1. Here are other very cool areas of application of Commutative Algebra.

• (Graph Theory) With any labeled graph $G$ on the set of vertices $V = \{x_1, ..., x_n\}$ we associate the polynomial

$$P_G = \prod_{i < j, x_i, x_j \in E(G)} (x_i - x_j).$$

A graph $G$ is $k$-colorable if $k$ colors are sufficient to color its vertices in such a way that no two vertices of the same color are adjacent. Let $f(x) = x^k - 1$ and consider the polynomial ideal $C(f, n, k)$ in $C[x_1, ..., x_n]$ generated by the polynomials $f(x_1), f(x_2), ..., f(x_n)$. Prove that the graph $G$ is not $k$-colorable if and only if $P_G$ belongs to the ideal $C(f, n, k)$.

• (Algebraic Geometry) The twisted cubic curve $V$ is given in parametrized form $x = t, y = t^2, z = t^3$. Describe the curve implicitly, namely find polynomials $F_i(x, y, z), i = 1..m$ such that $V = \{a \in C^3 : F_i(a) = 0 i = 1...m\}$. Prove that the set of polynomials that vanish at all points of $V$ is an ideal. Is this last statement true for any subset $V$ of $C^n$ or just for the twisted cubic?

2. Using your favorite book remind yourself of the definitions of Euclidean rings, Integral domains, Principal ideal domains (PID), Unique factorization domains (UFD) and prove any implications among these notions (if any). Find counterexamples when implications do not hold.

3. Prove or disprove: a) If $A$ is a UFD then $A[t]$ (the ring of polynomials in variable $t$ with coefficients in $A$) is a UFD.

b) Let $K$ be a field, $K[t]/<t^n>$ is isomorphic to $Z/<p^n>$ for $n \geq 2$.

c) Every ring has a quotient field.

4. Describe the ideals of the rings $Z[\sqrt{-3}], C[x, y]/<y^2 - x^3>$. 

1