{pmatrix}
  -2x_3 - x_5 \\
  -2x_3 + x_5 \\
  x_3 \\
  -2x_5 \\
  x_5
\end{pmatrix} =
\begin{pmatrix}
  -2 \\
  1 \\
  0 \\
  0 \\
  0
\end{pmatrix} x_3 +
\begin{pmatrix}
  -1 \\
  1 \\
  0 \\
  -2 \\
  1
\end{pmatrix} x_5
\]

Every vector of the Nullspace is of this form!! A linear combination of TWO special solutions. NOTE they are linearly independent!!
• The number of FREE variables = number of special solutions of the NULLSPACE

• Note the special solutions of the NULLSPACE are linearly independent (because of they have zeros and ones in right spots).

• Note the number of special solutions of the NULLSPACE equals the number of free variables.

• The special Solutions $v_1, v_2, \ldots, v_k$ of $N(A)$ have two key properties
  
  • Every vector of $N(A)$ is a linear combination of them ($span(v_1, \ldots, v_k)$ is $R(A)$)
  • The special solutions are linearly independent.

  They form a BASIS for the NULLSPACE.

• The size of the basis is the DIMENSION OF the Nullspace
We can do the same with the ROW SPACE of the matrix!!

When we row reduce the rows of the Echelon form that have a pivot OR the rows of the reduced row echelon form give a set of SPECIAL SOLUTIONS:

\[
\begin{bmatrix}
1 & 0 & 2 & 0 & 1 \\
0 & 1 & 2 & 0 & -1 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Every vector of the ROW SPACE of \( A \) is a linear combination

\[s \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 1 & 2 & 0 & -1 \end{bmatrix} + p \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]
Thus the **rank** is the number of pivot variables = number of special solutions of the ROW SPACE.

Note the special solutions of the ROW SPACE are linearly independent (because of echelon shape!).

The special Solutions of $R(A)$ have two key properties:
- Every vector of $R(A)$ is a linear combination of them (SPAN is $R(A)$)
- The special solutions are linearly independent

They form a BASIS for the ROW SPACE. Their size is the DIMENSION of the ROW SPACE, which equals the RANK.

**NOTEWORTHY:** Recall the number of special solutions of the NULLSPACE equals the number of free variables.

We see then for an $m \times n$ matrix $A$:
- Number of special solutions of $R(A)$ +
- Number of special solutions of $N(A) = n$.

If you know the rank, then you know the dimension of the Nullspace!!!
Example

Consider the matrix 

\[ A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ 3 & 6 & -9 \end{pmatrix} \]

- Find the special solutions of the (right) Nullspace \( N(A) \). Describe it (what is its dimension? In which space is a subspace?)
- Is \((-3, 0, 1)\) in \( N(A) \)?
- Find the special solutions the ROW Space \( R(A) \). Describe it (what is its dimension? In which space is a subspace?)
- Is \((3, 3, 3)\) in the Row space?
- Describe the COLUMN Space \( C(A) \). Describe it (what is its dimension? In which space is a subspace?)
- What is the LEFT Nullspace \( N(A^T) \)? Describe it (what is its dimension? In which space is a subspace?)

Every answer can be read from the matrix or the RREF.
PLEASE GET AN 8x11 PIECE OF PAPER READY.
WRITE YOUR LAST NAME....
FOLLOW THE INSTRUCTIONS
RELAX...HANDING IN, WINS YOU A POINT
1. **Which of these are subspaces?**
   - Set of all vectors \( \{ (x, y, z, w, v) \in \mathbb{R}^5 | -x - 3y - 3z + 3w + v = 0, 2x + 6y + z + 4w + 3v = 0 \} \)
   - Set of all vectors \( \{ (x, y, z, w, v) \in \mathbb{R}^5 | -x - 3y - 3z + 3w + v = 1, 2x + 6y + z + 4w + 3v = 2 \} \)
   - The set of all continuous functions with \( \int_0^1 f(t)dt = f(1/2) \).

2. **Find all the vector subspaces associated to the matrix**
   \[
   A = \begin{pmatrix}
   0 & 1 & 2 & 3 & 0 & -7 \\
   0 & 1 & 2 & 3 & 0 & -7 
   \end{pmatrix}
   \]
   Describe as well as possible.
   Can you find an example of a vector in each \( C(A), R(A), N(A), N(A^T) \)?

