Vectors and Matrices (1.1-1.3).
CHAPTER 1
Vectors and Linear Combinations 1.1

Fig. 3. Parallelogram of forces.
We can think of VECTORS, as ordered \( n \)-tuples of real numbers. Each entry is a **component**. We often write them vertically! They are points in \( \mathbb{R}^n \).

\[
U = \begin{pmatrix} 1 \\ 12 \\ -7 \end{pmatrix} \quad U^T = (1 \quad 12 \quad -7)
\]

- **Vectors can be added** Get a vector again!!
- **Vector can be multiplied by a number** Get a vector again!!
- **Definition** For a finite set of vectors \( v_1, v_2, \ldots, v_k \), and numbers \( \lambda_1, \lambda_2, \ldots, \lambda_k \) a **linear combination** is the vector obtained as

\[
\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k
\]

The result is obtained **operating one component** at a time!!

\[
\begin{pmatrix} 1 \\ 12 \\ -7 \end{pmatrix} + 7 \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \\ -7 \end{pmatrix} + \begin{pmatrix} 21 \\ -7 \\ 7 \end{pmatrix} = \begin{pmatrix} 22 \\ 5 \\ 0 \end{pmatrix}
\]
Addition of vectors is then a commutative operation: 
\[ A + B = B + A. \]

There is a geometric interpretation of adding two vectors:

What is \( A - B \) in the picture?
Addition of vectors is then a commutative operation: \( A + B = B + A \).

There is a geometric interpretation of adding two vectors:

What is \( A - B \) in the picture?
Addition of vectors is then a commutative operation:
\[ A + B = B + A. \]

There is a geometric interpretation of adding two vectors:

What is \( A - B \) in the picture?
QUESTION What happens if we take ONE vector \( A \) and compute all its linear combinations \( \lambda A \)? What is the picture?

QUESTION Take two vectors \( A, B \) in the plane, compute all its linear combinations \( \lambda A + \mu B \) with coefficients \( 0 \leq \lambda, \mu \leq 1 \). What is the resulting picture?

QUESTION Think of the 12 vectors that go from the center of a clock to the hours 1:00, 2:00, \ldots, 12:00. What is the sum of these vectors?

QUESTION Take two vectors \( A, B \) in the plane all its linear combinations \( \lambda A + \mu B \) with coefficients \( \lambda, \mu \) INTEGER numbers, how the linear combinations look like?

QUESTION Take two vectors \( A, B \) in the plane all its linear combinations \( \lambda A + \mu B \) with coefficients \( \lambda, \mu \) any real numbers? What is the end result?
QUESTION Take three vectors $A, B, C$ in the SPACE all its linear combinations $\lambda A + \mu B + \gamma C$ with coefficients $0 \leq \lambda, \mu, \gamma \leq 1$, what is the picture?

QUESTION Take three vectors $A, B, C$ in the SPACE all its linear combinations $\lambda A + \mu B + \gamma C$ with coefficients $0 \leq \lambda, \mu, \gamma \leq 1$, what is the picture?

QUESTION What do all linear combinations of the vectors
\[
\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\] in 3-D generate ??

QUESTION What happens if you add the vector $\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$, How does your answer change?
- **Wish:** to measure the length of a vector, or the angle between two vectors

- **Definition** The dot product of two vectors

\[ p = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}, \quad q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} \]

equals

\[ p \cdot q = p_1q_1 + p_2q_2 + \cdots + p_nq_n = q \cdot p \]

- **Definition** The length of a vector is defined as

\[ ||v|| = \sqrt{v \cdot v} \]
Proposition 1: When \( v, u \) are perpendicular vectors, their dot product equals zero. **WHY?** Give a justification!!

We can use the dot product to compute the angle between two vectors (**WHY?**):

\[
A \cdot B = ||A|| ||B|| \cos(\theta)
\]

In other words, \( A \cdot B = ||A|| ||B|| \cos(\theta) \).

- Dot product is NEGATIVE angle is above \( 90^\circ \).
- Dot product is POSITIVE angle is below \( 90^\circ \).

