Algebra, Math 250B
Final (Take-home) Due Friday March 24th 10:00AM

1. You can discuss the problems with me or with MAPLE, but not with other students without me being present too. You may use your notes or any book you wish, but cite references used, and include *complete* proofs (the only exception is stuff I proved in class and it is in your notes).

2. Write each problem in a separately, do not use same sheet for two problems. Try to use numbering of pages and both sides of paper!

3. Write neatly. Justify your answers very well! In particular, attach any MAPLE code you used to solve the problems.

Material in this test covers all the course.

1. Prove or disprove:

   a) If A is a Boolean ring, ($a^2 = a$ for all $a \in A$), the ring A is a commutative.

   b) $\mathbb{Z}[\sqrt{-5}]$ is a PID.

   c) The fields $\mathbb{Q}[\sqrt{7}]$ and $\mathbb{Q}[\sqrt{11}]$ are isomorphic.

   d) For any prime p, when F_{p^n} is a subfield of F_{p^m} then n must divide m.

   e) There are at least 4 ideals inside $F_5[x]/(x^2 + x - 6)$.

   f) Every primary ideal is indecomposable.

   g) Every (non-zero) homomorphism image of a local ring is local

   h) Every subring of $\mathbb{Q}[x, y]$ is a unique factorization domain.

   i) The polynomial $x^4 + x^3 + x + 1$ is irreducible over $F_3[x]$.

   j) Let R be a PID and $I \subset R$ and ideal. Then every ideal in the quotient R/I is a principal ideal.

2. For this exercise set $f = x^9 - 1$.

   a) Let E be the splitting field of f over $\mathbb{Q}$. What is $[E : \mathbb{Q}]$?

   b) Let F be the splitting field of f over $\mathbb{Q}[i]$ What is $[F : \mathbb{Q}[i]]$?

   c) Let L be the splitting field over F_3. What is $[L : F_3]$?

3. For every prime p, find the Galois group over $\mathbb{Q}$ of $x^5 - 5p^4x + p$. Comment on what could happen if p is not prime.

4. Ideals on polynomial rings: a) What are the ideals for the ring $\mathbb{R}[x, y]/ < x^2, y^2 >$? Is that ring isomorphic to the ring $\mathbb{R}[x, y]/ < x, y^3 >$? Are they isomorphic as real vector spaces?

   b) Suppose I is an ideal such that $K[x_1, \dots, x_n]/I$ is a finite K-vector space. Prove that $dim_K(K[x_1, \dots, x_n]/\sqrt{I})$ is finite too.

c)Let I be an ideal of $K[x, y]$ generated by some polynomials of degree at most d. Prove that, as one changes monomial orders, I always has finitely many distinct reduced Gröbner bases.

5. Using the univariate polynomial $f(x) = x^k - 1$ construct the polynomial ideal $C(f, n, k)$ in $\mathbb{C}[x_1, ..., x_n]$ generated by the polynomials $f(x_1), f(x_2), ..., f(x_n)$.

   a) Prove the ideal $C(f, n, k)$ is radical. Is it prime?

   b) What is $dim_{\mathbb{C}}(\mathbb{C}[x_1, ..., x_n]/C(f, n, k))$?

   c) With any labeled graph G on the set of vertices $V = \{x_1, ..., x_n\}$ we associate the polynomial

$$P_G = \prod_{i<j, x_i x_j \in E(G)} (x_i - x_j).$$

   A graph G is *k-colorable* if k colors are sufficient to color its vertices in such a way that no two vertices of the same color are adjacent. Prove that a graph G is not k-colorable if and only if P_G belongs to the ideal $C(f, n, k)$ and provide and algorithm to check this.

6. a) Let $A : V \to V$ is a linear map on a finite dimensional vector space V. Suppose A has characteristic polynomial $x^2(x-1)^4$ and minimal polynomial $x(x-1)^2$. What is the dimension of V? What are the possible Jordan forms of A?

   b) For the integer matrix

$$\begin{bmatrix} -2 & 162 & 162 \\ -51 & 89 & 89 \\ 129 & 47 & 47 \end{bmatrix}$$

   Determine exactly which integer vectors (c_1, c_2, c_3) are in the image of the associated module map from $\mathbb{Z}^3$ into itself.