3. **Explain or construct the following:**
   - Find a new vector that is in the subspace spanned by \((2, 1, -1), (0, 0, -1)\). For which values of \(a\) is \((2, a, 10)\) part of that subspace?
   - Construct a matrix \( A \) whose nullspace consists of all linear combinations of \((2, 2, 1, 0)\) and \((3, 1, 0, 1)\).
   - If the nullspace of \( A \) consists of all multiples of \( u = (2, 1, 0, 1) \),
The solutions \( \{ x : Ax = b \} \) is NOT a subspace unless \( b = 0 \).
But it is the TRANSLATION of a subspace!

**NOTE:** if \( x_1, x_2 \) are solutions of \( Ax = b \), then
\[
A(x_1 - x_2) = b - b = 0,
\]
so \( x_1 - x_2 \in N(A) \).

**NOTE:** If \( Au = 0 \) and \( Ax = b \), then
\[
A(x + u) = Ax + Au = b + 0 = b.
\]
Thus \( x + u \) is another solution of \( Ax = b \).

**PUNCH LINE** ALL solutions of \( Ax = b \) can be written in the form \( x_p + u \) where \( x_p \) is one solution of \( Ax = b \) and \( u \) is in the Nullspace.

**Theorem:** Type of solutions of \( Ax = b \)? Read it from the rank of \( A \) of an \( m \times n \) matrix:

1. \( \text{rank} = m = n \) (A square invertible), ONE SOLUTION
2. \( \text{rank} = m \) and \( \text{rank} < n \), INFINITELY MANY SOLUTIONS, dimension \( n - \text{rank} \).
3. \( \text{rank} < m \) and \( \text{rank} = n \) has ZERO or ONE SOLUTION.
4. \( \text{rank} < m \) and \( \text{rank} < n \) has ZERO or INFINITE SOLUTIONS.
Linear Independence, Bases, and Dimension
Remember Definition Let $v_1, v_2, \ldots, v_k$ be vectors on vector space $V$. A **linear combination** is a sum of the form

$$c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$$

with scalars $c_1, \ldots, c_k$. The **span** of the vectors is the set of all possible linear combinations.

**Example:** Consider the subspace $S$ spanned by

$$f_0(x) = 1, \quad f_1(x) = \cos(x), \quad f_2(x) = \sin(x),$$

$$f_3(x) = \cos^2(x), \quad f_4(x) = \cos(x)\sin(x), \quad f_5(x) = \sin^2(x),$$

$$f_6(x) = \cos(2x), \quad f_7(x) = \sin(2x).$$

Is this the smallest **spanning set** for $S$?

**No!** Suffices with just $f_0, f_1, f_2, f_6, f_7$ because

$$\cos^2(x) = \frac{1}{2}\cos(2x) + 1/2, \quad \sin^2(x) = \frac{-1}{2}\cos(2x) + 1/2,$$

$$\cos(x)\sin(x) = \frac{1}{2}\sin(2x)$$

Not all vectors $f_0, \ldots, f_7$ are essential to span subspace! $f_3, f_4, f_5$, redundant.
Recall The vectors \( v_1, v_2, \ldots, v_k \) are linearly dependent if there exist scalars \( c_1, \ldots, c_k \) not all zero such that \( c_1 v_1 + \cdots + c_k v_k = 0 \). Vectors that are not dependent are called linearly independent.

**Example:** Are \( f_0(x) = 1, f_1(x) = \cos(x), f_2(x) = \sin(x), f_3(x) = \cos^2(x), f_4(x) = \cos(x)\sin(x), f_5(x) = \sin^2(x) \) linearly dependent? YES!!

\[
1 \cdot 1 + 0 \cdot \cos(x) + (-1)\cos^2(x) + (-1)\sin^2(x)
\]

**Example** Take \( u_1 = (1, 0, 0), u_2 = (1, 2, 0), u_3 = (0, 0, -3) \), Are they linearly dependent OR INDEPENDENT? Need to know the solutions of homogeneous system of equations \( x_1 u_1 + x_2 u_2 + x_3 u_3 = 0 \). Solve \( Ax = 0 \) with \( A = [u_1 \; u_2 \; u_3] \). Matrix is invertible! Thus, only solution is \( x = (0, 0, 0) \). Thus they are linearly independent.

**Are** \((1, 1, 1), (1, 0, 2)\) and \((4, 3, 5)\) are linearly independent?
How to check DEPENDENCIES practice? (A) Set up a homogeneous system of equations
(B) Check whether the only solution is $c_i = 0$.