**WHY IS THIS TRUE?** Basic trigonometry!!
The quantity $|\frac{A \cdot B}{||A||||B||}|$ never exceeds ONE, because the cosine never does!! thus

**Schwarz Inequality:** $|A \cdot B| \leq ||A||||B||$

The **Triangle inequality** Length of $(A + B)$ is less than or equal to the length of $A$ plus length of $B$

\[
||A + B||^2 = (A + B) \cdot (A + B) = A \cdot A + A \cdot B + A \cdot B + B \cdot B = ||A||^2 + 2(A \cdot B) + ||B||^2 \quad \text{because } A \cdot B = B \cdot A
\]

\[
\leq ||A||^2 + 2(||A|| ||B||) + ||B||^2 = (||A|| + ||B||)^2 \quad \text{Schwarz Ineq.}
\]

Thus $||A + B||^2 \leq (||A|| + ||B||)^2$. Taking (positive) square root gives

\[
||A + B|| \leq ||A|| + ||B||
\]
The Law of Cosines: Justify with vectors!!!

\[ c^2 = a^2 + b^2 - 2ab \cos(\gamma) \]

HOW CAN YOU SHOW THE LAW OF COSINES IS CORRECT USING VECTOR PROPERTIES?
LAST EPISODE OF THIS ADVENTURE WE SAW....
Lines, Planes, and HYPER-planes, a first taste!!

- Two points define a line (ONE dimensional object)
  - in $\mathbb{R}^2$ given by a single equation $ax + by = q$, but needs more in higher dimension!!

- Three non-collinear points define a plane (TWO dimensional object)
  - in $\mathbb{R}^3$ is given by a single equation $ax + by + cz = q$, again, you needs more equations to describe the same object in higher dimension!!

- Four non-collinear or co-planar points in 4-dimensions or more define a 3-hyperplane (THREE dimensional object)
  - inside $\mathbb{R}^4$ given by one equation $ax + by + cz + dw = q$!!

Challenge: Given the points how do you find the equation of the line, plane or hyperplane that contains them?

- NEED to find the coefficients a,b,d,c,q, etc.
You are given points (2,3,−4) and (3,−2,5) find the parametric line equations that describe the line $x = 2 + t, \ y = 3 - 5t, \ z = -4 + 9t$.

You are given points (2,−2,1), (−1,0,3), and (5,−3,4) what is the equation of the plane that contains them?

We are looking for $ax + by + cz = d$ that contains those points! Need to find $a, b, c, d$. HOW? Learn this in 21D!!

You can use cross-product

But suppose instead I give you (7,2,−3,9), (3,4,1,−2), (1,0,3,−5), (1,1,0,2). How to find $ax + by + cz + dw = q$??

KEY IDEA Each point must satisfy the equation $ax + by + cz + dw = q$, each point gives ONE linear equation!! Set up a system of linear equations with variables $a, b, c, d, q$ and then solve it the system.

The system is now:

$$7a + 2b - 3c + 9d - q = 0,$$
$$3a + 4b + c - 2d - q = 0,$$
$$a + 3c - 5d - q = 0,$$
$$a + b + 2d - q = 0.$$
NOTE: This is the same as saying: there are numbers $a, b, c, d$ such that

$$
a \begin{bmatrix} 7 \\ 3 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} -3 \\ 1 \\ 3 \\ 0 \end{bmatrix} + d \begin{bmatrix} 9 \\ -2 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} q \\ q \\ q \\ q \end{bmatrix}
$$

If the 4 points $(7, 2, -3, 9), (3, 4, 1, -2), (1, 0, 3, -5), (1, 1, 0, 2)$ are in a hyperplane the vector $(q, q, q, q)^T$ is a linear combination of the 4 vectors you get by looking at the first, second, third, fourth entries of the original points!!!
• To find a non-trivial solution we can add the condition that $q = 1!!$ The new system is

\[
\begin{align*}
7a + 2b - 3c + 9d - q &= 0, \\
3a + 4b + c - 2d - q &= 0, \\
a + 3c - 5d - q &= 0, \\
a + b + 2d - q &= 0, \quad \text{AND} \quad q = 1.
\end{align*}
\]

