Problem 2: If a permutation $a_1a_2\cdots a_n$ has inversion table $(b_1, b_2, \cdots b_n)$ what is the permutation that corresponds to the inversion table $(n-1-b_1, n-2-b_2, \cdots 0-b_n)$?

The answer is the "reverse" permutation $a_na_{n-1}\cdots a_1$, our original permutation written backwards.

Let $k = a_j$ in the permutation $a_1a_2\cdot a_n$. Then b_k denotes the number of elements in the set $\{a_i— a_i > k$ and $i < j\}$. Thus given k, there are b_k elements to the left of k that are bigger than k in the permutation $a_1a_2\cdots a_n$. Note that in total, there are $n-k$ elements which are greater than k in the permutation, since the entries of the permutation run from 1 to n. Thus, it is implied that there are $n-k-b_k$ elements to the right of k which are greater than k. Now consider what happens when we reverse this permutation. Every element that was to the left of k is now to the right of k, which implies that in the reverse permutation there are now $n-k-b_k$ elements to the left of k bigger than k which means that the k^{th} entry of the inversion table for the reverse permutation is $n-k-b_k$, which was to be shown. Note that this is the unique answer becuase each permutation has a unique inversion table, as was shown in class.

Problem 3

0.1. **solutions.** We have recurrence relation that implies that $A(x)-x = 4xA(x)-5x^2A(x)$, thus we get a rational generating function

$$A(x) = \frac{1}{1-4x+5x^2}$$

One needs to find the partial fraction decomposition of the right hand side. At the end one can recover an expression

$$A(x) = \frac{i}{2}\sum_{n\geq 0}\frac{x^n}{a^n} - \frac{i}{2}\sum_{n\geq 0}\frac{x^n}{b^n}$$

where $a = 0.4+0.2i$ and $b = 0.4-0.2i$. We now can find the coefficient of x^n in a_n to be

$$a_n = \frac{i}{2}(\frac{1}{a^n} - \frac{1}{b^n}).$$

Other formulas can be extracted from this one.

Problem 4

Using generating functions, find an explicit formula for $a_n = na_{n-1} + (-1)^n$ and $a_0 = 1$.

Let

$$f(x) = \sum_{n=0}^{\infty}\frac{a_nx^n}{n!}$$

$a_0 = 1$

$$f(x) = \sum_{n=1}^{\infty}\frac{a_nx^n}{n!} + 1$$

Substituting $na_{n-1} + (-1)^n$ for a_n, we have:

$$f(x) = \sum_{n=1}^{\infty} \frac{na_{n-1}x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{a_{n-1}x^n}{(n-1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

$$= x\sum_{n=0}^{\infty} \frac{a_n x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

$$= xf(x) + e^{-x}$$

So $f(x)$ is the exponential generating function times a geometric sum:

$$f(x) = \frac{e^{-x}}{1-x} = (\sum_{n=0}^{\infty} \frac{(-x)^n}{n!})(\sum_{n=0}^{\infty} x^n) = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} \frac{(-1)^k}{k!}) x^n$$

$$\frac{a_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$a_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

1. Problem 5

What is the number of Permutations in S_n that there is no triple $i < j < k$ with $\pi(j) < \pi(i) < \pi(k)$?

1.1. **Solution.** Let $\bar{\pi}$ denote the number of those integers $1 \leq j \leq i$ with $\pi(j) \geq \pi(i)$. We will use this variation of inversion to provide the answer. If there is no triple $i < j < k$ with $\pi(j) < \pi(i) < \pi(k)$, then each $i < j$ such that $\pi(i) > \pi(j)$ also satisfies $\pi(i) > \pi(j+1)$; thus $\bar{\pi}$ is a monotone increasing vector. Conversely if $\bar{\pi}$ is monotone, then there no triple $i < j < k$ with $\pi(j) < \pi(i) < \pi(k)$. To see this proceed by contradiction. Consider the portion $\pi(j), \pi(j+1), \ldots, \pi(k)$. In this sequence there must be two consecutive terms $\pi(l), \pi(l+1)$ such that $\pi(l) < \pi(i) < \pi(l+1)$. Then for any $v < l+1$ with $\pi(v) > \pi(l+1)$ also satisfies $\pi(v) > \pi(l)$. Moreover i itself satisfies $i < l$ and $\pi(i) < \pi(l+1)$. Thus $\bar{\pi}(l) > \bar{\pi}(l+1)$, a contradiction.

Thus, the number we seek is equal to the numbe of monotone mappings of $\{1, \ldots, n\}$ into itself such that $1 \leq \phi(i) \leq i$.

We have counted a similar number before, but with the second condition we get $\frac{1}{n+1}\binom{2n}{n}$.

2. PROBLEM 6

For two generating functions f, g let N, M be the least-order non-zero term indexes for f, g respectively. Then

$$f = x^N \sum_{n=0} a_{n+N} x^n,\ g(x) = x^M \sum_{m=0} b_{m+M} x^m$$

and a_N, b_M are non-zero. Therefore when we multiply fg.

$$fg := x^{N+M} \sum c_n x^n$$

where $c_n = \sum_{k=0}^{n} a_{k+N} b_{n-k+M}$. In particular $c_0 = a_N b_M$ so fg is not zero. Hence the ring of formal power series is an integral domain. Its quotient field is the field of formal Laurent series.

3. PROBLEM 7

Solutions for this problem are in most books in basic combinatorics. I recommend learning about Prüfer codes.

4. PROBLEM 8

4.1. **solutions.** We first we find a recurrence for the Bell numbers. Let S be the set to be partitioned and $x \in S$. If the class containing x has k elements, it can be chosen in $\binom{n-1}{k-1}$ ways and the remaining $n-k$ elements can be partitioned in B_{n-k} ways. So the number of partitions in which the class containing x has k elements is $\binom{n-1}{k-1} B_{n-k}$. This remains true for $k = n$ if we set $B_0 = 1$. Thus

$$B_n = \sum_{k=1}^{n} \binom{n-1}{k-1} B_{n-k} = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$$

Next we prove that the exponential generating function of B_n is

$$p(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = e^{e^x - 1}$$

We saw a fast solution using the composition of generating functions. But we can also use direct elementary calculations.

$p(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$

This is equal to

$1 + \sum_{k=0}^{\infty} \frac{B_k}{k!} \sum_{n=k+1}^{\infty} \frac{x^n}{n} \frac{1}{(n-k-1)!}$, and thus for the derivative

$p'(x) = \sum_{k=0}^{\infty} \frac{B_k}{k!} \sum_{n=k+1}^{\infty} \frac{x^{n-1}}{(n-k-1)!} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!} \sum_{r=0}^{\infty} \frac{x^r}{r!} = p(x)e^x$

In other words $(log(p(x)))' = \frac{p'(x)}{p(x)} = e^x$, and $p(x) = e^{e^x + c}$. For some constant c, which can be determined by setting $x = 0$:

$1 = p(0) = e^{e^0 + c}$, $c = -1$. Thus we have $p(x) = e^{e^x - 1}$.