**Theorem** Let $v_1, \ldots, v_n$ vectors in $K^m$ and $A = [v_1 \ldots v_k]$ the corresponding $m \times n$ matrix.

1. The vectors are linearly dependent $\iff$ if there is a non-zero solution to $Ax = 0$.
2. The vectors are linearly independent $\iff$ the only solution to the homogeneous system $Ax = 0$ is the trivial one.
3. The vector $b$ lies in the span of $v_1, v_2, \ldots, v_n$ $\iff$ the system $Ac = b$ has at least one solution.

Say we are given some vectors $v_1, v_2, \ldots, v_n$ in $K^m$ if $n > m$ then they must be linearly dependent.

**Definition** A **basis** of a vector space (subspace) $V$ is a set of vectors which are linearly independent and span $V$.

**Examples:** The standard vectors $e_i = (0, 0, \ldots, 1, \ldots, 0)$ form a basis for $\mathbb{R}^m$. The monomials $1, x, x^2, x^3, \ldots, x^r$ are a basis for the vector space of polynomials of degree $\leq r$. 

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MATH 22A: LINEAR ALGEBRA  Chapter 3
Finding a basis for a subspace

Consider the subspace \( W = \text{SPAN}(v_1, v_2, v_3, v_4, v_5) \) where \( v_i \) is the \( i \) column of \( A \)

\[
A = \begin{bmatrix}
1 & -3 & 2 & -3 & 9 \\
2 & 0 & 1 & 3 & 3 \\
-2 & -4 & 1 & -9 & 7 \\
1 & 3 & -1 & 6 & -6 \\
\end{bmatrix}
\]

**Task 1:** Find a basis for \( W \)

\[
\begin{bmatrix}
1 & 2 & -2 & 1 \\
-3 & 0 & -4 & 3 \\
2 & 1 & 1 & -1 \\
-3 & 3 & -9 & 6 \\
9 & 3 & 7 & -6 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & -2 & 1 \\
-3 & 0 & -4 & 3 \\
2 & 1 & 1 & -1 \\
-3 & 3 & -9 & 6 \\
9 & 3 & 7 & -6 \\
\end{bmatrix}
\]
Task 2:

Find a basis for $W$ using the ORIGINAL vectors.

\[
\begin{bmatrix}
1 & -3 & 2 & -3 & 9 \\
2 & 0 & 1 & 3 & 3 \\
-2 & -4 & 1 & -9 & 7 \\
1 & 3 & -1 & 6 & -6
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1/2 & 3/2 & 3/2 \\
0 & 1 & -1/2 & 3/2 & -5/2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

From $QA = RREF$ we see that the first two columns of the original matrix form a basis!!!

**Theorem** Let $S = \{v_1, v_2, \ldots, v_k\}$ be non-zero vectors of vector space $V$. If $W = \text{SPAN}(S)$, then some subset of $S$ is a basis for $W$. 
Important properties

- **Lemma**: There is one and only one way to write a vector in $V$ as a combination of basis vectors.
- **Theorem**: Any two bases of the same vector space $V$ contain the same number of vectors.
- **Definition**: The number of vectors on a basis of the vector (sub)space is called its **dimension**.
- The dimension is an ID number for vector spaces!
- $V$ has a **finite** basis then $V$ is a finite-dimensional vector space.
- **Theorem**: Any linearly set of vectors in $V$ can be extended to a basis, by adding more vectors if necessary.
- **Theorem**: Any spanning set can be reduced to a basis, by discarding unnecessary vectors.
Example

Suppose you are given two vectors in $\mathbb{R}^4$: $v_1 = (1, 0, 1, 0)$ and $v_2 = (-1, 1, -1, 0)$. Add two more vectors that will make a BASIS for $\mathbb{R}^4$, HOW?:

$$
\begin{bmatrix}
1 & -1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$
True or False??

1. $u = (1, 1, 1), \ v = (1, 0, -1)$ is a basis for $\mathbb{R}^3$.  
2. $u = (1, 1, 1), \ v = (1, 0, -1)$ is a basis for the subspace $W = \text{span}(u, v)$.  
3. Four vectors in $\mathbb{R}^2$ are always dependent.  
4. A subspace has just one basis.  
5. If the columns of a matrix are dependent, so are the rows.  
6. The columns of a matrix are a basis for the column space.  
7. The column space is the same as the row space for any 2 by 2 matrix.  
8. $(1, 2, 2) (-1, 2, 1), (0, 8, 6)$ is a basis for $\mathbb{R}^3$.  
9. Suppose $S$ is a 5-dimensional subspace of $\mathbb{R}^6$. Every basis of $S$ can be extended to become a basis of $\mathbb{R}^6$.  
10. Suppose $S$ is a 5-dimensional subspace of $\mathbb{R}^6$. Every basis of $\mathbb{R}^6$ can be reduced to become a basis of $S$. 