• We can solve any system of linear equations using MATLAB. Create a MATRIX for the system:

\[
\begin{bmatrix}
7 & 2 & -3 & 9 & -1 \\
3 & 4 & 1 & -2 & -1 \\
1 & 0 & 3 & -5 & -1 \\
1 & 1 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

• The right-hand side vector we want to solve for is $b = (0, 0, 0, 0, 1)^T$. Solution is $(0, 1/5, 1, 2/5, 1)$. 

Jesús De Loera, UC Davis MATH 22A: LINEAR ALGEBRA Chapter 1
• **Question:** If given two vectors in the plane, \((a, b)\) and \((c, d)\), when are they giving the same line through the origin? \((a, b)\) must be a multiple of \((c, d)\): \((a, b) = \lambda(c, d)\) or \((a, b) - \lambda(c, d) = 0!\)

• **Question:** Two vectors \((p, q, r)\) \((u, v, w)\) in \(\mathbb{R}^3\) define a plane \(P\) through the origin \((0, 0, 0)\), if I give you a new vector \((e, f, g)\) **How do I know when \((e, f, g)\) is in \(P\)?**

• Say the plane has equation \(aX + bY + cZ = 0\), for some unknown numbers, \(a, b, c\) (not all zero!!) then it must be the case that

\[
\begin{bmatrix}
  p \\
  u \\
  e
\end{bmatrix} + b\begin{bmatrix}
  q \\
  v \\
  f
\end{bmatrix} + c\begin{bmatrix}
  r \\
  w \\
  g
\end{bmatrix} = 0
\]
Definition We say that vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are **linearly dependent** if there are numbers $\lambda_1, \ldots, \lambda_k$, not all zero, such that

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_k \mathbf{v}_k = 0$$

There is a linear combination that gives the zero vector!!!

If such numbers are impossible to find then we say the vectors are **linearly independent** E.g., $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.

We just saw that vectors are all in the same line or plane precisely when they are linearly dependent!!

HOW to find such numbers $\lambda_i$? again we are solving a system of linear equations!
A **MATRIX** is an $m \times n$ array of numbers. It has $m$ **rows** and $n$ **columns**

\[
\begin{bmatrix}
1 & 3 & 2 & -1 & 0 \\
2 & 6 & 1 & 4 & 3 \\
-1 & -3 & -3 & 3 & 1 \\
3 & 9 & 8 & -7 & 2
\end{bmatrix}
\]

**NOTE:** Vectors are matrices ( $n \times 1$ matrices).

Matrices can be thought of as groups of vectors! E.g., the above matrix is made of FIVE vectors inside 4-dimensional space.
What is a linear combination in terms of matrices?
Multiplication of a matrix and a vector!!

We think of \( \lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_n p_n \) as a matrix with columns \( A = [p_1 \; p_2 \; \ldots \; p_n] \) multiplied by the vector.

Define \( Ax = \lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_n p_n \)

Example:

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{pmatrix}
\begin{pmatrix}
9 \\
8 \\
7
\end{pmatrix}
\]

final matrix will be 2x1

2x3 \hspace{0.5cm} 3x1

3=3 so proceed
Now we can use this to multiply two matrices $A, B$. Clearly some things must match!!

**Key Point** Multiplication of two matrices can be done vector by vector!! IF the sizes of the matrices match!!

For a $m \times n$ matrix $A$ and an $n \times p$ matrix $B$ HOW TO MULTIPLY THEM?

Say $B$ has column vectors $B = [v_1 \ v_2 \ v_3 \ \ldots \ v_p]$ The new matrix $AB$ is a matrix with columns (matching the right size for $A$).

$$AB = [Av_1 \ Av_2 \ Av_3 \ \ldots \ Av_p]$$

**Example**

NO! we cannot multiply any pair of matrices AND $AB$ not always equal to $BA$