Jesús De Loera, UC Davis MATH 22A: LINEAR ALGEBRA Chapter 3
The Four Fundamental Subspaces Strike Back
Recall we have 4 important spaces:

First we have the **column space** of linear combinations of columns:

\[
Ax = \begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix} = 
\]

\[x_1(\text{column 1}) + x_2(\text{column 2}) + \ldots + x_n(\text{column n}) = b.\]

**TWIN:** the **row space** of linear combinations of the rows of A.
• Nullspace: nullspace $N(A)$, a subspace of $\mathbb{R}^n$.

• Solutions to the system $Ax = 0$ are not changed by elimination operations.

$$Ax = \begin{bmatrix}
- - \text{row} & 1 - - \\
- - \text{row} & 2 - - \\
- - \text{row} & m - - \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_m \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix},$$

where $x \in N(A)$.

• Left Nullspace: nullspace of $A^T$, a subspace of $\mathbb{R}^m$; all solutions to $A^T y = 0$. 
Everything is a beautiful by-product of ROW REDUCTION!
Say \( A \) is an \( m \times n \) matrix and \( R \) is its RREF with \( r \) pivots (\( rank = r \))

The \( r \) pivots of \( R \) are a basis for the row space of \( A \). Row space of \( R \) same as that of \( A \).

The \( r \) pivot columns of \( A \) (!!!) are a basis of the column space.

The \( n - r \) special solutions are a basis for the Nullspace of \( A \) (\( R \) same thing)

If \( EA = R \), where \( E \) is the matrix that gives the reduction, then the last \( m - r \) rows of \( E \) are a basis for the LEFT nullspace of \( A \).
CHALLENGE 1  Find bases for each of the subspaces associated with $A$:

$$
A = \begin{bmatrix}
1 & 0 & 0 \\
6 & 1 & 0 \\
9 & 8 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & 1 & 2 & 3 \\
0 & 0 & 1 & 2
\end{bmatrix}
$$

Find the rank of the $8 \times 8$ checkerboard matrix (put a 0 in the white positions, 1 in the black positions)

True or False?

1. $A$ and $A^T$ have the same rank
2. If the row space equals the column space then $A^T = A$.
3. The matrices $A$, $-A$ share the same subspaces.
4. If $A$ and $B$ have the same subspaces then $A$ is a multiple of $B$. 
**Fundamental Theorem** Let $A$ be an $m \times n$ matrix. Then dimensions of the **fantastic 4 subspaces** obey the laws:

- $\dim \ C(A) = \dim \ C(A^T)$.
- Row space and nullspace in $\mathbb{R}^n$, and $\dim \ C(A^T) = r$ then $\dim \ N(A) = n - r$.

\[
\dim(C(A^T)) + \dim(N(A)) = n.
\]

- Similarly for the Column space and left nullspace in $\mathbb{R}^m$, so if $\dim \ C(A) = r$, then $\dim \ N(A^T) = m - r$.

\[
\dim(C(A)) + \dim(N(A^T)) = m.
\]
Everything is a beautiful by-product of ROW REDUCTION!
Say $A$ is an $m \times n$ matrix.

**Lemma:** The row space of $A$, $C(A^T)$, has the same dimension as the row space of the echelon form OR the reduced echelon form. Thus $\dim(C(A^T)) = \text{rank}(A)$.

**Lemma:** The Nullspace $N(A)$ has dimension $n - \text{rank}(A)$
Why?
Because the special solutions (a free variable has value 1 while the others free variables are zero) give linearly independent solutions that generate $N(A)$.

**Lemma:** The dimension of the column space $C(A)$ equals $\text{rank}(A)$ which also equals the dimension of the row space!
WHY?
Because the row reduction is same as multiplying $A$ with an invertible square $Q$. If $R$ is the RREF of $A$, then the columns corresponding to pivots of $R$ are mapped by $Q^{-1}$ to independent vectors!

**DONT FORGET:** YOU will find bases for each of these 4 spaces in the midterm!!!!!!!!